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Adiabatic theory: stability of systems with increasing gaps

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ABSTRACT. – For hamiltonians of the form $H(t) = H_0 + V(t)$, where H_0 has increasing gaps in its spectrum and $V(t)$ is bounded and smooth, we prove two results. First, for the case when H_0 has discrete spectrum and $V(t)$ is periodic in time we extend previous results concerning the absence of absolutely continuous spectrum of the corresponding Floquet operator. Our result cover the case of N-dimensional pulsed rotators. Second, we give upper bounds on the increase of the mean energy. In the particular case of N-dimensional pulsed rotator, if $V(t)$ is n times differentiable, $n \geq 2$, then

$$(\Psi_t, H(t)\Psi_t) \leq const.t^{2/n}.$$

Key words: Schrödinger operators, time dependent perturbation quantum stability.

RÉSUMÉ. – Nous démontrons deux résultats concernant les hamiltoniens de la forme : $H(t) = H_0 + V(t)$, où H_0 a des lacunes spectrales croissantes et $V(t)$ est un opérateur régulier et borné. Tout d'abord, si H_0 possède un spectre discret et si $V(t)$ est périodique en temps, nous étendons à l'opérateur de Floquet correspondant certains critères d'absence de spectre absolument continu. Le cas du rotateur pulsé en dimension N est couvert par ce résultat.

Ensuite, nous donnons une borne sur la croissance temporelle de l'énergie moyenne. Dans le cas particulier d'un rotateur pulsé de dimension N et d'un potentiel $V(t)$ n-fois différentiable en temps, nous obtenons, pour $(n \geq 2)$:

$$(\Psi_t, H(t)\psi_t) \leq const.t^{2/n}.$$

1. INTRODUCTION AND RESULTS

In this note we consider the problem of "stability" (see [1], [2], [3], [4], [6], [7] and references therein) for systems described by hamiltonians of the form

$$H(t) = H_0 + V(t) \tag{1.1}$$

where $V(t)$ is bounded and smooth and H_0 has increasing gaps in its spectrum. By stability we mean that some properties of the evolution given by H_0 are in some sense stable against time dependent perturbations.

Let H_0 be a positive (self-adjoint) operator in a Hilbert space $\mathcal{H}$. Suppose that

$$\sigma(H_0) = \cup_{j=1}^{\infty} \sigma_j$$

and denote

$$\Delta_j = dist(\sigma_{j+1}, \sigma_j)$$

$$m_j = multp\ \sigma_j\ (= dim\ Ran\ P_j)$$

where P_j are the spectral projections of H_0 corresponding to σ_j,

$$d_j = \sup_{\lambda,\mu \in \sigma_j} \mid \lambda - \mu \mid .$$

Let $V(t)$ be a uniformly bounded family of self-adjoint operators. By $V \in \mathcal{C}^n$ we shall mean that $V(t)$ is n times norm differentiable with continuous n-th derivative. (We impose norm differentiability for the sake of the reader not interested in mathematical subtelties; all the results below hold true if norm differentiability is replaced by strong differentiability.) In what follows $U(t)$ denotes the evolution generated by $H(t)$:

$$i\frac{d}{dt}U(t) = H(t)U(t); \quad U(0) = 1. \tag{1.2}$$

We shall prove two results. First, assume that H_0 has discrete spectrum (*i.e.* $m_j < \infty$) and that $V(t)$ is periodic:

$$V(t) = V(t + 2\pi). \tag{1.3}$$

Let M be the monodromy matrix (Floquet operator) associated with $H(t)$:

$$M = U(2\pi). \tag{1.4}$$

Obviously, if $V(t) \equiv 0$ then the corresponding monodromy matrix, $M_0 = e^{-2\pi i H_0}$ has (generically dense) pure point spectrum. To prove that for $V(t) \not\equiv 0$, M has also pure point spectrum is a difficult matter and it has been accomplished only in some particular cases (see [1], [2], [3], [4], [5], [6] [7] and references therein). The problem is that the pure point spectrum is very unstable against perturbations. On the contrary the singular singular spectrum is much more stable and the result below (generalising earlier results in [6], [8], [9], [10], [11]) shows that M has no absolutely continuous spectrum as far as $\Delta_j \sim j^\alpha$, $\alpha > 0$; $m_j \sim j^\beta$, $\beta < \infty$ and $V(t)$ is smooth enough.

THEOREM 1. – *Suppose* $V \in C^n$ (i.e. $\max_{l=1,2\dots,n}$ $sup_{t\in\mathbf{R}}$ $\| (d/dt)^l V(t) \| < \infty$), $n < \infty$),

$$\sum_{j=1}^{\infty} \frac{1}{\Delta_j} < \infty \tag{1.5}$$

and

$$\sum_{j=1}^{\infty} \frac{m_{j+1}}{\Delta_j^n} < \infty \tag{1.6}$$

Then M has no absolutely continuous spectrum.

In order to compare with previous results we give the following consequence of Theorem 1:

COROLLARY 1. – *Suppose*

$$\Delta_j \geq a j^\alpha, \tag{1.7}$$

$$m_j \leq A j^\beta \tag{1.8}$$

for some $a > 0$, $\alpha > 0$, $A < \infty$, $0 \leq \beta < \infty$, *and* $V \in C^n$, *with*

$$n \geq \left[\frac{1+\beta}{\alpha}\right] + 1 \tag{1.9}$$

where [ ] stands for the integer part. Then M has no absolutely continuous spectrum.

An example (which partly motivated this note, see the discussion in [11]) covered by Corollary 1 is the pulsed rotator in N dimensions. Let S^N be the unit sphere in $\mathbf{R}^{N+1}$,

$$S^N = \left\{ x_0, \dots, x_n \mid \sum_{i=0}^{N} x_i^2 = 1 \right\}$$

endowed with the metric induced by the standard metric in $\mathbf{R}^{N+1}$, and $-\Delta_N$ be the Laplace-Beltrami operator in $L^2(S^N)$. For $N = 1$, $-\Delta_1 = -\frac{d^2}{d\theta^2}$ in

$L^2(0, 2\pi)$ with periodic boundary conditions, and for $N = 2$, $-\Delta_2 = J^2$ where J^2 is the square of the usual angular momentum. The operators $-\Delta_N$ have discrete spectrum, more precisely the eigenvalues

$$\lambda_j^N = j(j + N - 1);\ j = 0, 1, ...$$

with multiplicities

$$m_j^N = \frac{1}{N!}((j+1)...(j+N) - (j-1)j...(j+N-2)).$$

It follows that as $j \to \infty$, $\lambda_j^N \sim j^2$, $m_j^N \sim j^{N-1}$, and as a consequence $-\Delta_N$ satisfy (1.7), (1.8) with

$$\alpha^N = 1,\ \ \beta^N = N - 1. \tag{1.10}$$

PROPOSITION. – *Let $V \in \mathcal{C}^{N+1}$. Then the monodromy matrix associated with $-\Delta_N + V$ has no absolutely continuous spectrum.*

Remarks.

(i) For the origin of the problem the reader is relegated to Howland [6], [8], [9]. In [8] he proved the result for $m_j = 1$. The generalisation of the proof to cover the case $\beta = 1$ is outlined in [6]. Some results for unbounded $V(t)$ are contained in [9].

(ii) Independently of [9], the problem of removing the nondegeneracy condition was considered in [10]. While following Howland in the main ideea of proof (the use of trace class perturbation theory) the main aim of [10] was to point out that the adiabatic iterations formalism (see e.g. [12], [13] and references therein) is quite effective in dealing with the degeneracy problem. Although not stated in full generality, the result in [10] is that if $V \in \mathcal{C}^2$ (*i.e.* one iteration can be performed) M has no absolutely continuous spectrum, as far as $2\alpha - \beta > 1$. This requires $\alpha > 1/2$, but notice that for $\alpha > 1$ some values $\beta > \alpha$ are allowed. It was also mentioned in [10] that if V is smoother (so that more adiabatic iterations can be performed) the value of α can be lowered, but no precise result was stated. Combining higher adiabatic iterations with some estimates in [8], Joye [11] proved that if $V \in \mathcal{C}^n$, then M has no absolutely continuous spectrum, as far as (1.9) holds true and in addition:

$$\beta < \alpha \tag{1.11}$$

and

$$d_j \leq Dj^\gamma,\ \ D < \infty,\ \gamma \leq \alpha. \tag{1.12}$$

Our improvement comes from a suitable grouping of bands which allows the use of Theorem 1 even in cases when (1.5) does not hold true. The second result (which does not follow from Theorem 1) we prove, gives upper bounds on the increase of the mean energy as $t \to \infty$. Let $\| \Psi_0 \| = 1$, $\Psi_0 \in \mathcal{Q}(H_0) = \mathcal{Q}(H(t))$ (= the form domain of H_0) and consider

$$< H > (t) \equiv (\Psi_t, H(t)\Psi_t)$$

where $\Psi_t = U(t)\Psi_0$.

THEOREM 2. – *Suppose:*

$$cj^\alpha \leq \Delta_j \leq Cj^\alpha; \quad c, \alpha > 0, \ C < \infty$$

$$d_j \leq dj^\alpha; \quad d < \infty$$

$$n \geq \left[\frac{1+\alpha}{2\alpha}\right] + 1, \quad n < \infty.$$

Then for $V \in C^n$ uniformly on $\mathbf{R}$ *(V is not supposed to be periodic) one has:*

$$\limsup_{t \to \infty} t^{-\frac{1+\alpha}{n\alpha}} < H > (t) < \infty. \tag{1.13}$$

COROLLARY. – *Let $H_0 = -\Delta_N$, and suppose $V \in C^n$, $n > 2$. Then*

$$\limsup_{t \to \infty} t^{-\frac{2}{n}} < H > (t) < \infty. \tag{1.14}$$

Remarks.

(i) There are no assumtions concerning the nature of the spectrum inside σ_j.

(ii) For $V \in C^1$ and Ψ_0 in the domain of H_0, there is a trivial bound (which does not depend upon the spectral properties of H_0):

$$\limsup_{t \to \infty} t^{-1} < H > (t) < \infty$$

which is a consequence of the fact that

$$\frac{d}{dt} < H > (t) = \left(\Psi_t, \left(\frac{d}{dt}V(t)\right)\Psi_t\right).$$

2. PROOFS

Proof of Theorem 1. A finite number of finite constants will appear during the proofs; for simplicity we shall denote all of them by the same letter c. Besides the usual notations:

$$\frac{d^k}{dt^k}A(t) = A^{(k)}(t), \quad AB - BA = [A, B],$$

we shall use

$$\| A \|_n \equiv max_{l=0,1,\dots,n} sup_{t\in[0,2\pi]} \| A^{(l)}(t) \|,$$

$$A^{(1,B)}(t) \equiv A^{(1)}(t) + i[B(t), A(t)]. \tag{2.1}$$

Notice that (2.1) is a derivation; in particular the Leibnitz rule holds.

Without restricting the generality one can suppose $\Delta_{j+1} \geq \Delta_j$. Indeed suppose that $\Delta_{k+l} < \Delta_k$ for $l = 1, 2, ...m$. Then (with the appropriate relabelling) one can take as σ_{k+1} the union $\cup_{p=k+1}^{k+m+1}\sigma_p$. In the same way one can suppose Δ_1 arbitrarily large.

A finite number of operators, $H_k(t)$, of the form

$$H_k(t) = H_0 + W_k(t)$$

with uniformly bounded $W_k(t)$ will appear during the proof. We suppose Δ_1 to be large enough such that all $H_k(t)$ have the same structure of the spectrum as H_0, with the same asymptotics of the gaps

$$\sigma(H_k(t)) = \cup_{j=1}^{\infty}\sigma_{k,j}(t), \ \Delta_{k,j}(t) \subset [\Delta_j - c, \Delta_j + c],$$

$$m_{k,j}(t) = m_j.$$

Some technical points will be stated as Lemmae; their proof will be given at the end of the section.

Let

$$H_0(t) = H_0 + V(t) \tag{2.2}$$

and let $P_{0,j}(t)$ be the spectral projections of $H_0(t)$ corresponding to $\sigma_{0,j}(t)$. By definition we set $\Delta_0 = \Delta_1$.

LEMMA 1.

$$P_{0,j}(t) \in \mathcal{C}^{n-1}$$

and

$$\| P^{(1)}_{0,j}(t) \| \leq c\frac{1}{\Delta_{j-1}};\quad j = 1, 2, ... \tag{2.3}$$

We write now the adiabatic iteration scheme (see [12], [13] for the case of a finite number of gaps and [10] for the first iteration in the general case; a different but essentially equivalent scheme was used in [11]). Starting from $H_0(t) = H(t)$ we define $H_k(t)$ for $k = 1, 2, ..., n - 1$ by

$$H_{k+1}(t) = H_k(t) + B_k(t) \tag{2.4}$$

$$B_k(t) = i\sum_{j=1}^{\infty} P_{k,j}(t)P^{(1,H(t))}_{k,j}(t) \equiv i\sum_{j=1}^{\infty} B_{k,j}(t), \tag{2.5}$$

where $P_{k,j}(t)$ are the spectral projections of $H_k(t)$ corresponding to $\sigma_{k,j}(t)$. Of course one has to prove that at each step, $B_k(t)$ is self-adjoint, uniformly bounded and smooth.

LEMMA 2. – *For* $k = 0, 1, ..., n - 1;\quad j = 1, 2, ...$

$$P_{k,j}(t) \in C^{n-k} \tag{2.6}$$

$$\| P^{(1)}_{k,j}(t) \|_{n-k-1} + \| P^{(1;H(t))}_{k,j}(t) \|_{n-k-1} \leq \frac{1}{\Delta_{j-1}} \tag{2.7}$$

$$B_k(t) \in C^{n-k-1} \tag{2.8}$$

$$B_k(t) = B_k(t)^* \tag{2.9}$$

The key point of the adiabatic iteration is that actually $\| B_{l,j}(t) \|_{n-l-1}$ decay much faster.

LEMMA 3.

$$\| B_{l,j}(t) \|_{n-l-1} \leq \frac{1}{\Delta^{l+1}_{j-1}}. \tag{2.10}$$

From this point the proof of Theorem 1 is a routine repetition of the proof of Theorem 1 in [10].

LEMMA 4.

$$||| B_{n-1}(t) ||| < c \tag{2.11}$$

where $||| \, . \, |||$ stands for the trace norm.

Consider $U_A(t)$ (see [14], [11] for the existence of $U_A(t)$) given by

$$iU_A(t)^{(1)} = (H(t) - B_{n-1}(t))U_A(t); \quad U_A(0) = 1. \tag{2.12}$$

LEMMA 5.

$$P_{n-1,j}(t) = U_A(t)P_{n-1,j}(0)U_A(t)^* \tag{2.13}$$

Consider the "Moller operator", $\Omega(t)$, corresponding to the pair $H(t),\ H(t) - B_{n-1}(t)$ *i.e.*

$$\Omega(t) = U_A(t)^* U(t). \tag{2.14}$$

By the usual computation:

$$\Omega(t) - 1 = i\int_0^t U_A(u)^* B_{n-1}(u)U_A(u)\Omega(u)du. \tag{2.15}$$

From Lemma 4 and (2.15):

$$||| \, \Omega(t) - 1 \, ||| \leq c, \quad t \in [0, 2\pi] \tag{2.16}$$

Due to (2.14) the monodromy matrix, M, can be written as

$$M = U_A(2\pi) + U_A(2\pi)(\Omega(2\pi) - 1). \tag{2.17}$$

From (2.13) and $P_{n-1,j}(0) = P_{n-1,j}(2\pi)$, one has $[U_A(2\pi), P_{n-1,j}(2\pi)] = 0$ which implies that $U_A(2\pi)$ has pure point spectrum. This together with (2.16) and the stability of absolutely continuous spectrum under trace class perturbations [15], finishes the proof of Theorem 1.

Proof of Corollary 1. – Consider the sequence $j_p = 2^p,\ p = 0, 1, \ldots$ and rewrite $\sigma(H_0)$ as

$$\sigma(H_0) = \cup_{p=1}^{\infty} \tilde{\sigma}_p \tag{2.18}$$

with

$$\tilde{\sigma}_1 = \sigma_1; \quad \tilde{\sigma}_p = \cup_{l=j_{p-2}+1}^{j_{p-1}} \sigma_l, \ p = 2, 3, \ldots \tag{2.19}$$

Notice that

$$\tilde{\Delta}_p = \Delta_{2^{p-1}} \geq \tilde{a}\ 2^{\alpha p}; \ \tilde{a} > 0 \tag{2.20}$$

$$\tilde{m}_p = \sum_{l=2^{p-2}+1}^{2^{p-1}} m_l \leq \tilde{A}\ 2^{(1+\beta)p}$$

With n given by (1.9), the conditions in Theorem 1 holds true and the proof of Corollary 1 is finished.

Proof of Theorem 2. – Let

$$\tilde{E}_p = \sup_{\lambda \in \tilde{\sigma}_p} \lambda; \quad \tilde{e}_p = \inf_{\lambda \in \tilde{\sigma}_p} \lambda \tag{2.21}$$

with $\tilde{\sigma}_p$ as in the proof of Corollary 1. Then

$$\tilde{E}_p \leq c 2^{(\alpha+1)p}; \quad \tilde{E}_p \leq c\, \tilde{e}_p. \tag{2.22}$$

Since $H_k(t) - H(t)$ are uniformly bounded we can consider

$$< H_{n-1} > (t) \equiv (\Psi_t, H_{n-1}(t)\Psi_t)$$

instead of $< H > (t)$. From (2.14), (2.15):

$$\Psi_t = \Psi_{A,t} + \Phi_t \tag{2.23}$$

with

$$\Psi_{A,t} = U_A(t)\Psi_0, \tag{2.24}$$

$$\Phi_t = iU_A(t) \int_0^t U_A(u)^* B_{n-1}(u) U_A(u) \Omega(u) du. \tag{2.25}$$

and we can rewrite $< H_{n-1} > (t)$ as follows:

$$< H_{n-1} > (t) = (\Psi_{A,t}, H_{n-1}(t)\Psi_{A,t}) + (\Phi_t, H_{n-1}(t)\Psi_{A,t}) +$$
$$+ (\Psi_{A,t}, H_{n-1}(t)\Phi_t) + (\Phi_t, H_{n-1}(t)\Phi_t). \tag{2.26}$$

Without restricting the generality we can assume that $H_{n-1}(t) \geq 0$. Then by Cauchy Schwartz inequality:

$$| (\Psi_{A,t}, H_{n-1}(t)\Phi_t) | \leq (\Psi_{A,t}, H_{n-1}(t)\Psi_{A,t})^{1/2} (\Phi_t, H_{n-1}(t)\Phi_t)^{1/2}. \tag{2.27}$$

and then

$$< H_{n-1} > (t) \leq 2(\Psi_{A,t}, H_{n-1}(t)\Psi_{A,t}) + 2(\Phi_t, H_{n-1}(t)\Phi_t). \tag{2.28}$$

Denoting:

$$H_{n-1,p}(t) = P_{n-1,p}(t) H_{n-1,}(t) P_{n-1,p}(t) \tag{2.29}$$

one has

$$H_{n-1}(t) = \sum_{p=1}^{\infty} H_{n-1,p}(t), \tag{2.30}$$

$$\tilde{e}_p - c \leq U_A^*(t) H_{n-1,p}(t) U_A(t) \leq \tilde{E}_p + c \tag{2.31}$$

which together with (2.13) and (2.22) gives:

$$\begin{aligned}
&(\Psi_{A,t}, H_{n-1}(t)\Psi_{A,t}) \\
&= \sum_{p=1}^{\infty} (P_{n-1,p}(0)\Psi_0 U_A^*(t) H_{n-1,p}(t) U_A(t) P_{n-1,p}(0)\Psi_0) \\
&\leq c + \sum_{p=1}^{\infty} E_p (P_{n-1,p}(0)\Psi_0, P_{n-1,p}(0)\Psi_0) \\
&\leq c(1 + \sum_{p=1}^{\infty} e_p (P_{n-1,p}(0)\Psi_0, P_{n-1,p}(0)\Psi_0)) \\
&\leq c(1 + (\Psi_0, H_{n-1}(0)\Psi_0).
\end{aligned} \tag{2.32}$$

Consider now the second term in the r.h.s. of (2.28). Since $\| \Psi_{A,t} \| = \| \Psi_t \| = 1$ one has $\| \Phi_t \| \leq 2$ which together with

$$\sum_{p=1}^{N(t)} H_{n-1,p}(t) \leq c + \tilde{E}_{N(t)}; \quad H_{n-1,p}(t) \leq c + \tilde{E}_p$$

and (2.30) gives

$$(\Phi_t, H_{n-1}(t)\Phi_t) \leq c(1 + \tilde{E}_{N(t)}) + \sum_{p=N(t)+1}^{\infty} \tilde{E}_p \| P_{n-1,p}(t)\Phi_t \|^2 . \tag{2.33}$$

Due to (2.13), (2.25) and (2.5)

$$P_{n-1,p}(t)\Phi_t = i U_A(t) \int_0^t U_A(u)^* B_{n-1,p}(u) U_A(u) \Omega(u) du$$

which together with (2.10) gives

$$\| P_{n-1,p}(t)\Phi_t \|^2 \leq t^2 \frac{1}{\tilde{\Delta}_{p-1}^{2n}},$$

so finally taking into account (2.20) and (2.22)

$$(\Phi_t, H_{n-1}(t)\Phi_t) \leq c(2^{(\alpha+1)N(t)} + t^2 2^{(\alpha+1-2n\alpha)N(t)}) \tag{2.34}$$

and chosing $N(t)$ as to minimize the r.h.s. of (2.34) one obtains

$$(\Phi_t, H_{n-1}(t)\Phi_t) \leq c\, t^{(\alpha+1)/n\alpha}$$

which together with (2.28) and (2.32) finishes the proof of Theorem 2.

Proof of Lemma 1. – Consider first $P_{0,j}(t)$ (in what follows we omit the variable t):

$$P_{0,j}^{(1)} = \frac{1}{2\pi i}\int_{\Gamma_j}(H-z)^{-1}V^{(1)}(H-z)^{-1}dz,$$

where Γ_j is a contour enclosing $\sigma_{0,j}$ and satisfying $\inf_{z\in\Gamma_j} dist(z,\sigma(H))\geq c$. It follows

$$\| P_{0,j}^{(1)} \| \leq c\int_{\Gamma_j}\frac{1}{dist(z,\sigma(H))^2}\mid dz\mid .$$

Deforming Γ_j to a union of two lines passing through the middle of the gaps above and below $\sigma_{0,j}$, one obtains:

$$\| P_{0,j}^{(1)} \|_0 \leq c\int_{\infty}^{\infty}\left(\frac{1}{\Delta_j^2+x^2}+\frac{1}{\Delta_{j-1}^2+x^2}\right)dx \leq c\frac{1}{\Delta_{j-1}}.$$

For higher derivations one notices that $P_{0,j}^{(k)}$ is always of the form

$$\int_{\Gamma_j}(H-z)^{-1}A(H-z)^{-1}dz \tag{2.35}$$

with uniformly bounded A, and then repeat the above estimation.

Proof of Lemma 2. – Induction. The case $k=0$ follows from Lemma 1 (notice that $P_{0,j}^{(1)} = P_{0,j}^{(1,H)}$) and (1.5). The equality (2.9) follows from

$$P_{0,j}P_{0,j}^{(1)} = P_{0,j}^{(1)} - P_{0,j}^{(1)}P_{0,j} \tag{2.36}$$

and

$$\sum_{j=1}^{\infty} P_{0,j}^{(1)} = 0. \tag{2.37}$$

Suppose now that (2.6-9) hold true for $l=0,1,...,k-1$. For (2.6) observe that $H_k = H_0 + W_k$ with $W_k \in \mathcal{C}^{n-k}$. Now, since

$$[H,P_{k,j}] = -\sum_{l=0}^{k-1}[B_l,P_{k,j}] \tag{2.38}$$

one has

$$P_{k,j}^{(1,H)} = P_{k,j}^{(1)} - i\sum_{l=0}^{k-1}[B_l,P_{k,j}-P_{l,j}] - i\sum_{l=0}^{k-1}[B_l,P_{l,j}] \tag{2.39}$$

The first term in the r.h.s. of (2.39) can be estimated as in the proof of Lemma 1 (with H replaced by H_k). For the second term write

$$P_{k,j} - P_{l,j} = -\frac{1}{2\pi i}\int_{\Gamma_j}(H_k - z)^{-1}(H_k - H_l)(H_l - z)^{-1}dz$$

and again estimate as in Lemma 1. For the last term use the identity

$$[B_l, P_{l,j}] = -iP_{l,j}^{(1,H)} \tag{2.40}$$

and the induction hypothesis. The identity (2.40) follows direectly from the definition of B_l, (2.36) and (2.37) written for $P_{l,j}$ and

$$\sum_{m=1}^{\infty}[P_{l,m}[H, P_{l,m}], P_{l,j}] = -[H, P_{l,j}].$$

Now (2.8) follows from (2.7) and (1.5). For (2.9) use again (2.36), (2.37) written for $P_{k,j}$ and

$$\begin{aligned}\sum_{j=1}^{\infty}P_{k,j}[H, P_{k,j}] &= \sum_{j=1}^{\infty}P_{k,j}[H - H_k, P_{k,j}]\\ &= \sum_{j=1}^{\infty}P_{k,j}(H - H_k)P_{k,j} - (H - H_k).\end{aligned}$$

Proof of Lemma 3. – The proof of Lemma 3 is based on the following identities. Define:

$$L_{k,j} = P_{k+1,j} - P_{k,j},$$

$$K_{k,j} = P_{k,j}L_{k,j},$$

$$D_{k,j} = P_{k+1,j}L_{k,j}.$$

Then (we suppose Δ_1 sufficiently large as to have $\| L_{k,j} \| < 1$)

$$D_{k,j} = P_{k+1,j}K_{k,j}(1 - L_{k,j})^{-1}, \tag{2.41}$$

$$B_{k+1,j} = iD_{k,j}P_{k+1,j}^{(1,H)} + P_{k+1,j}K_{k,j}^{(1,H-H_{k+1})}. \tag{2.42}$$

Proof of (2.41). – From $P_{k+1,j} = P_{k+1,j}^2$ one has

$$D_{k,j} = P_{k+1,j}(P_{k+1,j} - P_{k,j} + P_{k,j})L_{k,j} = D_{k,j}L_{k,j} + P_{k+1,j}K_{k,j}$$

which gives (2.41). In proving (2.42) we use the following identity:

$$P_{k,j}^{(1,H-H_{k+1})} = 0. \tag{2.43}$$

Proof of (2.43). –

$$\begin{aligned} P_{k,j}^{(1,H-H_{k+1})} &= P_{k,j}^{(1,H)} - i\left[H_{k+1}, P_{k,j}\right] \\ &= P_{k,j}^{(1,H)} - i\left[H_{k+1} - H_k, P_{k,j}\right] = P_{k,j}^{(1,H)} - i\left[B_k, P_{k,j}\right] = 0. \end{aligned}$$

In the last step we used (2.40).

$$\begin{aligned} B_{k+1,j} &= iP_{k+1,j}P_{k+1,j}^{(1,H)} = iP_{k+1,j}(P_{k+1,j} - P_{k,j} + P_{k,j})P_{k+1,j}^{(1,H)} \\ &= iD_{k,j}P_{k+1,j}^{(1,H)} + iP_{k+1,j}P_{k,j}P_{k+1,j}^{(1,H)}. \end{aligned} \tag{2.44}$$

Using $\left[H_{k+1}, P_{k+1,j}\right] = 0$, (2.43) and the Leibnitz rule one has

$$\begin{aligned} P_{k,j}P_{k+1,j}^{(1,H)} &= P_{k,j}P_{k+1,j}^{(1,H-H_{k+1})} = (P_{k,j}P_{k+1,j})^{(1,H-H_{k+1})} \\ &= (K_{k,j} + P_{k,j})^{(1,H-H_{k+1})} = K_{k,j}^{(1,H-H_{k+1})} \end{aligned}$$

which toghether with (2.44) gives (2.42). We finish now the proof of Lemma 3, by induction. The case $l = 0$ follows from the definition of $B_{0,j}$ and (2.3). Suppose (2.10) holds true for $l = 0, 1, \dots, k$. Since

$$\begin{aligned} K_{k,j} &= -P_{k,j}\frac{1}{2\pi i}\int_{\Gamma_j}(H_k - z)^{-1}B_k(H_{k+1} - z)^{-1}dz \\ &= -\frac{1}{2\pi i}\int_{\Gamma_j}(H_k - z)^{-1}B_{k,j}(H_{k+1} - z)^{-1}dz, \end{aligned}$$

an estimate as in the proof of Lemma 1 and the induction hypothesis gives:

$$\| K_{k,j} \|_{n-k-1} \leq c\frac{1}{\Delta_{j-1}^{k+2}}$$

which together with (2.41) and (2.42) finishes the proof of Lemma 3.

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REFERENCES

[1] J. A. BAETA SEGUNDO, H. HEY and W. WRESZINSKI, On quantum stability for systems under quasiperiodic perturbations. To appear in *J. Stat. Phys.*

[2] J. BELLISSARD, Stability and instability in Quantum Mechanics. In: *Schrodinger operators*, S. GRAFFI (ed), LNM, Vol. **1159**, Springer, Berlin 1985.

[3] L. BUNIMOVICI, H. R. JAUSLIN, J. L. LEBOWITZ, A. PELLEGRINOTTI and P. NIELABA, Diffusive Energy in classical and quantum driven oscillators. *J. Stat. Phys.,* Vol. **62**, 1991, pp. 793-817.

[4] M. COMBESCURE, Recurrent versus diffusive dynamics for a kicked quantum oscillator, *Ann. Inst. Henri Poincaré, Phys. Theor.*, Vol. **57**, 1992, p. 67.

[5] P. DUCLOS and P. STOVICEK, Floquet Hamiltonians with Pure Point Spectrum, *Commun. Math. Phys.*, Vol. **177**, 1996, pp. 327-347.

[6] J. HOWLAND, Quantum stability. In: *Schrodinger operators*. E. BALSLEV (ed), LNP, Vol. **403**, 1992, Springer, Berlin.

[7] H. R. JAUSLIN, Stability and chaos in classical and quantum hamiltonian systems. In: *II Granada seminar on computational physics.* P. GARRIDO and J. MARRO (eds), World Scientific, Singapore 1993.

[8] J. HOWLAND, Floquet operators with singular spectrum. I. *Ann. Inst. Henri Poincare, Phys. Theor.*, Vol. **50**, 1989, pp. 309-323, II. *Ann. Inst. Henri Poincaré: Phys. Theor.*, Vol. **50**, 1989, pp. 325-334.

[9] J. HOWLAND, Stability of quantum oscillators, *J. Phys.*, Vol. **A 25**, 1992, pp. 5177-5181.

[10] G. NENCIU, Floquet operators without absolutely continuous spectrum. *Ann. Inst. Henri Poincaré: Phys. Theor.*, Vol. **59**, 1993, pp. 91-97.

[11] A. JOYE, Absence of absolutely continuous spectrum of Floquet operators. *J. Stat. Phys.*, Vol. **75**, 1994, pp. 929- 952.

[12] G. NENCIU and RASCHE, Adiabatic theorem and Gell-Mann-Low formula. *H. P. A.*, Vol. **62**, 1989, pp. 372-388.

[13] G. NENCIU, Asymptotic invariant subspaces, adiabatic theorems and block diagonalisation. In: BOUTET DE MONVEL *et al.* (eds), *Recent developments in quantum mechanics,* pp. 133-149, Dordrecht: Kluver 1991.

[14] S. G. KREIN, Linear differential equations in Banach spaces, *Translations of Mathematical Monographs.*, Vol. **29**, 1971, Providence.

[15] M. S. BIRMAN and M. G. KREIN, On the theory of wave operators and scattering theory, *Sov. Math. Dokl.*, Vol. **3**, 1962, pp. 740-744.

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