ANNALES DE L'I. H. P., SECTION A

A. SHADI TAHVILDAR-ZADEH

Relativistic and nonrelativistic elastodynamics with small shear strains

Annales de l'I. H. P., section A, tome 69, nº 3 (1998), p. 275-307
<http://www.numdam.org/item?id=AIHPA_1998__69_3_275_0>

Ann. Inst. Henri Poincaré,

Vol. 69, n° 3, 1998, p. 275-307. *Physique théorique*

Relativistic and nonrelativistic elastodynamics with small shear strains

by

A. Shadi TAHVILDAR-ZADEH*

Princeton University

ABSTRACT. – We present a new variational formulation for relativistic dynamics of isotropic hyperelastic solids. We introduce the shear strain tensor and study the geometry of characteristics in the cotangent bundle for the relativistic equations, under the assumption of small shear strains, and obtain a result on the stability of the double characteristic manifold. We then focus on the nonrelativistic limit of the above formulation, and compare it to the classical formulation of elastodynamics via displacements. We obtain a global existence result for small-amplitude elastic waves in materials under a constant isotropic deformation and a result on the formation of singularities for large data. © Elsevier, Paris

RÉSUMÉ. – Nous présentons une nouvelle formulation variationnelle pour la dynamique relativiste des solides isotropes hyperélastiques. Nous introduisons le tenseur de déformation de cisaillement, nous étudions la géométrie des caractéristiques dans la boule cotangente pour les équations relativistes, sous l'hypothèse des petites déformations de cisaillement, et nous obtenons un résultat sur la stabilité de la variété doublement caractéristique. Puis nous nous concentrons sur la limite non-relativiste de cette formulation et nous la comparons à la formulation classique de l'élastodynamique obtenue à l'aide des déplacements. Nous démontrons l'existence globale pour les ondes élastiques de petite amplitude dans les matériaux soumis à une déformation homogène isotrope, et nous obtenons un théorème sur la formation de singularités pour de grandes données. © Elsevier, Paris

* Supported in part by National Science Foundation grant DMS-9504919.

1. FORMULATING RELATIVISTIC ELASTODYNAMICS

1.1. The Classical Formulation

Elastodynamics concerns itself with the time-evolution of elastic deformations in a solid body. Its classical formulation is based on the concept of *displacements*, i.e., elastic deformations of a solid away from a "totally relaxed" or "undeformed" reference state: Let $\phi : \mathbb{R}\times\mathcal{N} \to \mathbb{R}^3$, $\phi = \phi(t, y)$ be a smooth deformation evolving in time of a material which in its undeformed state occupies the region $\mathcal{N} \subset \mathbb{R}^3$. Let $F = D_y\phi$ be the deformation gradient. The stored energy W of an isotropic hyperelastic material depends on F through the principal invariants $\imath_1, \imath_2, \imath_3$ of the (left Cauchy-Green) strain tensor $B = FF^T$. In the absence of body forces, the equations of motion, which can be derived from a Lagrangian, are

$$\partial_t^2\phi^i - \partial_\ell \frac{\partial W}{\partial F_{i\ell}} = 0. \tag{1.1}$$

This is a second-order, quasi-linear, and (under the right assumptions on W) hyperbolic system of partial differential equations.

Now let $\psi(t, y) := \phi(t, y) - y$ be the displacement field. When ψ is *infinitesimal*, the equations of linear elasticity provide a good model for the dynamics of the body. These equations arise from the expansion around the trivial solution $\phi = y$ of equations (1.1) and ignoring the higher order terms:

$$L\psi := \partial_t^2\psi - c_2^2\Delta\psi - (c_1^2 - c_2^2)\nabla(\nabla \cdot \psi) = 0, \tag{1.2}$$

where $c_1 > c_2 > 0$ are the propagation speeds of longitudinal and transverse waves, respectively:

$$\begin{aligned} c_1^2 &:= 4(W_{11} + 4W_{12} + 2W_{13} + 4W_{22} + 4W_{23} + W_{33}) \\ c_2^2 &:= -2(W_2 + W_3), \end{aligned}$$

with W_k and W_{mn} denoting the first and second partial derivatives of W with respect to the invariants $\imath_k$, evaluated at the unstressed reference state $\imath_1 = 3, \imath_2 = 3, \imath_3 = 1$.

1.2. Nonlinear and Relativistic Dynamics

Nonlinear elasticity comes in when finite displacements are considered. For a typical solid three-dimensional body whose linear dimensions are comparable, say a metallic cube, the displacements cannot get too large without ceasing to be elastic displacements. This is because most solid materials, rubber being a notable exception, have a small *elastic limit*, accumulation of strains beyond which point will cause the material to crack, either immediately or eventually after going through a plastic phase, and in any case leaving the domain of applicability of the elasticity model. The hypothesis of small displacements is thus natural and reasonable when dealing with three-dimensional bodies with comparable linear dimensions. In the classical nonlinear theory of elasticity, large displacements are mostly considered when the elastic body in question has a special geometry that allows the displacements to accumulate and become large in a certain direction, while the strains are still small. This is the case for rods and plates, and more generally, for objects of modest size but with non-comparable linear dimensions.

One situation where it is natural to consider strains which are *not* uniformly small is in geophysics and astrophysics, where the solid bodies under study have comparable dimensions but are of a celestial scale in size, so that because of self-gravitation, the isotropic pressure in the interior of these bodies is enormous. Linear elasticity clearly does not apply to these bodies since they are very far from being relaxed. In studying the elastic response of such media to a disturbance, e.g. quakes on Earth or in a white dwarf star, one has to bear in mind that the shear forces produced by any feasible disturbance will be very small compared to the hydrostatic pressure in the background. Therefore the correct perturbative analysis involves linearizations not around a state of no stress, but rather around a state of very large isotropic pressure which is a nontrivial static or stationary solution of the equations of motion.

Astrophysical applications of elastodynamics necessitate the inclusion of relativistic effects in the theory. While special-relativistic effects in mechanics are commonly thought of as becoming apparent only when material velocities reach an appreciable fraction of the speed of light, there is another way for these effects to show which is perhaps more relevant to the dynamics of a solid body: It is when the material *density* becomes so high that the speed of sound waves travelling in the body begins to approach the speed of light. This is indeed the situation in a typical neutron star, with a radius of only about 10 km and a mass equal to 1.4 solar masses. In this case of course, the gravitational field is also very strong and general-relativistic effects must be taken into account.

In a truly relativistic theory of elastodynamics, the classical notion of displacement has to be put aside, since there is no invariant way of defining either a displacement vector or an undeformed state. This is because there is no reference frame in which gravity can be switched off. Starting with

Souriau [15], there have been many attempts, successful and otherwise, to present a Lagrangian formulation of relativistic elastodynamics which does not refer to displacements and is free of any assumptions on the existence of a globally relaxed state [1]. Recently, Kijowski & Magli [10] proposed their formulation, which is based on mappings from the spacetime into an abstract material manifold. Independently, D. Christodoulou succeeded in generalizing his formulation of relativistic fluid mechanics [4] to the completely general case of the adiabatic dynamics of perfectly elastic, aeolotropic media in presence of dislocations and electromagnetic fields [5]. When reduced to the special case of isotropic solids, Christodoulou's formulation, described below, agrees with that of Kijowski & Magli up to the definition of the strain tensor, but is more general since it includes some thermodynamics as well.

1.3. The Variational Formulation

The dynamics of a body are described by a mapping f from the 4-dimensional spacetime manifold $(\mathcal{M}, g)$ into an abstract 3-dimensional Riemannian manifold $(\mathcal{N}, m)$, called the *material manifold.* The differential of the mapping is required to have maximal rank, with the 1-dimensional null spaces being *timelike* at every point. The inverse image of a point $y \in \mathcal{N}$ under f is thus a timelike curve in $\mathcal{M}$ which is the world-line of the particle labeled y in the material. The *material velocity* u is the unit future-directed tangent vector field of these curves. The orthogonal complement of u_x in $T_x\mathcal{M}$ is a hyperplane Σ_x, the *simultaneous space at* x, on which the metric of spacetime g induces a Riemannian metric γ,

$$\gamma_{\mu\nu} = g_{\mu\nu} + u_\mu u_\nu .$$

The *(relativistic) strain* h is defined to be the pullback of the material metric m under the mapping f,

$$h_{\mu\nu} := (f^* m)_{\mu\nu} = \partial_\mu f^a \partial_\nu f^b m_{ab}, \tag{1.3}$$

and is thus a symmetric 2-tensor living on Σ_x. The velocity u_x is an eigenvector of h with eigenvalue zero. The other three eigenvalues, h_1, h_2, h_3 have to be positive since m is positive definite and $D_x f$ is an isomorphism between Σ_x and $T_{f(x)}\mathcal{N}$, which is taken to be orientation-preserving. The Lagrangian density $\mathcal{L}$ of a homogeneous, isotropic, perfectly elastic solid in general must be a function of h which is invariant under the

[1] See Maugin [12] for a critique of some of these formulations.

orthogonal group $O(3)$ acting on Σ_x, i.e. a function of the three invariants q_1, q_2, q_3 of h:

$$\begin{aligned} q_1 &:= h_1 + h_2 + h_3 = \mathrm{tr}_\gamma h = \mathrm{tr}_g h \\ q_2 &:= h_1 h_2 + h_2 h_3 + h_3 h_1 = \frac{1}{2}[(\mathrm{tr} h)^2 - \mathrm{tr}(h^2)] \\ q_3 &:= h_1 h_2 h_3 = \det{}_\gamma h. \end{aligned}$$

Thus for a solid we have

$$\mathcal{L} = \mathcal{L}(s, q_1, q_2, q_3),$$

where $s : \mathcal{M} \to \mathbb{R}^+$ is the *entropy per particle*, defined as a function on $\mathcal{M}$ which is independent of the spacetime metric g. Fluid dynamics can be thought of as a subcase, where $\mathcal{L}$ depends only on q_3 and s.

The *energy-momentum-stress* tensor T of the material can now be computed from the formula

$$T_{\mu\nu} = 2\frac{\partial \mathcal{L}}{\partial g^{\mu\nu}} - \mathcal{L} g_{\mu\nu}.$$

Letting H_x^+ be the hyperboloid of unit future-directed timelike vectors at x, we define ρ, the *(proper) energy density* at x by

$$\rho(x) = \inf_{v \in H_x^+} T(v, v).$$

Then it is easy to see that in our case we have $\rho = \mathcal{L}$, the velocity u is the eigenvector of T corresponding to eigenvalue ρ, and that

$$T_{\mu\nu} = \rho u_\mu u_\nu + S_{\mu\nu}, \tag{1.4}$$

where S is the *stress* tensor,

$$S_{\mu\nu} = 2\frac{\partial \mathcal{L}}{\partial q_1} h_{\mu\nu} + 2\frac{\partial \mathcal{L}}{\partial q_2}(q_1 h_{\mu\nu} - h^2{}_{\mu\nu}) + (2\frac{\partial \mathcal{L}}{\partial q_3} q_3 - \mathcal{L})\gamma_{\mu\nu}. \tag{1.5}$$

The stress tensor satisfies $S_{\mu\nu} u^\mu = 0$ and thus is a symmetric bilinear form living on Σ_x. The eigenvalues of S relative to γ are called the *principal pressures* p_1, p_2, p_3, which unlike the case of a perfect fluid, do not have to be positive or equal. The positivity condition for the energy momentum tensor implies that $\rho \geq \max\{|p_1|, |p_2|, |p_3|\}$.

The equations of motion for a solid body are [2]

$$\nabla^\nu T_{\mu\nu} = 0, \tag{1.6}$$

which express the conservation laws of energy and momentum. (1.6) is a direct consequence of the spacetime metric g satisfying the Einstein equations,

$$G_{\mu\nu} := R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = \frac{2\kappa}{c^4} T_{\mu\nu}, \tag{1.7}$$

where $R_{\mu\nu}$ is the Ricci curvature of the spacetime metric $g_{\mu\nu}$, R is its scalar curvature, κ is 4π times Newton's constant of gravitation and c the speed of light. This is because $\nabla^\nu G_{\mu\nu} = 0$, in view of the twice-contracted Bianchi identities.

To equations (1.6) we need to append the equivalent of the equation of continuity for solids, which here takes the form

$$\mathfrak{L}_u h = 0, \tag{1.8}$$

where $\mathfrak{L}$ denotes the Lie derivative. This is a consequence of (1.3). In a coordinate frame (1.8) reads

$$u^\lambda \nabla_\lambda h_{\mu\nu} + h_{\lambda\nu} \nabla_\mu u^\lambda + h_{\mu\lambda} \nabla_\nu u^\lambda = 0. \tag{1.9}$$

Let

$$n := \sqrt{q_3}$$

denote the *number density* of the material in physical space, i.e., the number of particles (or equivalently, the number of flow lines) per unit volume of Σ_x. Let ω denote the volume form of $(\mathcal{N}, m)$, and let v be the volume 3-form induced by g on Σ_x, i.e., $v_{\alpha\beta\gamma} = u^\lambda \epsilon_{\lambda\alpha\beta\gamma}$ where ϵ is the volume 4-form of $(\mathcal{M}, g)$. Then it is clear that $f^*\omega = nv$, and hence $d(nv) = d(f^*\omega) = f^*(d\omega) = 0$. We thus have the law of conservation of particle number

$$\nabla_\mu (n u^\mu) = 0. \tag{1.10}$$

(This can also be derived from (1.8) and therefore is not a new equation.) If we regard the spacetime metric g as a fixed background metric (the *test-relativistic* case), then (1.6) and (1.8) together form a complete system

[2] Throughout this paper Greek indices run from 0 to 3 and Latin ones from 1 to 3. All up-and-down repeated indices are summed over the range.

of 10 equations, where the 10 unknowns are the entropy s, the velocity field u which is subject to the normalizing condition

$$u_\mu u^\mu = -1,$$

and the strain tensor h which is subject to the condition

$$h_{\mu\nu} u^\mu = 0.$$

If the effects of the dynamics of the solid on the gravitational field are not negligible, then g is also unknown and we need to add (1.7) to the system of equations.

Finally, it can be shown (see [4]) that for C^1 solutions, the component of (1.6) in the direction of u is equivalent to the *adiabatic condition*:

$$\nabla_u s = u^\mu \partial_\mu s = 0. \tag{1.13}$$

1.4. The shear strain tensor

Let σ be the symmetric 2-covariant tensor defined by

$$\sigma_{\mu\nu} := q_3^{-1/3} h_{\mu\nu} - \gamma_{\mu\nu}. \tag{1.14}$$

Thus σ measures the deviation from isotropy. We call it the *shear strain* tensor. From the definition it is clear that σ is like γ and h a tensor living on Σ_x. Let $\sigma_1, \sigma_2, \sigma_3$ be the eigenvalues of σ restricted to Σ_x. Since

$$h_i = n^{2/3}(1 + \sigma_i),$$

we must have

$$(1 + \sigma_1)(1 + \sigma_2)(1 + \sigma_3) = 1,$$

and thus only two of the σ_i are independent. Let $|\sigma| := \max\{|\sigma_1|, |\sigma_2|, |\sigma_3|\}$ and let

$$\tau = \mathrm{tr}\sigma, \quad \kappa = \frac{1}{2}[(\mathrm{tr}\sigma)^2 - \mathrm{tr}\sigma^2], \quad \delta = \det{}_\gamma \sigma,$$

be the invariants of σ. Then $\tau + \kappa + \delta = 0$ and thus we only need to consider τ and δ. Moreover,

$$\tau = O(|\sigma|^2), \qquad \delta = O(|\sigma|^3).$$

The invariants of h and σ are related in the following way:

$$q_1 = n^{2/3}(3+\tau), \quad q_2 = n^{4/3}(3+\tau-\delta), \quad q_3 = n^2.$$

Let

$$e := \frac{\rho}{n}$$

be the *energy per particle*. The *constitutive equation* of the material is the specification of e as a smooth function of the thermodynamic variables:

$$e = e(s, n, \tau, \delta) \tag{1.15}$$

Substituting this in (1.4) and (1.5) we obtain

$$T_{\mu\nu} = \rho u_\mu u_\nu + p\gamma_{\mu\nu} + q\sigma_{\mu\nu} + w{\sigma^2}_{\mu\nu}, \tag{1.16}$$

where

$$p := n^2\frac{\partial e}{\partial n} - \frac{2}{3}n\tau\frac{\partial e}{\partial \tau} - \frac{4}{3}n(\tau+\delta)\frac{\partial e}{\partial \delta}, \quad q := 2n\frac{\partial e}{\partial \tau} - 2n\tau\frac{\partial e}{\partial \delta}, \quad w := 2n\frac{\partial e}{\partial \delta}.$$

In astrophysical applications, e.g. the case of a white dwarf or a neutron star, the shear strains are much smaller than the hydrostatic compression, which is due to self-gravitation. To study small deviations from an isotropic background state, we can expand e around $\sigma = 0$:

$$e(s, n, \tau, \delta) = e_0(s, n) + \tau e_2(s, n) + \delta e_3(s, n) + O(|\sigma|^4), \tag{1.17}$$

where e_i are smooth functions on $\mathbb{R}^+ \times \mathbb{R}^+$. In particular, e_0 is the energy per particle of the background isotropic state, and e_2 can be thought of as the modulus of rigidity.

We make the following assumption regarding the constitutive equation:

For every fixed $s > 0$ and $n > 0$, e has a unique local minimum at $\sigma = 0$. (1.18)

This is already reflected in (1.17) by the absence of a first order term in σ, and (since $\tau > 0$ for σ_i small) by the requirement that

$$e_2 > 0. \tag{1.19}$$

However, we do not need to impose this *a priori* since, as we shall see later, it is a necessary condition for the *hyperbolicity* of the system of equations.

2. THE GEOMETRY OF CHARACTERISTICS

2.1. The Characteristic Set

The *symbol* σ_ξ of the system (1.6, 1.8) at a given covector $\xi \in T^*_x\mathcal{M}$ is a linear operator on the space of variations $(\dot{s}, \dot{u}, \dot{h})$. It is obtained by replacing all terms of the form $\nabla_\mu F^a$ in the principal part of the equations with $\xi_\mu \dot{F}^a$, where F^a is any one of the unknowns. The set of covectors ξ such that the null space of σ_ξ is nontrivial is by definition the *characteristic set* C^*_x in the cotangent bundle.

We note that the algebraic constraints (1.11, 1.12) on u and h imply the following constraints on the variations $\dot{u}, \dot{h}$:

$$u_\mu \dot{u}^\mu = 0, \qquad h_{\mu\nu}\dot{u}^\nu + \dot{h}_{\mu\nu}u^\nu = 0. \tag{2.1}$$

To calculate the components of the symbol, we choose as a basis for $T_x\mathcal{M}$ the orthonormal frame of vectors $\{E_0, E_1, E_2, E_3\}$, with $E_0 = u_x$ and E_i the eigenvectors of σ. We take the dual basis for $T^*_x\mathcal{M}$. In this frame we have

$$g_{00} = -1,\ g_{0i} = 0,\ g_{ij} = \gamma_{ij},\ \gamma_{0\nu} = 0,\ \sigma_{0\nu} = 0,\ \sigma_{ij} = 0 \text{ for } i \neq j,$$
$$u^0 = 1,\ u_0 = -1,\ u^i = 0,\ \dot{u}^0 = 0.$$

Hence the symbol of (1.9) is

$$\xi_0 \dot{h}_{\mu\nu} + (h_{\mu\lambda}\xi_\nu + h_{\lambda\nu}\xi_\mu)\dot{u}^\lambda. \tag{2.2}$$

We will also need the symbol of the adiabatic condition (1.13):

$$u^\mu \xi_\mu \dot{s}. \tag{2.3}$$

Let ξ_u denote the component of ξ in the direction of (the cotangent vector dual to) u. Thus $\xi_u = \xi_\mu u^\mu = \xi_0$ in the frame we have chosen. Let $\xi \in C^*_x$ and assume $\xi_u = 0$. Then from (2.2) we see that $\dot{u}^i = 0$. Moreover, (2.3) implies that $\dot{s}$ is arbitrary. This means that the 3-dimensional spacelike plane $\Pi_0 := \{\xi \in T^*_x\mathcal{M} \mid \xi_u = 0\}$ is part of the characteristic set. The remaining equations, obtained by setting the symbol of (1.6) equal to zero, will be three equations for the six variations $\dot{h}_{ij}$, and it is easy to see that the null space of σ_ξ for $\xi \in \Pi_0$ is four-dimensional.

Now suppose $\xi \in C^*_x$ and $\xi_u \neq 0$. Then $\dot{s} = 0$ and from (2.2) we have

$$\dot{h}_{00} = \dot{u}^0 = 0, \qquad \dot{h}_{0j} = -h_{kj}\dot{u}^k, \qquad -\xi_0\dot{h}_{ij} = (h_{kj}\xi_i + h_{ik}\xi_j)\dot{u}^k. \tag{2.4}$$

In studying the symbol of (1.6) we can ignore the terms with $\mu = 0$ since we already know that this component of the equation is equivalent to the adiabatic condition. Consider first the term $\nu = 0, \mu = i$. This gives us

$$\xi^0\{-(\rho + p)\gamma_{ik} - q\sigma_{ik} - w{\sigma^2}_{ik}\}\dot{u}^k,$$

while the terms $\nu = j, \mu = i$ give

$$\xi^j\{\dot{p}\gamma_{ij} + \dot{q}\sigma_{ij} + q\dot{\sigma}_{ij} + \dot{w}{\sigma^2}_{ij} + w\dot{\overline{\sigma^2}}_{ij}\},$$

where $\dot{p} = \frac{\partial p}{\partial n}\dot{n} + \frac{\partial p}{\partial \tau}\dot{\tau} + \frac{\partial p}{\partial \delta}\dot{\delta}$, etc. To calculate $\dot{n}$, we take the symbol of (1.10), which is $\xi_\mu(\dot{n}u^\mu + n\dot{u}^\mu)$. Thus we have

$$-\xi_0\dot{n} = n\xi_k\dot{u}^k.$$

From this, (1.14) and (2.4) we can calculate $\dot{\sigma}$ and $\dot{\overline{\sigma^2}}$ in terms of $\dot{u}$. Let us define $\zeta, \eta \in \mathbb{R}^3$ as follows:

$$\zeta := (\sigma_1\xi_1, \sigma_2\xi_2, \sigma_3\xi_3), \qquad \eta := (\sigma_1^2\xi_1, \sigma_2^2\xi_2, \sigma_3^2\xi_3).$$

We then have

$$\begin{aligned}
\dot{\sigma}_{00} &= 0, \qquad \dot{\sigma}_{0j} = -\sigma_{kj}\dot{u}^k\\
-\xi_0\dot{\sigma}_{ij} &= \{-\frac{2}{3}(\gamma_{ij} + \sigma_{ij})\xi_k + (\gamma_{kj} + \sigma_{kj})\xi_i + (\gamma_{ik} + \sigma_{ik})\xi_j\}\dot{u}^k\\
\dot{\overline{\sigma^2}}_{00} &= 0, \qquad \dot{\overline{\sigma^2}}_{0j} = -{\sigma^2}_{kj}\dot{u}^k\\
-\xi_0\dot{\overline{\sigma^2}}_{ij} &= \{-\frac{4}{3}(\sigma_{ij} + {\sigma^2}_{ij})\xi_k + (\sigma_{kj} + {\sigma^2}_{kj})\xi_i + (\sigma_{ik} + {\sigma^2}_{ik})\xi_j\\
&\quad + (\gamma_{kj} + \sigma_{kj})\zeta_i + (\gamma_{ik} + \sigma_{ik})\zeta_j\}\dot{u}^k.
\end{aligned}$$

These formulas in turn allow us to compute

$$\begin{aligned}
-\xi_0\,\dot{\tau} &= -\xi_0\,\mathrm{tr}\dot{\sigma} = (-\frac{2}{3}\tau\xi_k + 2\zeta_k)\dot{u}^k\\
-\xi_0\,\dot{\delta} &= \xi_0\,(\dot{\tau} + \dot{\kappa}) = (-\frac{4}{3}(\tau + \delta)\xi_k - 2\tau\zeta_k + 2\eta_k)\dot{u}^k.
\end{aligned}$$

Thus substituting in the above for all the variations in terms of $\dot{u}$ and regrouping terms, we have that for a $\xi \in C_x^*$ with $\xi_0 \neq 0$, a vector $\dot{u} \in \mathrm{Null}(\boldsymbol{\sigma}_\xi)$ satisfies

$$P_{ik}\dot{u}^k = 0, \tag{2.5}$$

where $P = P(x, \xi)$ is the following matrix-valued quadratic form in ξ:

$$
\begin{aligned}
P_{ik} = {} & a\xi_i\xi_k + b(\xi_i\zeta_k + \xi_k\zeta_i) + c(\xi_i\eta_k + \xi_k\eta_i) \\
& + d\zeta_i\zeta_k + f(\zeta_i\eta_k + \zeta_k\eta_i) + l\eta_i\eta_k \\
& + \gamma_{ik}\{-(\rho + p)\xi_0^2 + q|\xi|^2 + w\xi \cdot \zeta\} + \sigma_{ik}\{-q\xi_0^2 + q|\xi|^2 \\
& + w(|\xi|^2 + \xi \cdot \zeta)\} + \sigma^2{}_{ik}\{-w\xi_0^2 + w|\xi|^2\},
\end{aligned}
$$

where $\xi \cdot \zeta := \xi_k\zeta^k$, $|\xi|^2 := \xi_k\xi^k$, and

$$
\begin{aligned}
a &:= n\frac{\partial p}{\partial n} - \frac{2}{3}\tau\frac{\partial p}{\partial \tau} - \frac{4}{3}(\tau + \delta)\frac{\partial p}{\partial \delta} + \frac{1}{3}q, & b &:= 2\frac{\partial p}{\partial \tau} - 2\tau\frac{\partial p}{\partial \delta} + q + w \\
c &:= 2\frac{\partial p}{\partial \delta} + w & d &:= 2\frac{\partial q}{\partial \tau} - 2\tau\frac{\partial q}{\partial \delta} + w \\
f &:= 2\frac{\partial q}{\partial \delta} & l &:= 2\frac{\partial w}{\partial \delta}.
\end{aligned}
$$

Using now that the frame we have chosen consists of eigenvectors of σ, we can rewrite P in a more concise way:

$$P_{ik} = C_{ik}^{mn}(\sigma)\xi_m\xi_n - D_i(\sigma)\gamma_{ik}\xi_0^2, \tag{2.6}$$

where

$$
\begin{aligned}
C_{ik}^{mn}(\sigma) := {} & \delta_i^m\delta_k^n\{a + b(\sigma_i + \sigma_k) + c(\sigma_i^2 + \sigma_k^2) + d\sigma_i\sigma_k \\
& + f(\sigma_i^2\sigma_k + \sigma_i\sigma_k^2) + l\sigma_i^2\sigma_k^2\} \\
& + \gamma^{mn}\gamma_{ik}\{q + w\sigma^m + \sigma_i(q + w(1 + \sigma^m)) + \sigma_i^2 w\},
\end{aligned}
$$

and

$$D_i(\sigma) := \rho + p + \sigma_i q + \sigma_i^2 w.$$

The system (2.5) has a nontrivial solution $\dot{u}$ if and only if $\det P = 0$. Let $Q(\xi) = \det P$. Then Q is a homogeneous polynomial of degree six in ξ_μ, which is actually cubic in the squares of the ξ_μ, with coefficients that depend on x, and we have

$$C_x^* = \{\xi \in T_x^*M \mid \xi_0^4 Q(\xi, x) = 0\}.$$

2.2. Hyperbolicity

Let $\mathcal{P}$ denote the vector space of quadratic forms in $\mathbb{R}^4$ with real 3×3 matrix values. $P \in \mathcal{P}$ is *hyperbolic* with respect to $E_0 = (1, 0, 0, 0)$ if

$P(E_0) \neq 0$ and the zeros of $\lambda \mapsto \det P(\lambda E_0 + \xi)$ are real for every $\xi \in \mathbb{R}^4$. It is clear that if P is hyperbolic then

$$P(\frac{\partial}{\partial t}, \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3})$$

is a linear, second-order, constant-coefficient differential operator acting on vector-valued functions $f : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}^3$ and is hyperbolic in the sense of the characteristic speeds being real in every direction (see e.g. [8]).

For the system of partial differential equations (1.6–1.8) to be hyperbolic the quadratic form $P_{ik}(x, \xi)$ in (2.6) must be hyperbolic for all x. We can obtain necessary conditions for hyperbolicity by looking at the zero-order terms in P: If we set $\sigma_i = 0$ in (2.6) we obtain

$$P^0_{ik} := C^{mn}_{ik}(0)\xi_m\xi_n - D_i(0)\gamma_{ik}\xi_0^2, \tag{2.7}$$

where

$$C^{mn}_{ik}(0) = a^0\delta^m_i\delta^n_k + q^0\gamma^{mn}\gamma_{ik}, \qquad D_i(0) = \rho^0 + p^0.$$

Here $\rho^0 = ne_0$ and $p^0 = n\dfrac{\partial\rho^0}{\partial n} - \rho^0$ are respectively the energy density and hydrostatic pressure of the background isotropic state, $q^0 = 2ne_2$ and $a^0 = n\dfrac{\partial p^0}{\partial n} + \dfrac{1}{3}q^0$. The characteristic surface for P^0 is the set of $\xi \in T^*_x\mathcal{M}$ satisfying

$$Q^0(\xi) := \det P^0(\xi) = (c_2^2|\xi|^2 - \xi_0^2)^2(c_1^2|\xi|^2 - \xi_0^2) = 0, \tag{2.8}$$

with

$$c_2^2 = \frac{q^0}{\rho^0 + p^0} = 2\frac{e_2}{\partial\rho^0/\partial n},$$
$$c_1^2 = \frac{a^0 + q^0}{\rho^0 + p^0} = \frac{\partial p^0/\partial n + 8e_2/3}{\partial\rho^0/\partial n} = \eta_0^2 + \frac{4}{3}c_1^2,$$

where $\eta_0^2 = \dfrac{dp^0}{d\rho^0}$ (at constant s). If real, η_0 is the sound speed in the background state. Since $P = P^0 + O(|\sigma|)$, it is clear that a necessary condition for the hyperbolicity of P is that c_1 and c_2 be real. Thus we arrive at the following natural assumptions about the equation of state (1.17):

$$\frac{\partial\rho^0}{\partial n} > 0, \qquad \frac{\partial p^0}{\partial n} > 0, \qquad e_2 > 0. \tag{2.9}$$

For the above theory to be consistent with Relativity, we also need *causality* to be satisfied, i.e., the characteristic set C^*_x must lie *outside* the light cone

$\xi_0^2 = |\xi|^2$ in the cotangent space $T_x^*\mathcal{M}$. This implies that $c_1 < 1$ and $c_2 < 1$. Thus we have the additional condition

$$e_2 < \frac{3}{8}\left(\frac{\partial \rho^0}{\partial n} - \frac{\partial p^0}{\partial n}\right). \tag{2.10}$$

From (2.8) we see that the nonplanar portion of the characteristic set C_x^*, up to first order terms in σ, consists of three spherical cones, two of which coincide. The coincident cones carry the shear waves (transverse waves) travelling with speed c_2, and the single cone carries the compression waves (longitudinal waves) travelling with speed $c_1 > \sqrt{4/3}c_2$. This is the classical picture of *linearized* elasticity, i.e., the characteristics of (1.2).

Conditions (2.9) guarantee the hyperbolicity of P^0. However, the presence of a double root for Q^0 in (2.8) indicates that P, considered as a perturbation of P^0, might still fail to be hyperbolic, since a double root can in general be perturbed into two roots or into none at all. In other words, we are looking at a perturbation of a *non-strictly* hyperbolic operator P^0. If this were a general, non-symmetric perturbation, there would be no reason for P to be hyperbolic.

However, from (2.6) it is clear that both P and P^0 are *symmetric*: $P_{ik} = P_{ki}$. This is a consequence of the equations of motion being derivable from a Lagrangian. If we let D denote the diagonal matrix with entries D_i and define $\tilde{P} = D^{-1/2}PD^{-1/2}$, then on the one hand for $|\sigma|$ small $\det P = 0$ if and only if $\det \tilde{P} = 0$, and on the other hand

$$\tilde{P}_{ik} = \tilde{C}_{ik}^{mn}(\sigma)\xi_m\xi_n - \gamma_{ik}\xi_0^2,$$

so that for each fixed $\bar{\xi} \in \mathbb{S}^2$, the roots of the polynomial $\tilde{Q}(\lambda) := \det \tilde{P}(\lambda, \bar{\xi})$ are the square roots of the *eigenvalues* of the symmetric bilinear form

$$M_{ik}(\sigma) := \tilde{C}_{ik}^{mn}(\sigma)\bar{\xi}_m\bar{\xi}_n.$$

Now,

$$M_{ik}(0) = a^0\bar{\xi}_i\bar{\xi}_k + q^0\gamma_{ik},$$

is positive definite provided $q^0 > 0$ and $q^0 + a^0 > 0$, which are guaranteed to hold by the assumptions (2.9). Thus for $|\sigma|$ small enough, $M_{ik}(\sigma)$ is positive definite as well, which implies that P is hyperbolic.

Having shown the hyperbolicity of P, we can now move on to the question of the geometry of the characteristic surface $Q = 0$. Here the kind of perturbative analysis around P^0 done in the above is no longer helpful, since the characteristic surface corresponding to P^0, although of very simple geometry, is highly degenerate in the sense that because of the coincident shear cones there is a double characteristic in every direction. We expect this picture to be very unstable with respect to perturbations. In the next section we introduce another operator, P^1, which corresponds to the case where $e = e_0 + \tau e_2$. We will obtain the characteristic surface for P^1 and then show that its shape is *stable* under further perturbations.

2.3. The slowness surface in John materials

Consider a hyperelastic solid material with the following constitutive equation:

$$e(s, n, \tau, \delta) = e_0(s, n) + \tau e_2(s, n),$$

where e_0, e_2 are smooth functions satisfying the hyperbolicity (2.9) and causality (2.10) conditions, but otherwise arbitrary. I propose to call materials with a constitutive equation of the above form *John Materials*, in honor of Fritz John's (1910–1994) fundamental contributions to mathematical elasticity. The energy tensor of a John material is therefore of the form

$$T_{\mu\nu} = \rho u_\mu u_\nu + p\gamma_{\mu\nu} + q\sigma_{\mu\nu},$$

where

$$\rho = ne, \quad p = n^2\frac{\partial e_0}{\partial n} + n\tau(n\frac{\partial e_2}{\partial n} - \frac{2}{3}e_2), \quad q = 2ne_2.$$

Let $\xi \in C_x^*$ with $\xi_u \neq 0$ and let $\dot{u} \in \mathrm{Null}(\sigma_\xi)$. Then $\dot{u}$ satisfies $P^1_{ik}\dot{u}^k = 0$ where

$$P^1_{ik}(x, \xi) = (a + b(\sigma_i + \sigma_k))\xi_i\xi_k + (q|\xi|^2 - (\rho + p)\xi_0^2 + \sigma_i q(|\xi|^2 - \xi_0^2))\gamma_{ik}, \tag{2.11}$$

with

$$a = n\frac{\partial p}{\partial n} + \frac{1}{3}q - \frac{2}{3}\tau\frac{\partial p}{\partial \tau}, \quad b = 2\frac{\partial p}{\partial \tau} + q.$$

We note that $P^1|_{\sigma=0} = P^0$, hence P^1 is hyperbolic for σ small enough. Let $Q^1 = \det P^1$. We will now show that the characteristic set

$$C_x^*(P^1) := \{\xi \in \mathbb{R}^4 \mid Q^1(x, \xi) = 0\},$$

contains only four double characteristics, by deriving the *Kelvin form* of the equation (see [6]). Let

$$\begin{aligned}
a_i &:= \frac{(a+b(\sigma_i+\sigma_j))(a+b(\sigma_i+\sigma_k))}{a+b(\sigma_j+\sigma_k)}, \qquad i,j,k \text{ distinct}\\
b_i &:= a+2b\sigma_i - a_i = \frac{-b^2(\sigma_i-\sigma_j)(\sigma_i-\sigma_k)}{a+b(\sigma_j+\sigma_k)}, \qquad i,j,k \text{ distinct}\\
d_i &:= \rho+p+\sigma_i q\\
f_i &:= q(1+\sigma_i),
\end{aligned}$$

and we define

$$\alpha_i := \sqrt{a_i}\xi_i, \qquad \delta_i := b_i\xi_i^2 - d_i\xi_0^2 + f_i\xi_j\xi^j.$$

Then

$$P^1_{ik} = \alpha_i\alpha_k + \gamma_{ik}\delta_i,$$

so that the equation for the characteristic surface is

$$Q^1(\xi) = \alpha_1^2\delta_2\delta_3 + \alpha_2^2\delta_1\delta_3 + \alpha_3^2\delta_1\delta_2 + \delta_1\delta_2\delta_3 = 0. \tag{2.12}$$

Note that $\delta_i = 0$ is the equation for an elliptical cone in $T^*_x\mathcal{M}$, whose trace on the hyperplane $\xi_0 = 1$ is an ellipsoid of revolution with semiaxes r_i and s_i,

$$\frac{\xi_i^2}{s_i^2} + \frac{\sum_{j\neq i}\xi_j^2}{r_i^2} = 1,$$

where

$$r_i^2 := \frac{d_i}{f_i} = \frac{\dfrac{\rho+p}{q}+\sigma_i}{1+\sigma_i}, \qquad s_i^2 := \frac{d_i}{f_i+b_i} = r_i^2 + O(|\sigma|^2)$$

Let us first assume that we are in the generic case where the principal shears $\sigma_1, \sigma_2, \sigma_3$ are distinct, and in particular, $\sigma_1 < \sigma_2 < \sigma_3$. Then since r_i is a monotone function of σ_i, we have $r_3 < r_2 < r_1$ and $s_3 < s_2 < s_1$. Moreover, $b_1 < 0$, $b_2 > 0$ and $b_3 < 0$, so that $s_1 > r_1$, $s_2 < r_2$ and $s_3 > r_3$. Also

$$s_2^2 - r_3^2 \geq r_2^2 - r_3^2 - C(\sigma_3-\sigma_2)(\sigma_2-\sigma_1) \geq C'(\sigma_3-\sigma_2) > 0$$

and similarly $r_2 > s_3$. We therefore conclude that the three ellipsoids $\delta_i = 0$ are mutually disjoint, so that for any ξ, at most one of the $\delta_i(\xi)$ can be zero.

Since Q^1 is a homogeneous polynomial, $C_x^*(P^1)$ is a conic subset of $T_x^*\mathcal{M} \cong \mathbb{R}^4$, and its geometry is therefore determined by a slice

$$\mathcal{S}_x := C_x^*(P^1) \cap \{\xi_0 = 1\}.$$

$\mathcal{S}_x$ is called the *slowness surface* at x. Now let $\xi \in \mathcal{S}_x$. Then $Q^1(\xi) = 0$. Suppose that $\delta_i(\xi) = 0$ for some $1 \leq i \leq 3$. Then $\alpha_i \delta_j \delta_k = 0$ for i, j, k distinct, and thus by the above we have $\alpha_i = 0$, which implies (since $a_i > 0$) that $\xi_i = 0$, and hence the three disjoint and mutually orthogonal circles

$$C_i := \{\xi \in \mathbb{R}^3 \mid \xi_i = 0, \xi_j^2 + \xi_k^2 = r_i^2,\ i, j, k \text{ distinct}\}$$

are part of the slowness surface.

If, on the other hand, $\delta_i \neq 0$ for $i = 1, 2, 3$, then from (2.12) we have that

$$1 + \sum_{k=1}^{3} \frac{\alpha_k^2(\xi)}{\delta_k(\xi)} = 0, \tag{2.13}$$

Let $\lambda := 1/|\xi|^2$ and $\bar{\xi} := \xi/|\xi|$. Define

$$A_i(\bar{\xi}) := \frac{a_i}{d_i}\bar{\xi}^2, \qquad B_i(\bar{\xi}) := \frac{b_i \bar{\xi}_i^2 + f_i}{d_i}.$$

Then (2.13) is equivalent to

$$\sum_{i=1}^{3} \frac{A_i}{\lambda - B_i} = 1. \tag{2.14}$$

For a given $\bar{\xi} \in \mathbb{S}^2$, the values of λ for which (2.14) is satisfied are the abscissae of the intersection points of the line $\varphi = 1$ with the curve

$$\varphi(\lambda) = \frac{A_1}{\lambda - B_1} + \frac{A_2}{\lambda - B_2} + \frac{A_3}{\lambda - B_3}. \tag{2.15}$$

We note that

$$A_i = \left(\frac{a^0}{\rho^0 + p^0} + O(|\sigma|)\right)\bar{\xi}_i^2, \tag{2.16}$$

$$B_i = r_i + O(\Pi_{j \neq i}|\sigma_i - \sigma_j|)\bar{\xi}_i^2, \tag{2.17}$$

and in particular $B_1 > B_2 > B_3$. It is now clear that if none of the A_i are zero, then the curve $\varphi = \varphi(\lambda)$ has three vertical asymptotes at $\lambda = B_i$

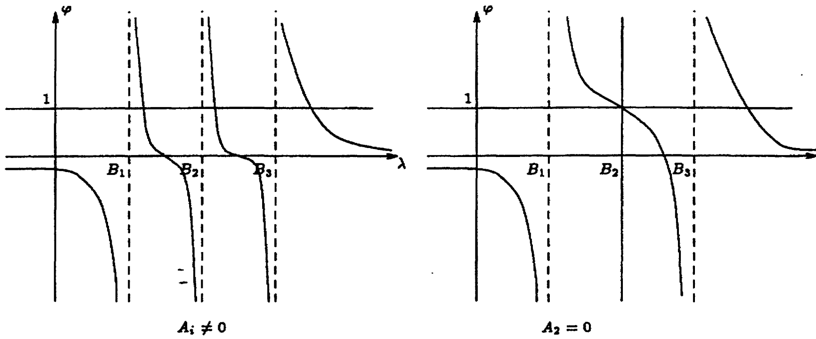

Fig. 1. – The intersections of $\varphi = 1$ with $\varphi = \varphi(\lambda)$

which separate the intersection points with $\varphi = 1$, and in this case there can be no multiple roots for (2.14) (see Figure 1).

Thus if $\mathcal{S}_x$ has multiple points, they have to be on a coordinate plane, where one of the A_i vanishes. If $A_i = 0$, then the vertical line $\lambda = B_i$ is no longer an asymptote but a branch of the curve $\varphi = \varphi(\lambda)$. Thus the intersection points with the line $\varphi = 1$ are still separated, unless $\lambda = B_i$ is *between* the other two asymptotes, in which case a multiple point is again possible (See Figure 1). To be more precise, on a coordinate plane $\xi_i = 0$, the characteristic equation (2.12) factors into a quadratic and a quartic:

$$Q^1\big|_{\xi_i=0} = \delta_i(\alpha_j^2\delta_k + \alpha_k^2\delta_j + \delta_j\delta_k), \qquad i, j, k \text{ distinct.}$$

The quadratic curve $\delta_i = 0$ is the circle C_i. The quartic curve

$$Q_{jk} := (\alpha_j^2\delta_k + \alpha_k^2\delta_j + \delta_j\delta_k)\big|_{\xi_i=0} = 0, \tag{2.18}$$

cannot have a multiple point, at least for small σ, since

$$Q_{jk}\big|_{\sigma=0} = (c_1^2(\xi_j^2 + \xi_k^2) - 1)(c_2^2(\xi_j^2 + \xi_k^2) - 1).$$

Thus the only possibility for a multiple point is that the quartic curve $Q_{jk} = 0$ intersect the circle C_i. Moreover, only one of the two branches of the quartic can intersect the circle, since at $\sigma = 0$, $1/c_1^2 < 1/c_2^2 = r_i^2$, and thus there are at most four double characteristics in this case. In fact it is not hard to see that

$$Q_{jk}\big|_{\xi_j^2+\xi_k^2=r_i^2} = aq(q-\rho-p)\{(\rho+p)(\sigma_j-\sigma_i)+q(\sigma_k-\sigma_j)\xi_j^2\}+O(|\sigma|^2)\xi_j^4,$$

so that the directions in which we can have a double root are close to the following four directions in the coordinate plane $\xi_i = 0$:

$$(\pm\sqrt{\frac{\sigma_i - \sigma_j}{\sigma_k - \sigma_j}}, \pm\sqrt{\frac{\sigma_i - \sigma_k}{\sigma_j - \sigma_k}}, 0).$$

For these directions to be real, it is necessary for σ_i to be the *middle* eigenvalue of σ, i.e., $i = 2$.

In the exceptional case when two of the principal shears coincide, the slowness surface $\mathcal{S}_x$ is a surface of revolution. To see this, suppose that $\sigma_2 = \sigma_3 = \sigma \neq 0$, so that $\sigma_1 = 1/(1+\sigma)^2 - 1$. We then have

$$a_2 = a_3 = \tilde{a}, \quad b_2 = b_3 = 0, \quad d_2 = d_3 = d, \quad f_2 = f_3 = f, \quad \delta_2 = \delta_3 = \delta,$$

where $\tilde{a} := a + 2b\sigma$, $d := \rho + p + \sigma q$, $f := q(1+\sigma)$ and $\delta := f|\xi|^2 - d\xi_0^2$. Therefore,

$$Q^1 = \delta(\alpha_1^2\delta + (\alpha_2^2 + \alpha_3^2)\delta_1 + \delta\delta_1) := \delta\tilde{Q},$$

where, introducing cylindrical coordinates $R^2 = \xi_2^2 + \xi_3^2$, $Z = \xi_1$ we have

$$\begin{aligned}\tilde{Q}\Big|_{\xi_0=1}(R,Z) =& a_1 Z^2[(R^2+Z^2)f - d] \\ &+ [f_1 R^2 + (b_1 + f_1)Z^2 - d_1][(\tilde{a}+f)R^2 + fZ^2 - d],\end{aligned}$$

and thus the surface $\tilde{Q} = 0$ is cylindrically symmetric with respect to the ξ_1-axis. Once again, this quartic surface cannot have any multiple points since at $\sigma = 0$ its sheets are well-separated, hence the only possible multiple points would be the intersections of the outer sheet of this surface with the sphere $\delta = 0$. But then from the above we must have $R = 0$ at the multiple points, so that in this case there are only two double characteristics, located at $(\pm\sqrt{d/f}, 0, 0)$.

The following theorem summarizes all that can be concluded from the above analysis regarding the geometry of the characteristic set of P^1:

THEOREM 2.1. –

1. *The characteristic set $C_x^*(P^1) = \{\xi \mid Q^1(\xi) = 0\}$ is a conic set in $T_x^*\mathcal{M}$, the slice $\xi_u = 1$ of which is a 2-dimensional algebraic surface in $\mathbb{R}^3$ with three real sheets. The innermost sheet of this surface is strictly convex and is separated from the two outer ones.*
2. *In the generic case when the three principal shears are distinct, the two outer sheets touch at only four points. These points lie on a plane $\xi \cdot E = 0$, where E is the eigenvector corresponding to the middle eigenvalue. Moreover, in the coordinate frame given by the eigenvectors of σ, the slowness surface possesses reflectional symmetry with respect to the three coordinate planes. The trace of the surface on a coordinate plane $\xi_i = 0$ is comprised of a circle C_i of radius r_i centered at the origin, together with a non-self-intersecting quartic*

curve which is a small perturbation of two concentric circles with radii $1/c_1$ *and* $1/c_2$ *centered at the origin.*

3. *In the exceptional cases when two of the principal shears coincide, the slowness surface is a surface of revolution whose axis of symmetry is the eigendirection corresponding to the other eigenvalue. The two outer sheets, one of which is a sphere, touch only at two points on that axis.*

Figure 2 shows the positive octant of the surface in the generic and the exceptional cases.

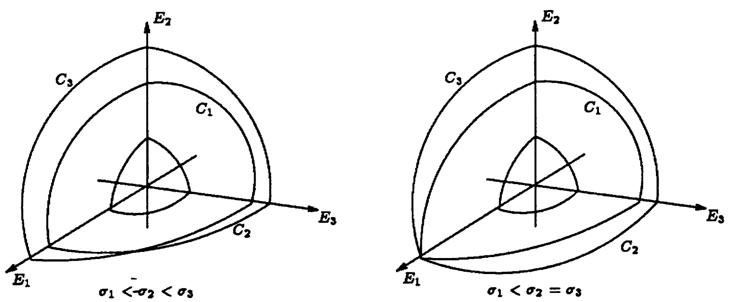

Fig. 2. – The Slowness Surface.

2.4. Stability of the characteristic surface

We now return to the general case of a material with an arbitrary constitutive equation (1.17) which is subject only to conditions (2.9–2.10). The corresponding operator P, defined in (2.6), can be thought of as a (symmetric) perturbation of the operator P^1 introduced in the previous section: $P = P^1 + O(|\sigma|)$. We want to show that for small $|\sigma|$, the characteristic set of P has the same qualitative shape as that of P^1, i.e., P has only four non-degenerate double characteristics close to those of P^1.

The study of perturbations of non-strictly hyperbolic 3×3 systems was taken up by F. John in [8], where he studies the neighborhood, in the space of all second-order, linear, constant coefficient operators, of a specific non-strictly hyperbolic operator J with four non-degenerate double characteristics:

$$J_{ik}(\lambda, \xi) := (\lambda^2 - \epsilon_i|\xi|^2)\delta_{ik} - (1 - \epsilon_i)\xi_i\xi_k, \qquad \text{for } 0 < \epsilon_1 < \epsilon_2 < \epsilon_3 < 1,$$

which he calls the "modified elasticity operator." He shows that for a general (nonsymmetric) perturbation of J to be hyperbolic, it has to satisfy certain *identities*, since the space of all hyperbolic operators in that neighborhood has a positive codimension. From his proof it follows that any hyperbolic perturbation of J must necessarily be non-strictly hyperbolic, and in fact has exactly four non-degenerate double characteristics close to those of J. We note that the slowness surface of John's elasticity operator J looks just as in Figure 2, which justifies naming the material after him [3].

Expanding on John's work, the following general result on the stability of non-degenerate double characteristics was later obtained by Hörmander [7], and also independently by Bernardi and Nishitani [1]. We recall that a nonzero vector $\xi^0 \in \mathbb{R}^4$ is called a *characteristic* of $P \in \mathcal{P}$ if $\det P(\xi^0) = 0$. ξ^0 is *simple* if the zero of $\det P(\xi)$ at $\xi = \xi^0$ is of order one, and *double* if it is of order two. It is easy to see that at a double characteristic, the rank of the Hessian of $\det P$ is at most 3. If this rank is 3 then the double characteristic is called *non-degenerate.*

THEOREM (Hörmander). – *Let $P \in \mathcal{P}$ be hyperbolic and have a non-degenerate double characteristic at ξ^0, with* $\dim\ker P(\xi^0) = 2$. *Let $\tilde{P} \in \mathcal{P}$ be close enough to P. If $\tilde{P}$ is hyperbolic, then it also has a non-degenerate double characteristic, which is close to ξ^0.*

Thus in our case, we need to examine the Hessian of Q^1 at the multiple points. It will turn out that in the generic case, the rank of the Hessian is indeed 3, and thus we can conclude that P has four double characteristics close to those of P^1. In the exceptional case however, the rank of the Hessian is one, and thus the addition of higher-order terms to P^1 can change the shape of the characteristics.

We recall that $Q^1(\xi)$ is a homogeneous polynomial of degree six in ξ_μ, $\mu = 0, \ldots, 3$. Since the Hessian $\nabla^2 Q^1$ is also homogeneous, it is enough to find the rank of its restriction to $\xi_0 = 1$. Let

$$H_{\mu\nu}(\xi_1, \xi_2, \xi_3) := (\partial_\mu \partial_\nu Q^1)\big|_{\xi_0 = 1}.$$

[3] The origin of John's operator remains a mystery. John himself is silent about this matter in [8], except for mentioning that in the excluded case where the ϵ_i are equal, J becomes the linear elasticity operator P^0. Hörmander believes [7] that John arrived at this operator by taking the Lamé operator of crystal optics for the case of a biaxial crystal and replacing the planar portion of the characteristic set with a light cone. Indeed, the Fresnel surface does come up in [8]. However, in view of the similarities between J and P^1, a direct motivation from 3-D elasticity, or perhaps from crystal acoustics, cannot be ruled out. It is interesting to note that the relativistic constitutive equation $\rho = q_2$ for an elastic solid will also give rise to the same structure, i.e. the Fresnel surface with a unit sphere added inside [3].

Let us assume that we are in the generic case, where the principal shears σ_i are distinct. By renumbering the axes we can arrange that σ_3 is the middle one, so that the double characteristics set is

$$D_x^* = \{\xi \in C_x^*(P^1) \mid \xi_3 = 0,\ \xi_1^2 + \xi_2^2 = r_3^2,\ Q_{12} = 0\},$$

where Q_{12} is the quartic defined in (2.18). Since Q^1 is in fact cubic in the squares of the ξ_μ, there are no terms in H which are linear in ξ_3, so that

$$H(\xi_1, \xi_2, 0) = \begin{pmatrix} \tilde{H}(\xi_1, \xi_2) & 0 \\ 0 & H_{33} \end{pmatrix}.$$

Thus to show that H is rank three on D_x^* it is enough to show that at the double characteristics, $H_{33} \neq 0$ and that the rank of $\tilde{H}$ is at least two.

Let $\bar{H}$ be the 2×2 submatrix in the lower right corner of $\tilde{H}$. Then $\bar{H}_{ij} = \partial_i \partial_j (Q(1, \xi_1, \xi_2, 0))$, $i, j = 1, 2$. It then suffices to show that $\bar{H}$ is nonsingular. Now,

$$Q(1, \xi_1, \xi_2, 0) = \delta_3(1, \xi_1, \xi_2, 0) Q_{12}(1, \xi_1, \xi_2, 0),$$

Therefore

$$\bar{H}_{ij}\big|_{\delta_3=0, Q_{12}=0} = 2 f_3 (\xi_i \partial_j Q_{12} + \xi_j \partial_i Q_{12}),$$

which is rank 2 if ξ and ∇Q_{12} are linearly independent, i.e., if the intersection of the quartic curve $Q_{12} = 0$ with the circle C_3 is transverse. It is easy to compute that at an intersection point,

$$\xi_1 \partial_2 Q_{12} - \xi_2 \partial_1 Q_{12} = a^0 q^0 (\rho^0 + p^0 - q^0)(\sigma_1 - \sigma_2) + O(|\sigma|^2),$$

which is nonzero since we have assumed $\sigma_1 \neq \sigma_2$. Hence $\bar{H}$ is nonsingular.

To calculate H_{33}, we note

$$H_{33} = \partial_3^2 Q^1(1, \xi_1, \xi_2, \xi_3)\big|_{\xi_3=0, \xi_1^2+\xi_2^2=r_3^2},$$

so that, from the expression (2.12) for Q^1 we have

$$\begin{aligned} H_{33} &= 2\big[a_3 \delta_1 \delta_2 + (b_3 + f_3)(\alpha_2^2 \delta_1 + \alpha_1^2 \delta_2 + \delta_1 \delta_2)\big]_{\xi_1^2+\xi_2^2=r_3^2, \xi_3=0} \\ &= 2 \frac{a_3}{f_3^2} \left[f_3 b_1 \xi_1^2 + f_1 d_3 - d_1 f_3 \right] \left[b_2 (d_3 - f_3 \xi_1^2) + f_2 d_3 - d_2 f_3 \right] \\ &= 2 a^0 (\rho^0 + p^0 - q^0)^2 (\sigma_1 - \sigma_3)(\sigma_2 - \sigma_3) + O(|\sigma|^3), \end{aligned}$$

which is again nonzero since we are in the generic case. We have thus shown that the four double characteristics of P^1 are non-degenerate, so that by the above theorem of Hörmander, any hyperbolic perturbation of P^1 must also have exactly four double characteristics which are close to those of P^1. We have thus established the following:

THEOREM 2.2. – *At a point $x \in \mathcal{M}$ where the three principal shears σ_i are distinct, the hyperbolic operator whose symbol is P defined in (2.6) has exactly four non-degenerate double characteristics which are $O(|\sigma|)$-close to the following four directions*

$$\left(\frac{q^0}{\rho^0 + p^0}, \pm\sqrt{\frac{\sigma_3 - \sigma_1}{\sigma_2 - \sigma_1}}, \pm\sqrt{\frac{\sigma_3 - \sigma_2}{\sigma_1 - \sigma_2}}, 0\right).$$

3. NONRELATIVISTIC ELASTODYNAMICS

3.1. Nonrelativistic dynamics and the relationship between the two formulations

Corresponding to the system (1.6, 1.8), there is a first-order system of equations for the nonrelativistic dynamics of a solid body, which can be obtained through the following well-known procedure: One begins with the Minkowski metric for the spacetime $\mathcal{M}$:

$$ds^2 = -(dx^0)^2 + \sum_{i=1}^{3}(dx^i)^2,$$

and introduces the Newtonian time coordinate $t := x^0/c$, the Newtonian velocity $v^i := cu^i/u^0$, and the internal energy [4] $\varepsilon := \rho - nc^2$, where c denotes the speed of light in vacuum. Taking the limit $c \to \infty$ in (1.6, 1.8), one then obtains:

$$(\partial_t v^i + v^k \partial_k v^i) + \partial_j S^{ij} = 0 \tag{3.1}$$

$$\partial_t h_{ij} + \mathfrak{L}_v h_{ij} = 0 \tag{3.2}$$

$$\partial_t s + v^k \partial_k s = 0 \tag{3.3}$$

where S is the stress tensor

$$S_{ij} := 2\varepsilon_1 h_{ij} + 2\varepsilon_2(q_1 h_{ij} - {h^2}_{ij}) + (2q_3\varepsilon_3 - \varepsilon)\delta_{ij},$$

[4] Here we are assuming that the material is made up of only one kind of (macroscopic) particle, the rest mass of which is normalized to be one.

with $\varepsilon_k := \partial\varepsilon/\partial q_k$. The equation of continuity is a consequence of (3.2):

$$\partial_t n + \partial_i(nv^i) = 0. \tag{3.4}$$

Using it, the momentum equation (3.1) can also be written in the form

$$\partial_t(nv^i) + \partial_j(nv^iv^j + S^{ij}) = 0. \tag{3.5}$$

The adiabatic condition (3.3) is satisfies by C^1 solutions. More generally, the energy equation is satisfied:

$$\partial_t(\varepsilon + \frac{1}{2}n|v|^2) + \partial_i\{(\varepsilon + \frac{1}{2}n|v|^2)v^i + S^i_jv^j\} = 0.$$

The above nonrelativistic equations can also be put in the same context as the relativistic ones by letting f be a mapping from the Galilean spacetime $\mathcal{G}$, with coordinates (t, x^i), into the material manifold $\mathcal{N}$, which is now taken to be a region the three-dimensional Euclidean space, and let f_t denote its restriction to the time slice $\{t\} \times \mathbb{R}^3 \subset \mathcal{G}$. We define h to be the pullback of the Euclidean metric δ under f_t, i.e.

$$h_{ij}(t,x) = \delta_{ab}(y)\frac{\partial y^a}{\partial x^i}\frac{\partial y^b}{\partial x^j}, \tag{3.6}$$

where $y(t,x) = f_t(x) \in \mathcal{N}$. The *internal energy per unit deformed volume* ε is then assumed to be a function of the three principal invariants q_1, q_2, q_3 of h, as well as depending on the entropy s.

We thus have two different formulations for nonrelativistic elastodynamics: The one given in the beginning of this paper, which leads to a second-order system for the displacement field, and the one described above which gives a first-order system for the velocity and strain fields (reminiscent of the Euler equations of fluid dynamics). The relationship between the two formulations becomes clear when we note that the deformation $\phi : \mathbb{R} \times \mathcal{N} \to \mathbb{R}^3$ is the inverse of the mapping f:

$$x = \phi(t,y) = f_t{}^{-1}(y).$$

Thus, with F denoting the deformation gradient as before, $F^i_a = \partial x^i/\partial y^a$, from (3.6) we have that

$$h = F^{-1}(F^{-1})^T = (F^TF)^{-1}.$$

Therefore, the matrix h is the inverse of the classical right Cauchy-Green strain matrix. Since FF^T and F^TF have the same eigenvalues, we conclude that the eigenvalues of h are the reciprocals of the eigenvalues of B, and hence the following relations hold between the principal invariants of B and h:

$$q_1 = \frac{\imath_2}{\imath_3}, \qquad q_2 = \frac{\imath_1}{\imath_3}, \qquad q_3 = \frac{1}{\imath_3}. \tag{3.7}$$

Moreover, by comparing the expressions for the energy integral in the two formulations we can find the relationship between W and ε:

$$\int_{\mathcal{N}} W(y)\, d^3y = \int_{f_t^{-1}(\mathcal{N})} (W \circ f_t)(x)\, |D_x f_t|\, d^3x = \int_{\mathbb{R}^3} (W \circ f_t)(x) n(x) d^3x.$$

The integrand in the last integral above must be equal to the energy per unit deformed volume $\varepsilon(x)$, and hence $W \circ f_t = \varepsilon / n$. In other words

$$W(\imath_1, \imath_2, \imath_3) = \frac{1}{\sqrt{q_3}} \varepsilon(q_1, q_2, q_3). \tag{3.8}$$

It should be stressed that in the general-relativistic case one cannot use a formulation based on displacements, and thus the variational formulation is the only one available, while for nonrelativistic dynamics, either of the two formulations can be used, and it is conceivable that for a given problem, one is more appropriate than the other. Later on in this section we will have the opportunity to use both of these formulations.

3.2. Characteristics in the nonrelativistic case

We can calculate the symbol of the nonrelativistic equations (3.1–3.3) in a similar way. By going into a Galilean frame moving with the body we can set $v^k = 0$ at the point that the symbol is calculated. Taking variations we then obtain, with (τ, ξ) in the cotangent bundle to the Galilean spacetime,

$$\begin{aligned} \tau(n\dot{v}^i) + \xi_j \dot{S}^{ij} &= 0 \\ \tau\dot{h}_{ij} + (h_{jk}\xi_i + h_{ik}\xi_j)\dot{v}^k &= 0 \\ \tau\dot{s} &= 0 \end{aligned}$$

Moreover, the symbol of the continuity equation (3.4) is

$$\tau\dot{n} + \xi_k n \dot{v}^k = 0.$$

Once again, $\tau = 0$ is a characteristic, and if $\tau \neq 0$ then $\dot{s} = 0$ and we obtain that $\dot{n} = -n\xi_k\dot{v}^k/\tau$. We can now introduce the shear tensor σ by setting

$h_{ij} = n^{2/3}(\delta_{ij} + \sigma_{ij})$. Once again it follows that $S_{ij} = p\delta_{ij} + q\sigma_{ij} + w\sigma^2{}_{ij}$, with p, q, w as before. Continuing the calculation as in the relativistic case, we obtain that if (τ, ξ) belong to the characteristic set, then $M_{ik}\dot{v}^k = 0$ for some nonzero $\dot{v}$, where

$$M_{ik} = C_{ik}^{mn}(\sigma)\xi_m\xi_n - n\delta_{ik}\tau^2$$

where C_{ik}^{mn} is exactly as in the relativistic case. Thus the only difference is that the factor $D_i(\sigma)$ of the relativistic case is here replaced by n. This does not affect the above analysis concerning the geometry of characteristics, and all the above conclusions are thus valid in the nonrelativistic case as well.

3.3. Elastic waves in pre-stressed materials

In order to show that the formulation developed in the above for relativistic dynamics is not without application to the nonrelativistic case, we present here two results on the propagation of elastic waves in materials which are under a constant isotropic deformation. The results are easy extensions of two theorems by Sideris, one on the global existence of small-amplitude waves in hyperelastic materials [14], and the other on the formation of singularities for compressible Euler equations [13].

3.3.1. *Small-Data Global Existence*

In [14], Sideris considers the problem of global existence of nonlinear, isentropic elastic waves in an isotropic, homogeneous, hyperelastic material which is initially stress-free and is filling the whole space. He shows that if the stored energy function W satisfies certain conditions —called the *null conditions*, then the Cauchy problem for the displacement admits globally regular solutions, provided the initial data is sufficiently small in some high Sobolev norm. The null conditions are algebraic relations between the derivatives of W evaluated at the stress-free reference state. Previously, John had shown [9] that if these conditions do not hold, then at least in the spherically symmetric case singularities will develop from arbitrary small data after a very long time.

Here we provide an extension of Sideris's result to the case where the solid material under study is not initially stress-free but instead subjected to an initial constant isotropic deformation $y \mapsto \lambda y$ for some fixed $\lambda > 0$. The proof consists of repeating Sideris's argument while keeping track of the factors of λ, and thus will be largely omitted. It suffices to say that, following his method, analogous null conditions can be derived in which λ appears as a parameter. If we accept that global existence of nonlinear waves should not depend on the exact amount of pre-stress, then these conditions have to hold for all λ in some interval, which makes them differential equations in λ. By making suitable assumptions about the stored energy function, we can solve these equations to obtain the general class of materials for which the global existence result holds. In this way we hope to shed some light on the null conditions of Sideris.

We use the classical formulation of elastodynamics, as in [14], and start with the system (1.1). We consider small displacements off a pre-stressed state by letting

$$u(t,y) := \phi(t,y) - \lambda y$$

denote the perturbation. Here $\lambda < 1$ corresponds to initial compression and $\lambda > 1$ to the initial extension of the material. Let $G := D_y u = F - \lambda I$ and $C := B - \lambda^2 I = \lambda(G + G^T) + GG^T$. Let $\jmath_1, \jmath_2, \jmath_3$ denote the principal invariants of C. We have

$$\begin{aligned} \imath_1 &= 3\lambda^2 + \jmath_1 \\ \imath_2 &= 3\lambda^4 + 2\lambda^2 \jmath_1 + \jmath_2 \\ \imath_3 &= \lambda^6 + \lambda^4 \jmath_1 + \lambda^2 \jmath_2 + \jmath_3. \end{aligned}$$

We can thus regard W as a function of the $\jmath_k$ instead of the $\imath_k$, defining

$$\Sigma(\jmath_1, \jmath_2, \jmath_3) := W(\imath_1, \imath_2, \imath_3).$$

Hence

$$\frac{\partial W}{\partial F} = \frac{\partial W}{\partial G} = \frac{\partial \Sigma}{\partial \jmath_k} \frac{\partial \jmath_k}{\partial G}.$$

Moreover, up to fourth-order terms in G,

$$\begin{aligned} \jmath_1 &= 2\lambda \mathrm{tr} G + \mathrm{tr} GG^T \\ \jmath_2 &= 2\lambda^2 (\mathrm{tr} G)^2 - \lambda^2 \mathrm{tr} G^2 - \lambda^2 \mathrm{tr} GG^T + 2\lambda \mathrm{tr} G \mathrm{tr} GG^T - 2\lambda \mathrm{tr} G^2 G^T + \dots \\ \jmath_3 &= \frac{4}{3}\lambda^3 (\mathrm{tr} G)^3 - 2\lambda^3 \mathrm{tr} G (\mathrm{tr} G^2 + \mathrm{tr} GG^T) + \frac{2}{3}\lambda^3 \mathrm{tr} G^3 + 2\lambda^3 \mathrm{tr} G^2 G^T + \dots \end{aligned}$$

so that, upon Taylor-expanding Σ in powers of $\jmath_k$ around $\jmath_k = 0$ and substituting in (1.1), keeping only terms of up to second order in u and its derivatives [5], we obtain the following equation for the displacement u:

$$Lu = N(u, u), \tag{3.9}$$

[5] It was shown by Klainerman [11] that cubic terms in the nonlinearity do not affect the small-data global existence in elastodynamics.

where N denotes the quadratic terms, and L is the linear elasticity operator defined in (1.2), with the two propagation speeds

$$c_1^2 = 2(\Sigma_1 + 2\lambda^2\Sigma_{11}), \qquad c_2^2 = 2(\Sigma_1 - \lambda^2\Sigma_2). \tag{3.10}$$

Here and later on Σ_k, Σ_{mn}, etc. denote the partial derivatives of Σ with respect to the $\jmath_k$'s, evaluated at the isotropic background state $\jmath_1 = \jmath_2 = \jmath_3 = 0$.

In Sideris's work [14, §4], the quadratic nonlinearity $N(u, u)$ is analyzed in order to identify the terms in it with bad decay. Those terms are then eliminated by putting conditions on Σ. Following that procedure, it is not hard to see that in our case, the null conditions of Sideris have the form

$$\Sigma_{11} = \lambda^2\Sigma_{12}, \qquad 3\Sigma_{12} + 2\Sigma_{111} = 0, \qquad \forall\lambda > 0. \tag{3.11}$$

We note that here, $\Sigma_{11}, \Sigma_{12}, \Sigma_{111}$ are functions of the parameter λ. The extension of Sideris's result is therefore the following:

THEOREM 3.1. – *Assume that the stored energyfunction of a hyperelastic solid material satisfies the null conditions (3.11). Then the corresponding Cauchy problem for (3.9) with initial data that are small enough* [6] *has a unique global solution.*

Our task is now to identify a class of materials where the null conditions are satisfied. In the above we have assumed the perturbation u to be small, while no smallness assumption has been made about the background isotropic pressure. We are thus in a situation where the strains are not uniformly small. This motivates us to use the nonrelativistic version of the shear strain tensor σ:

$$\sigma = (\det h)^{-1/3}h - I = (\det B)^{1/3}(F^TF)^{-1} - I,$$

so that $\sigma = 0$ if F is a multiple of the identity. The invariants of B and σ are related as follows:

$$\tau = \imath_3^{-2/3}\imath_2 - 3, \qquad \delta = \imath_3^{-2/3}\imath_2 - \imath_3^{-1/3}\imath_1.$$

and we have $n = \imath_3^{-1/2}$ as before. Note that $\jmath_1 = \jmath_2 = \jmath_3 = 0$ corresponds now to $\tau = \delta = 0$, i.e. the isotropic background state. Moreover, by (3.8),

$$\Sigma(\jmath_1, \jmath_2, \jmath_3) = W(\imath_1, \imath_2, \imath_3) = e(n, \tau, \delta),$$

[6] See [14, Thm. 5.1.] for the precise smallness condition.

where $e$is again the energy per particle [7]. If we assume as before that e has an expansion of the form (1.17), then we can calculate explicitly the partial derivatives of Σ that appear in (3.10) and (3.11) in terms of the parameter λ. The remainder term in the expansion (1.17) will not affect this calculation since only derivatives of up to the third order are needed:

$$\begin{aligned}
\Sigma_1 &= -\frac{1}{2}\lambda^{-5}e_0'(\lambda^{-3})\\
\Sigma_2 &= -\frac{1}{2}\lambda^{-7}e_0'(\lambda^{-3}) - \lambda^{-4}e_2(\lambda^{-3})\\
\Sigma_{11} &= \frac{1}{4}\lambda^{-10}e_0''(\lambda^{-3}) + \frac{3}{4}\lambda^{-7}e_0'(\lambda^{-3}) + \frac{2}{3}\lambda^{-4}e_2(\lambda^{-3})\\
\Sigma_{12} &= \frac{1}{4}\lambda^{-12}e_0''(\lambda^{-3}) + \frac{3}{4}\lambda^{-9}e_0'(\lambda^{-3}) + \frac{4}{3}\lambda^{-6}e_2(\lambda^{-3}) + \frac{1}{2}\lambda^{-9}e_2'(\lambda^{-3})\\
&\quad + \frac{1}{3}\lambda^{-6}e_3(\lambda^{-3})\\
\Sigma_{111} &= -\frac{1}{8}\lambda^{-15}e_0'''(\lambda^{-3}) - \frac{9}{8}\lambda^{-12}e_0''(\lambda^{-3}) - \frac{15}{8}\lambda^{-9}e_0'(\lambda^{-3})\\
&\quad - \frac{20}{9}\lambda^{-6}e_2(\lambda^{-3}) - \lambda^{-9}e_2'(\lambda^{-3}) - \frac{4}{9}\lambda^{-6}e_3(\lambda^{-3}).
\end{aligned}$$

Setting $n = \lambda^{-3}$, the null conditions (3.11) thus take the form of two ordinary differential equations for three unknown functions:

$$\begin{cases} 3ne_2'(n) + 4e_2(n) + 2e_3(n) = 0\\ 9n^3e_0'''(n) + 54n^2e_0''(n) + 54ne_0'(n) + 18ne_2'(n) + 16e_2(n) - 4e_3(n) = 0. \end{cases}$$

Thus e_2 and e_3 can be found in terms of e_0:

$$e_2(n) = cn^{-1} - \frac{3}{8}n^2e_0''(n) - \frac{9}{8}ne_0'(n) \tag{3.12}$$

$$e_3(n) = -\frac{c}{2}n^{-1} + \frac{9}{16}n^3e_0'''(n) + \frac{57}{16}n^2e_0''(n) + \frac{63}{16}ne_0'(n), \tag{3.13}$$

with c an arbitrary constant. The arbitrary function e_0 is subject to further conditions to ensure the hyperbolicity of equations, or equivalently, the linear stability: The propagation speeds c_1 and c_2, have to be real, with $c_1 > c_2 > 0$. This implies that there must exist a constant $C > 0$ such that

$$(n^3e_0'(n))' < C \tag{3.14}$$

$$3n^{4/3}(n^{5/3}e_0'(n))' > -C \tag{3.15}$$

[7] Since we are in the isentropic case the dependence on the entropy is suppressed throughout this section.

In addition, it is reasonable to assume that the stress-free state $n = 1, \tau = \delta = 0$ is a minimum of the energy, so that

$$e_0(1) = 0, \qquad e_0'(1) = 0. \tag{3.16}$$

We thus arrive at the following:

PROPOSITION 1. – *If the stored energy function* Σ *of a homogeneous, isotropic, hyperelastic solid satisfies* (1.18), *then the null conditions* (3.11) *hold for* Σ *if and only if the coefficients* e_i *in the expansion* (1.17) *satisfy* (3.12–3.15).

Remark 2. – It should be noted that conditions (3.14–3.15) are overly restrictive. In particular, they do not allow for $e_0(n) \to \infty$ as $n \to \infty$, contrary to what is often assumed about stored energy functions. In fact, by integrating (3.14) it is easy to see that the following is a necessary condition for e_0:

$$e_0(n) \le \frac{C}{2}(1 - \frac{1}{n})^2.$$

Remark 2. – As a simple example of materials satisfying the null conditions, we may consider the class of Hadamard materials, where W has the following form

$$W(\imath_1, \imath_2, \imath_3) = a\imath_1 + b\imath_2 + f(\imath_3).$$

It is then easy to see that W satisfies the null conditions (3.11) if and only if

$$f(\imath_3) = c\imath_3 + d_1\sqrt{\imath_3} + d_2,$$

where

$$d_1 := -2(a + 2b + c), \qquad d_2 := b + c - a.$$

The corresponding coefficients in the expansion (1.17) are as follows:

$$\begin{aligned} e_3(n) &= -an^{-2/3} \\ e_2(n) &= an^{-2/3} + bn^{-4/3} \\ e_0(x) &= 3an^{-2/3} + 3bn^{-4/3} + cn^{-2} + d_1n^{-1} + d_2. \end{aligned}$$

The two speeds of propagation (measured with respect to coordinates in the undeformed state) are thus

$$c_2^2 = 2n^{2/3}e_2(n) = 2a + 2bn^{-2/3},$$
$$c_1^2 = n^{2/3}[(n^2e_0'(n))' + \frac{8}{3}e_2(n)] = 2a + 4bn^{-2/3} + 2cn^{-2},$$

and hence the hyperbolicity requirement is

$$a > 0, \qquad b > 0, \qquad c > 0.$$

3.3.2. *Large-Data Blowup*

In this section we provide an example of a Cauchy problem for the nonrelativistic system (3.1–3.3), whose C^1 solutions cannot have infinite lifespan. This is an extension to elastodynamics of Sideris's large-data blowup result [13] for compressible Euler equations, and the proof follows his original argument very closely. We consider solid materials with a constitutive equation of the form

$$e = e_0 + \tau e_2 + \delta e_3, \tag{3.17}$$

where e_i satisfy the following conditions:

(M1) $\partial e_2/\partial n \equiv 0$, i.e., $e_2 = e_2(s)$.

(M2) $e_2(s) > 0$ for all $s > 0$.

(M3) $\partial e_3/\partial n \equiv 0$, i.e., $e_3 = e_3(s)$.

Let S denote the corresponding stress tensor. We define the "pressure":

$$\bar{p} := \frac{1}{3}\mathrm{tr}S = n^2\frac{\partial e}{\partial n} = n^2\frac{\partial e_0}{\partial n}.$$

and assume that

(M4) $\bar{p}(n, s)$ is an increasing, convex function of n, for each $s > 0$.

(M5) $\bar{p}(n, s)$ is a non-decreasing function of s, for each $n > 0$.

We propose initial data that represent a compactly-supported perturbation of some quiet background isotropic state, i.e., we assume that

(D1) There exists $R_0 > 0$ such that $v(0, x) \equiv 0$, $h(0, x) \equiv n_0^{2/3} I$, and $s(0, x) = s_0$, for $|x| > R_0$, where n_0 and s_0 are positive constants.

By a Domain of Dependence argument, it follows that as long as the solution remains C^1,

$$v(t, x) \equiv 0, \qquad h(t, x) \equiv n_0^{2/3} I, \qquad s(t, x) \equiv s_0, \quad \text{for } |x| > R(t), \tag{3.18}$$

where

$$R(t) := R_0 + c_1^0 t,$$

and c_1^0 is the speed of propagation of longitudinal waves in the quiet background, i.e.

$$c_1^0 := \sqrt{\eta_0^2 + \frac{8}{3}e_2(s_0)}, \quad \text{where } \eta_0^2 := \frac{\partial \bar{p}}{\partial n}(n_0, s_0).$$

Let

$$Q(t) := \int_{\mathbb{R}^3} x^i v_i(t,x) n(t,x)\, d^3x \tag{3.19}$$

be the total radial momentum, which is finite by (3.18). Let p_0 denote the background pressure,

$$p_0 := \bar{p}(n_0, s_0),$$

and let S_0 denote the background stress tensor,

$$S_0^{ij} := p_0 \delta^{ij}.$$

By (3.5) we then have

$$\begin{aligned}\frac{dQ}{dt} &= -\int_{|x|\le R(t)} x_i \partial_j (nv^i v^j + S^{ij})\, d^3x = -\int_{\mathbb{R}^3} x_i \partial_j (nv^i v^j + S^{ij} - S_0^{ij})\, d^3x \\ &= \int_{\mathbb{R}^3} \delta_{ij}(nv^i v^j + S^{ij} - S_0^{ij})\, d^3x = \int_{\mathbb{R}^3} n|v|^2 + 3(\bar{p} - p_0),\end{aligned}$$

Let

$$M(t) := \int_{\mathbb{R}^3} (n(t,x) - n_0)\, d^3x.$$

Then by (3.4) we have $dM/dt \equiv 0$, so that M is conserved: $M(t) \equiv M(0) =: M_0$. Our further assumptions on the initial data are that

(D2) $M_0 \ge 0$.

(D3) $s(0,x) \ge s_0$ for all x.

The adiabatic condition (3.3) shows that the entropy s remains constant along the flow lines, and thus (D3) implies that $s(t,x) \ge s_0$ for $t > 0$ as well. Hence, by (M4) and (M5),

$$\bar{p}(n,s) - p_0 = \bar{p}(n,s) - \bar{p}(n,s_0) + \bar{p}(n,s_0) - \bar{p}(n_0,s_0) \ge \eta_0^2\,(n - n_0).$$

It then follows that

$$\int_{\mathbb{R}^3} (\bar{p} - p_0)\, d^3x \ge \eta_0^2 M_0 \ge 0,$$

so that we have

$$\frac{dQ}{dt} \ge \int_{\mathbb{R}^3} n|v|^2\, d^3x.$$

Therefore Q is an increasing function of t. In particular $Q(t) \geq 0$ for all $t > 0$ if $Q_0 := Q(0) \geq 0$. On the other hand

$$|Q|^2 \leq \int n|x|^2 d^3x \int n|v|^2 d^3x \leq R^2(t)[M_0 + \frac{4\pi n_0}{3} R^3(t)] \int n|v|^2 d^3x,$$

so that we arrive at the following differential inequality:

$$\frac{dQ}{dt} \geq \frac{Q^2}{M_0 R^2(t) + \frac{4\pi}{3} n_0 R^5(t)}.$$

Integrating this, we have

$$\frac{1}{Q(t)} \leq \frac{1}{Q_0} - \frac{1}{c_1^0} \int_{R_0}^{R_0 + c_1^0 t} \frac{dR}{M_0 R^2 + \frac{4\pi}{3} n_0 R^5}.$$

This contradicts the positivity of Q provided our last assumption on the initial data holds, i.e., if

(D4) $Q_0 > c_1^0 \left(\int_{R_0}^{\infty} \frac{dR}{M_0 R^2 + \frac{4\pi}{3} n_0 R^5} \right)^{-1}$. The contradiction implies that there exists a certain $T^* < \infty$ by which time a C^1 solution has to develop a singularity. In particular, the domain-of-dependence may break down at an earlier time, perhaps because a shock discontinuity forms. We have thus proved:

THEOREM 3.2. – *Suppose that the constitutive equation of a hyperelastic solid is of the form* (3.17) *and satisfies* (M1–M5). *Then the corresponding Cauchy problem for* (3.1)–(3.3) *with initial data satisfying* (D1–D4) *cannot have a global-in-time C^1 solution.*

ACKNOWLEDGEMENT

The author wishes to thank D. Christodoulou for many illuminating discussions and for being a constant source of ideas and encouragement.

REFERENCES

[1] E. BERNARDI and T. NISHITANI, Remarks on symmetrization of 2×2 systems and the characteristic manifolds. *Osaka J. Math.*, Vol. **29**, 1992, pp. 129–134.

[2] B. CARTER and H. QUINTANA, Foundations of general relativistic high-pressure elasticity theory. *Proc. R. Soc. Lond.*, A. 331, 1972, pp. 57–83.

[3] D. CHRISTODOULOU, Lecture notes of course taught at Princeton University, Fall 1995.

[4] D. Christodoulou, Self-gravitating relativistic fluids: A two-phase model. *Arch. Rational Mech. Anal.*, Vol. **130**, 1995, pp. 343–400.
[5] D. Christodoulou, On the geometry and dynamics of crystalline continua. *Ann. Inst. H. Poincaré Phys. Théor*, to appear.
[6] G. F. D. Duff, The Cauchy problem for elastic waves in an anisotropic medium. *Phil. Trans. Roy. Soc. A.*, Vol. **252**, 1960, pp. 249–273.
[7] L. Hörmander, Hyperbolic systems with double characteristics. *Comm. Pure. Appl. Math.*, Vol. **46**, 1993, pp. 261–301.
[8] F. John, Algebraic conditions for hyperbolicity of systems of partial differential equations. *Comm. Pure. Appl. Math.*, Vol. **31**, 1978, pp. 89–106.
[9] F. John, Formation of singularities in elastic waves. *Lec. Notes in Phys.*, Vol. **195**, 1984, pp. 190–214.
[10] J. Kijowski and G. Magli, Relativistic elastomechanics as a Lagrangian field theory. *Journal of Geometry and Physics*, Vol. 9, 1992, pp. 207–223.
[11] S. Klainerman, Long-time behavior of solutions to nonlinear evolution equations. *Arch. Rational Mech. Anal*, Vol. 78(1), 1982, pp. 73–98.
[12] G. A. Maugin, Exact relativistic theory of wave propagation in prestressed nonlinear elastic solids. *Ann. Inst. H. Poincaré Phys. Théor.*, Vol. 28, 1978, p. 155–185.
[13] T. C. Sideris, Formation of singularities in three-dimensional compressible fluids. *Comm. Math. Phys.*, Vol. **101**, 1985, pp. 475–485.
[14] T. C. Sideris, The null condition and global existence of nonlinear elastic waves. *Invent. Math.*, Vol. **123**, 1996, pp. 323–342.
[15] J. M. Souriau, *Géométrie et Relativité*. Hermann, Paris, 1964.

(Manuscript received February 18th, 1997.)