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Section B :
Calcul des Probabilités et Statistique.

Some properties of martingale integrals

by

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RÉSUMÉ. — Les intégrales stochastiques de la forme $\int_0^t H_u dM_u$ ont été longtemps étudiées sous diverses hypothèses sur $H = (H_t, \mathbf{F}_t)$ et sur $M = (M_t, \mathbf{F}_t)$: par exemple H est prévisible et M est une martingale de carré intégrable etc. En 1966, P. A. Meyer [*4*] a donné une définition des intégrales stochastiques par rapport à une martingale locale sous la supposition que la famille $(\mathbf{F}_t)$ est quasi-continue à gauche et H est prévisible. La même année, W. Millar, dans sa thèse, a fait indépendamment une étude des intégrales stochastiques par rapport à une martingale, qui n'est pas de carré intégrable, à partir de résultats de D. L. Burkholder. Ensuite, en 1968, C. Doléans-Dade et P. A. Meyer ont écarté l'hypothèse de la quasi-continuité à gauche [*3*].

Maintenant nous désignerons par $\mathbf{M}^2_{\text{loc}}$ l'ensemble des martingales localement de carré intégrable, nulles pour $t = 0$. Dans cet article, nous allons faire des recherches sur quelques propriétés des intégrales stochastiques par rapport à $M \in \mathbf{M}^2_{\text{loc}}$ supposant que la famille $(\mathbf{F}_t)$ est quasi-continue à gauche. D'abord, nous prouvons que tout $M \in \mathbf{M}_{\text{loc}}$ peut être déduit de quelque martingale de carré intégrable par un changement de temps. Ensuite, nous utiliserons les changements de temps avec lesquels nous pouvons ramener les intégrales stochastiques par rapport à $M \in \mathbf{M}^2_{\text{loc}}$ aux intégrales stochastiques ordinaires par rapport à une martingale de carré intégrable.

Soit $M = (M_t, \mathbf{F}_t) \in \mathbf{M}^2_{\text{loc}}$ et soit $\mathbf{H}_M$ l'ensemble de tous les processus prévisibles $H = (H_t, \mathbf{F}_t)$ tel que $P\left(\int_0^t H_a^2 d < M, M >_a < \infty\right) = 1$ pour tout $t > 0$.

Maintenant, nous pouvons vérifier un résultat suivant : si une suite $H^{(n)}$, des éléments de $\mathbf{H}_M$, converge presque sûrement uniformément vers un processus H, alors H appartient à $\mathbf{H}_M$ et $H^{(n)} \circ M$ converge uniformément en probabilité vers l'intégrale stochastique $H \circ M$ de H par rapport à M.

INTRODUCTION

Stochastic integrals of the form $\int_0^t H_u dM_u$ have long been studied under various hypotheses on $H = (H_t, \mathbf{F}_t)$ and $M = (M_t, \mathbf{F}_t)$: for example, H is predictable and M is a square integrable martingale, etc. In 1966, P. A. Meyer [4] has given a definition of stochastic integrals for local martingales under the assumption that the family $(\mathbf{F}_t)$ is quasi-left continuous and H is predictable, and W. Millar, in his thesis, has made an independent study of stochastic integrals for non-square integrable martingales by applying the idea of D. L. Burkholder. In 1968, C. Doléans-Dade and P. A. Meyer have removed the hypothesis of quasi-left continuity [3].

In this paper we are going to investigate some properties of stochastic integrals for locally square integrable martingales under the assumption that the family $(\mathbf{F}_t)$ is quasi-left continuous. For this purpose we shall use changes of time to reduce the stochastic integral for locally square integrable martingales to ordinary stochastic integrals for square integrable martingales.

1. BASIC DEFINITIONS TO RECALL.

We assume here that we are given on the basic P-complete probability space $(\Omega, \mathbf{F}, P)$ a right continuous, increasing family $(\mathbf{F}_t)$ of sub σ-fields of $\mathbf{F}$. We may assume that each $\mathbf{F}_t$ contains all F-sets of P-measure zero. In addition, suppose the family $(\mathbf{F}_t)$ is quasi-left continuous; that is, for every stopping time τ and every sequence (τ_n) of stopping times such that $\tau_n \uparrow \tau$, the σ-field $\mathbf{F}_\tau$ is generated by the field $\bigcup_{n=1}^{\infty} \mathbf{F}_{\tau_n}$. A stochastic process $X = (X_t, \mathbf{F}_t)$ is said to be predictable if the function $(t, \omega) \to X_t(\omega)$ is measurable with respect to the σ-field on $[0, \infty[x\Omega$, generated by all adapted stochastic processes almost all sample functions of which are left continuous. A right continuous martingale $M = (M_t, \mathbf{F}_t)$ is said to be square integrable if for each t $E[M_t^2] < \infty$. A right continuous stochastic process $M = (M_t, F_t)$ is said to be a locally square integrable (resp. local) martingale if there exists an increasing sequence (τ_n) of stopping times with respect to the family (F_t) such that $P(\tau_n \uparrow \infty, \tau_n < \infty) = 1$ and for each n, $(M_{t \wedge \tau_n}, F_t)$ is an L^2-bounded (resp. L^1-bounded) martingale: that is to say, $\sup_t E[M_{t\wedge\tau_n}^2] < \infty$ (resp. $\sup_t E[|M_{t\wedge\tau_n}|] < \infty$). We shall designate by **M** (resp. M_{loc}^2) the set of all square integrable (resp. all locally square integrable) martingales M adapted to the family (F_t) such that $M_0 = 0$.

2. EXAMPLES.

In this section we shall give two examples which imply that a local martingale is not always a locally square integrable martingale. It is Mrs. C. Doléans-Dade, undoubtedly, who has given such an example for the first time. The example 2 below is much the same as her [*2*].

Example 1. — Let (Ω, F, P) be the Wiener probability space with $\Omega = [0, 1]$, F the class of all linear Borel sets in Ω and P the Lebesgue measure. We put:

$$F_t = \begin{cases} F_0 & \text{if} \quad t < 1 \\ F & \text{if} \quad t \geqslant 1. \end{cases}$$

where F_0 is the σ-field generated by the sets of P-measure zero. Clearly $F_1 \neq F_{1-}$: in other words, the family (F_t) is not quasi-left continuous. If τ is a positive F-measurable function, we have $P(\tau < 1) = 1$ or 0. Consequently any locally square integrable martingale is a square integrable martingale. Indeed, if $M = (M_t, F_t)$ is a locally square integrable martingale, then there exists an increasing sequence (τ_n) of stopping times with respect to the family (F_t) such that each $(M_{t\wedge\tau_n}, F_t)$ is a square integrable martingale. It follows from $P(\lim_n \tau_n = \infty) = 1$ that we have $P(\tau_n \geqslant 1) = 1$ for n sufficiently large. Thus for each n, $E[M_{t\wedge\tau_n} \mid F_1] = M_1$ $(t > 1)$ and so M_1 is square integrable. Since $M_t = M_1$ for every $t \geqslant 1$, it is now clear that M is a square integrable martingale.

Now we put:

$$M_t = \begin{cases} 0 & \text{if} \quad t < 1 \\ X & \text{if} \quad t \geqslant 1 \end{cases}$$

where X is any integrable, but not square integrable, random variable such that $E[X] = 0$. Then the martingale (M_t, F_t) is a desired example.

Example 2. — Let $\Omega = [0, \infty[$, $\mathbf{F}^0$ the class of all linear Borel sets in Ω and P a probability measure on Ω. We designate by S the identity function of Ω into $[0, \infty[$ and let $\mathbf{F}_t^0$ be the σ-field generated by $S \wedge t$. It is easy to see that the family $(\mathbf{F}_t)$ is increasing and right continuous. Let $\mathbf{F}_t$ be the P-completed σ-field of $\mathbf{F}_t^0$. Note that the family $(\mathbf{F}_t)$ is quasi-left continuous when the measure P satisfies the condition: $P(\omega) = 0$ for each $\omega \in \Omega$. The next lemma is very interesting. Since it is proved in [*1*] and in fact the proof is not difficult, we shall omit the proof.

LEMMA 1. — A random variable τ is a stopping time with respect to the family $(\mathbf{F}_t)$ if and only if it satisfies the following condition: there exists some $s \in [0, \infty]$ such that

$$\begin{cases} \text{(i)} & \tau(\omega) \geqslant S(\omega) \quad a.s \text{ if } \quad S(\omega) \leqslant s \\ \text{(ii)} & \tau(\omega) = s \quad a.s \text{ if } \quad S(\omega) > s. \end{cases}$$

Now we define the probability measure P on Ω by $P(A) = m(A \cap [0, 1])$, $A \in \mathbf{F}$ where m is the Lebesgue measure on the real line. Then obviously the family $(\mathbf{F}_t)$ is quasi-left continuous. We put:

$$M_t = E[X \mid \mathbf{F}_t]$$

where $X(\omega) = S(\omega)^{-1/2} I_{]0,1]}(\omega)$. Clearly $M = (M_t, \mathbf{F}_t)$ is a martingale which is uniformly integrable. Suppose now $M \in \mathbf{M}^2_{loc}$. It follows from the definition of a locally square integrable martingale that there exists a stopping time $\tau > 0$ a.s such that the martingale $(M_{t \wedge \tau}, \mathbf{F}_t)$ is square integrable. By Lemma 1 we can choose some constant $s > 0$ such that the properties (*i*) and (*ii*) are satisfied. Since

$$M_t \geqslant E[X I_{\{S \leqslant t\}} \mid \mathbf{F}_t] = X I_{\{S \leqslant t\}} \geqslant 0,$$

we have

$$\begin{aligned} E[M^2_{s \wedge S \vee \tau}] &= E[M^2_{s \wedge S}] \\ &\geqslant E[M^2_{s \wedge S \wedge 1}] \\ &\geqslant E[X^2 I_{\{S \leqslant s \wedge 1\}}] = \int_0^{s \wedge 1} \frac{1}{t} \, dt = \infty. \end{aligned}$$

On the other hand we have $E[M^2_{s \wedge S \wedge \tau}] \leqslant E[M^2_{s \wedge \tau}] < \infty$. This is a contradiction. Consequently M is not a locally square integrable martingale.

3. SOME PROPERTIES OF A CHANGE OF TIME.

DEFINITION. — A family $(\tau_t)_{0 \leqslant t < \infty}$ is said to be a change of time with respect to the family $(\mathbf{F}_t)$ if the following conditions are satisfied:

$$\begin{cases} (1) \text{ for each } t,\ \tau_t \text{ is a finite stopping time with respect to the family } (\mathbf{F}_t), \\ (2) \text{ the function: } t \to \tau_t \text{ is continuous and strictly increasing,} \\ (3)\ \tau_0 = 0 \text{ and } \tau_\infty = \infty. \end{cases}$$

For each stopping time τ_t we can define a σ-field $\mathbf{F}_{\tau_t}$ as usual. If a process $X = (x_t, \mathbf{F}_t)$ is progressively measurable, we can define a new process $TX = (x_{\tau_t}, \mathbf{F}_{\tau_t})$. This process is said to be a process obtained from X by a change of time.

PROPOSITION 2. — The family $(\mathbf{F}_{\tau_t})$ is right continuous and quasi-left continuous.

Proof. — Let t be any fixed real number $\geqslant 0$. If

$$A \in \mathbf{F}_{\tau_{t+}} = \bigcap_{h>0} \mathbf{F}_{\tau_{t+h}},$$

then for each $h > 0$ we have

$$(\forall u \geqslant 0),\ A \cap [\tau_{t+h} \leqslant u] \in \mathbf{F}_u.$$

From (2)

$$A \cap [\tau_t < u] = \bigcup_{\substack{h>0 \\ h\,:\,\text{rational}}} (A \cap [\tau_{t+h} \leqslant u]) \in \mathbf{F}_u$$

Consequently $A \cap [\tau_t \leqslant u] \in \mathbf{F}_{u^+} = \mathbf{F}_u$. Thus $A \in \mathbf{F}_{\tau_t}$. The converse inclusion is obvious. Hence the family $(\mathbf{F}_{\tau_t})$ is right continuous.

Next we shall verify the second assertion. Let (β_n) be an increasing sequence of stopping times with respect to the family $(\mathbf{F}_{\tau_t})$, and let

$$\lambda_t = \inf \{ u > 0;\ \tau_u > t \}.$$

Then clearly $[\lambda_t \leqslant u] = [t \leqslant \tau_u] \in \mathbf{F}_{\tau_u}$. Thus each λ_t is a stopping time with respect to the family $(\mathbf{F}_{\tau_t})$. Therefore it follows from

$$[\tau_{\beta_n} \leqslant u] = [\beta_n \leqslant \lambda_u] \in \mathbf{F}_{\tau_{\lambda_u}} = \mathbf{F}_u$$

that (τ_{β_n}) is an increasing sequence of stopping times with respect to the family $(\mathbf{F}_t)$. According to the quasi-left continuity of the family $(\mathbf{F}_t)$, we have

$$\bigvee_n \mathbf{F}_{\tau_{\beta_n}} = \mathbf{F}_{(\lim_n \tau_{\beta_n})} = \mathbf{F}\tau_{(\lim_n \beta_n)}.$$

Consequently the family $(\mathbf{F}_{\tau_t})$ is quasi-left continuous. This completes the proof.

COROLLARY. — If the family $(\mathbf{F}_t)$ is not quasi-left continuous, then the family $(\mathbf{F}_{\tau_t})$ is not quasi-left continuous for any change of time (τ_t).

Proof. — Suppose there exists a change of time (τ_t) such that the family $(\mathbf{F}_{\tau_t})$ is quasi-left continuous, and put

$$\lambda_t = \inf \{ u > 0 : \tau_u > t \}.$$

Then it is clear that (λ_t) is a change of time with respect to the family $(\mathbf{F}_{\tau_t})$ and $\tau_{\lambda_t} = t$. Consequently, by Proposition 2, the family $(\mathbf{F}_t) = (\mathbf{F}_{\tau_{\lambda_t}})$ is quasi-left continuous. This is a contradiction.

The following lemma is obvious. We shall omit the proof.

LEMMA 3. — (1) Let (Λ_t) and (γ_t) be changes of times with respect to the family $(\mathbf{F}_t)$. Then $(\Lambda_t \wedge \gamma_t)$ and $(\Lambda_t \vee \gamma_t)$ are also changes of times with respect to the same family.

(2) Let (Λ_t) be a change of time with respect to the family $(\mathbf{F}_t)$ and let (φ_t) be a change of time with respect to the family $(\mathbf{F}_{\Lambda_t})$. Then (Λ_{φ_t}) is a change of time with respect to the family $(\mathbf{F}_t)$.

Note that for each change of time (Λ_t) there exists a change of time (φ_t) such that $\Lambda_{\varphi_t} \leqslant t$ for each t. Indeed, if we put

$$\varphi_t = \inf \{ u > 0 : u + \Lambda_u > t \},$$

then (φ_t) is a desired change of time.

Now we shall designate by $\mathcal{M}$ the set of all locally square integrable martingales $M = (M_t, \mathbf{F}_t)$ with $M_0 = 0$ such that for some change of time (τ_t) (of course, with respect to the family $(\mathbf{F}_t)$!) $(M_{\tau_t}, \mathbf{F}_{\tau_t})$ is a square integrable martingale. Clearly $\mathbf{M} \subset \mathcal{M} \subset \mathbf{M}^2_{\text{loc}}$. If $M = (M_t, \mathbf{F}_t) \in \mathcal{M} \backslash \mathbf{M}$, then for every change of time (τ_t) such that $(M_{\tau_t}, \mathbf{F}_{\tau_t})$ is a square integrable martingale, there exists some positive real number c (independent of (τ_t)) satisfying the condition: for each t, $\text{ess}_\omega \inf \tau_t(\omega) \leqslant c$.

If $H = (H_t, \mathbf{F}_t)$ is a left continuous process, for every change of time (τ_t) $(H_{\tau_t}, \mathbf{F}_{\tau_t})$ is also left continuous. This fact implies the following lemma.

LEMMA 4. — Let $H = (H_t, \mathbf{F}_t)$ be a predictable process. Then for any change of time (τ_t), $(H_{\tau_t}, \mathbf{F}_{\tau_t})$ is predictable.

4. In the following we shall investigate some properties of stochastic integrals for locally square integrable martingales under the assumption that the family $(\mathbf{F}_t)$ is quasi-left continuous.

THEOREM 5. — For every $M \in \mathbf{M}^2_{\text{loc}}$ there exists a continuous increasing process (A_t) such that $(M_t^2 - A_t)$ is a local martingale. (A_t) is unique, and we denote it by $\langle M, M \rangle = (\langle M, M \rangle_t)$.

This theorem is fundamental for the theory of martingale integrals. Since it is proved in [5], we omit its proof. The next theorem will play an essential role in the following.

THEOREM 6. — $\mathbf{M}^2_{\text{loc}} = \mathcal{M}$; that is to say, any locally square integrable martingale can be deduced from some square integrable martingale by a change of time.

Proof. — It is sufficient to verify that $\mathbf{M}^2_{\text{loc}} \subset \mathcal{M}$. Now let

$$M = (M_t, \mathbf{F}_t) \in \mathbf{M}^2_{\text{loc}};$$

that is, there exists an increasing sequence (τ'_n) of stopping times with respect to the family $(\mathbf{F}_t)$ such that $P(\tau'_n \uparrow \infty,\ \tau'_n < \infty,\ \forall n) = 1$ and for each n $(M_{t \wedge \tau'_n}, \mathbf{F}_t)$ is an L^2-bounded martingale. Let $\lambda_t = t + \langle M, M \rangle_t$ and $\tau_t = \inf\{ u > 0 : \lambda_u > t \}$. Then it is not difficult to see that (τ_t) (resp. (λ_t)) is a change of time with respect to the family $(\mathbf{F}_t)$ (resp. the family $(\mathbf{F}_{\tau_t})$). Obviously $\langle M, M \rangle_{\tau_t} \leqslant t$ and $\lambda_{\tau_t} = \tau_{\lambda_t} = t$. Since for each n $\left(M^2_{\tau_t \wedge \tau'_n} - \langle M, M \rangle_{\tau_t \wedge \tau'_n}, \mathbf{F}_{\tau_t}\right)$ is a martingale, we have

$$E\left[M^2_{\tau_t \wedge \tau'_n}\right] = E\left[\langle M, M \rangle_{\tau_t \vee \tau'_n}\right] \leqslant t,$$

from which it follows that $\sup_n E\left[M^2_{\tau_t \wedge \tau'_n}\right] \leqslant t$. Thus the sequence $\left(M_{\tau_t \wedge \tau'_n}\right)_{n=1,2}$ is uniformly integrable. Consequently $(M_{\tau_t}, \mathbf{F}_{\tau_t})$ is a martingale and for each t M_{τ_t} is square integrable. Hence the theorem is established.

Remark. — I dont know whether this relation $\mathbf{M}^2_{\text{loc}} = \mathcal{M}$ is true or not in the case that the family $(\mathbf{F}_t)$ is not quasi-left continuous.

Note that, by the uniqueness of $\langle M, M \rangle$, we have

$$T\langle M, M \rangle = \langle TM, TM \rangle \quad , \quad (TM = (M_{\tau_t}, \mathbf{F}_{\tau_t}))$$

for any change of time $T = (\tau_t)$ with respect to the family $(\mathbf{F}_t)$.

Let $M \in \mathbf{M}^2_{loc}$. We shall designate by $\mathbf{H}_M$ the set of all predictable processes $H = (H_t, \mathbf{F}_t)$ such that $P\left(\int_0^t H_s^2 d\langle M, M\rangle_s < \infty\right) = 1$ for all $t \geqslant 0$.

PROPOSITION 7. — Let $M = (M_t, \mathbf{F}_t) \in \mathbf{M}^2_{loc}$ and $H = (H_t, \mathbf{F}_t) \in \mathbf{H}_M$, then there exists a change of time $T = (\tau_t)$ such that the process TM is a square integrable martingale and $E\left[\int_0^t H_{\tau_s}^2 d\langle TM, TM\rangle_s\right] < \infty$ for all $t \geqslant 0$.

Proof. — Let $\lambda_t = t + \langle M, M\rangle_t + \int_0^t H_s^2 d\langle M, M\rangle_s$. Clearly λ_t is $\mathbf{F}_t$-measurable, $\lambda_0 = 0$, $\lambda_\infty = \infty$, $\lambda_s < \lambda_t$ $(s < t)$ and $t \to \tau_t$ is continuous. Put $\tau_t = \inf\{u > 0 : \lambda_u > t\}$. Then $T = (\tau_t)$ is a change of time such that $\langle M, M\rangle_{\tau_t} = \langle TM, TM\rangle_t \leqslant t$ for each t and

$$\int_0^{\tau_t} H_s^2 d\langle M, M\rangle_s = \int_0^t H_{\tau_s}^2 d\langle TM, TM\rangle_s \leqslant t.$$

Consequently this T is a desired change of time. This completes the proof.

Together with each pair (M, N) of elements of $\mathbf{M}^2_{loc}$, there is a process defined by the relation

$$\langle M, N\rangle = \frac{1}{2}(\langle M+N, M+N\rangle - \langle M, M\rangle - \langle N, N\rangle).$$

Put $H \circ \langle M, N\rangle = \left(\int_0^t H_s d\langle M, N\rangle_s, \mathbf{F}_t\right)$ where $M, N \in \mathbf{M}^2_{loc}$ and $H \in \mathbf{H}_M$.

THEOREM 8. — Let $M \in \mathbf{M}^2_{loc}$ and $H \in \mathbf{H}_M$. Then there exists one and only one $H \circ M \in \mathbf{M}^2_{loc}$ such that for every $N \in \mathbf{M}^2_{loc}$

$$\langle H \circ M, N\rangle = H \circ \langle M, N\rangle.$$

Proof. — Since the uniqueness of $H \circ M$ is clear, we shall show the existence. Now let

$$\lambda_t = t + \langle M, M\rangle_t + \int_0^t H_s^2 d\langle M, M\rangle_s \quad \text{and} \quad \tau_t = \inf\{u > 0 : \lambda_u > t\}.$$

It follows from Proposition 7 that we have

$$E\left[\int_0^t H_{\tau_s}^2 d\langle TM, TM\rangle_s\right] < \infty$$

and TM is a square integrable martingale. Furthermore, as $\tau_t \leqslant t$, TN is a square integrable martingale whenever $N \in \mathbf{M}$. Then there exists one and only one square integrable martingale $TH \circ TM$ such that

$$\langle TH \circ TM, TN \rangle = TH \circ \langle TM, TN \rangle.$$

Since $\Lambda = (\lambda_t)$ is a change of time with respect to the family $(\mathbf{F}_{\tau_t})$ and $\tau_{\lambda_t} = t$, we have

$$\langle \Lambda(TH \circ TM), N \rangle = H \circ \langle M, N \rangle$$

for every $N \in \mathbf{M}$ and $\Lambda(TH \circ TM)$ belongs to $\mathbf{M}^2_{\text{loc}}$. Consequently we have

$$\langle \Lambda(TH \circ TM), N \rangle = H \circ \langle M, N \rangle$$

for every $N \in \mathbf{M}^2_{\text{loc}}$. This implies that $\Lambda(TH \circ TM)$ does not depend on T. Therefore this process is the desired element $H \circ M$ of $\mathbf{M}^2_{\text{loc}}$. This completes the proof.

In the remainder of this paper we shall give a topological property of $H \circ M$. Let $H^{(n)} = (H_t^{(n)}, \mathbf{F}_t)$ and $H = (H_t, \mathbf{F}_t)$ be stochastic processes. If for each t $P(\lim_n \sup_{0 \leqslant u \leqslant t} | H_u^{(n)} - H_u | = 0) = 1$, then we say that $H^{(n)}$ converges uniformly almost surely to H. Next if for each t and $\varepsilon > 0$ $\lim_n P(\sup_{0 \leqslant u \leqslant t} | H_u^{(n)} - H_u | \geqslant \varepsilon) = 0$, then we say that $H^{(n)}$ converges uniformly in probability to H.

THEOREM 9. — Let $H^{(n)} = (H_t^{(n)}, \mathbf{F}_t)$, $n = 1, 2, \ldots$, be a sequence of elements of $\mathbf{H}_M$. If for each t $P(\lim_{m,n} \sup_{0 \leqslant u \leqslant t} | H_u^{(m)} - H_u^{(n)} | = 0) = I$, then $\sup_n | H^{(n)} |$ belongs to $\mathbf{H}_M$ and $H^{(n)}$ converges uniformly almost surely to some $H \in \mathbf{H}_M$.

Proof. — Put $H_t = \lim_n H_t^{(n)}$ a.s for each t. Clearly $H = (H_t, \mathbf{F}_t)$ is a stochastic process satisfying $P(\lim_n \sup_{0 \leqslant u \leqslant t} | H_u^{(n)} - H_u | = 0) = 1$, and so H is predictable. In addition, for each t we have

$$\int_0^t H_u^2 d\langle M, M \rangle_u \leqslant 2 \left\{ \int_0^t H_u^{(n)2} d\langle M, M \rangle_u + \int_0^t (H_u - H_u^{(n)})^2 d\langle M, M \rangle_u \right\}$$
$$\leqslant 2 \left\{ \int_0^t H_u^{(n)2} d\langle M, M \rangle_u + \sup_{0 \leqslant u \leqslant t} (H_u - H_u^{(n)})^2 \langle M, M \rangle_t \right\},$$

from which $P\left(\int_0^t H_u^2 d\langle M,M\rangle_u < \infty\right) = 1$. Consequently $H \in \mathbf{H}_M$. Similary we can prove that $\sup_n | H^{(n)} | \in \mathbf{H}_M$. Thus the theorem is established.

THEOREM 10. — Let $H^{(n)} = (H_t^{(n)}, \mathbf{F}_t)$, $n = 1, 2, \ldots$, be a sequence of elements of $\mathbf{H}_M$. If $H^{(n)}$ converges uniformly almost surely to H, then $H^{(n)} \circ M$ converges uniformly in probability to $H \circ M$.

Proof. — Let

$$\lambda_t = t + \int_0^t \sup_n H_u^{(n)2}\, d\langle M, M\rangle_u \quad \text{and} \quad \tau_t = \inf\{ u > 0 : \lambda_u > t \}.$$

It is easy to verify that (τ_t) is a change of time with respect to the family $(\mathbf{F}_t)$. Since $\int_0^{\tau_t} (H_u^{(n)} - H_u)^2 d\langle M, M\rangle_u \leqslant 4t$, the bounded convergence theorem shows that

$$\lim_n E\left[\int_0^{\tau_t} (H_u^{(n)} - H_u)^2 d\langle M, M\rangle_u\right] = 0.$$

Furthermore since $\{ (H^{(n)} \circ M)_{\tau_t} - (H \circ M)_{\tau_t} \}$ is a martingale, we have for any $\varepsilon > 0$

$$\begin{aligned}\varepsilon^2 P\{ \sup_{0 \leqslant u \leqslant t} | (H^{(n)} \circ M)_{\tau_u} - (H \circ M)_{\tau_u} | \geqslant \varepsilon \} \\ \leqslant E[\{ (H^{(n)} \circ M)_{\tau_t} - (H \circ M)_{\tau_t} \}^2] \\ \leqslant E\left[\int_0^{\tau_t} (H_u^{(n)} - H_u)^2 d\langle M, M\rangle_u\right]\end{aligned}$$

by using the extension of Kolomogorov's inequality to martingales, from which

$$\lim_n P\{ \sup_{0 \leqslant u \leqslant t} | (H^{(n)} \circ M)_{\tau_u} - (H \circ M)_{\tau_u} | \geqslant \varepsilon \} = 0.$$

It follows from $\tau_\infty = \infty$ that for each $\delta > 0$ there exists some integer $N \geqslant 0$ such that $P(t > \tau_N) < \delta$. Then we have

$$\begin{aligned}P\{ \sup_{0 \leqslant u \leqslant t} | (H^{(n)} \circ M)_u - (H \circ M)_u | \geqslant \varepsilon \} \\ \leqslant P\{ \sup_{0 \leqslant u \leqslant t} | (H^{(n)} \circ M)_u - (H \circ M)_u \geqslant \varepsilon, t \leqslant \tau_N \} + P(t > \tau_N) \\ \leqslant P\{ \sup_{0 \leqslant u \leqslant N} | (H^{(n)} \circ M)_{\tau_u} - (H \circ M)_{\tau_u} | \geqslant \varepsilon \} + \delta\end{aligned}$$

Consequently $\lim_n P\{ \sup_{0 \leqslant u \leqslant t} | (H^{(n)} \circ M)_u - (H \circ M)_u | \geqslant \varepsilon \} = 0$. This completes the proof.

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