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Semi-Groups of Markov Operators

by

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SUMMARY. — The paper states and proves some theorems for semi-groups of Markov operators (contractions on L_1) analogous to theorems known for a single operator:

(*i*) Let $\{ P_t \}$ be a semi-group. Q is said to be a convex combination of $\{ P_t \}$ if

$$Qf(x) = \left(\int_0^\infty \phi(t) P_t f dt \right)(x) \qquad (f \in L_\infty)$$

where $\phi(t) > 0$, $\phi(t) \searrow$,

$$\int_0^\infty \phi(t) dt = 1 \qquad \text{and} \qquad \int_0^\infty t\phi(t) dt < \infty.$$

(*ii*) $\{ P_t \}$ is defined to be conservative, ergodic, a Harris process or quasi-compact if Q has this property. Some theorems for such semi-groups analogous to theorems for single operator are proved.

(*iii*) A necessary and sufficient condition for the existence of a σ-finite invariant measure is given.

1. PRELIMINARIES

Let (X, Σ, m) be a finite measure space. A Markov operator is a positive linear contraction P on $L_1(X, \Sigma, m)$. P will be written to the right of its variable while its adjoint, acting on $L_\infty(X, \Sigma, m)$ will be denoted by P and written to the left of its variable. Thus $\langle uP, f \rangle = \langle u, Pf \rangle$ for $u \in L_1$ and $f \in L_\infty$.

The operator P acts on the space of the measures absolutely continuous with respect to m, which is isometric to $L_1(m)$ as follows

$$\mu P(A) = \int P1_A d\mu.$$

The same formula is defined for σ-finite measures. Our reference for ergodic theory of a single Markov operator is [*6*].

DÉFINITION 1.1. — A *Markov Process* is a strongly measurable semigroup $\{ P_t \mid t \geqslant 0 \}$ of Markov operators.

By slight modifications of theorem 1.1 of [*11*] we have:

THEOREM 1.1. — Let $\{ P_t \}$ be a Markov process, then for every $f \in L_\infty(m)$ there exists a function $g(t, x)$ measurable on $[0, \infty)_x X$ (and uniquely defined with respect to the product of Lebesgue measure and m), such that for every function $\phi(t) \geqslant 0$ on $[0, \infty)$ with

$$\int_0^\infty \phi(t)dt < \infty, \quad \int_0^\infty \phi(t)g(t, x)dt = \left(\int_0^\infty \phi(t)P_t f dt \right)(x) \quad \text{a. e. } m \quad \text{on X.}$$

DÉFINITION 1.2. — A Markov process is said to be conservative if for every $0 \leqslant f$ we have

$$\lim_{T \to \infty} \int_0^T P_t f dt = \begin{cases} 0 \\ \infty \end{cases} \quad \text{a. e.}$$

DÉFINITION 1.3. — A measure μ is said to be *invariant* under $\{ P_t \}$ if $\mu P_t = \mu$, $\forall t$.

2. CONVEX COMBINATION OF MARKOV PROCESSES

DÉFINITION 2.1. — Let $\phi(t) > 0$ be a decreasing function on $[0, \infty)$ with

$$\int_0^\infty \phi(t)dt = 1 \qquad \text{and} \qquad \int_0^\infty t\phi(t)dt < \infty,$$

Q is called a *convex combination* of the Markov processes $\{ P_t \}$ if

$$Qf(x) = \left(\int_0^\infty \phi(t) P_t f dt \right)(x).$$

LEMMA 2.1. — Let Q be a convex combination with the function $\phi(t)$ as in the definition 2.1 then for every $f \in L_\infty(m)$ and for every real number T we have

$$\left| \int_0^T P_t(I - Q) f dt \right|_\infty \leqslant 4 \| f \|_\infty \cdot \int_0^\infty t\phi(t) dt$$

Proof

$$\begin{aligned}
&\left\| \int_0^T P_t(I - Q) f dt \right\|_\infty \\
&\quad = \left\| \int_0^T \left(P_t - P_t \int_0^\infty \phi(s) P_s ds \right) f dt \right\|_\infty \\
&\quad = \left\| \int_0^\infty \phi(s) \int_0^T (P_t - P_{t+s}) f dt ds \right\|_\infty \\
&\quad \leqslant \left\| \int_0^T \phi(s) \int_0^T (P_t - P_{t+s}) f dt ds \right\|_\infty + \left\| \int_T^\infty \phi(s) \int_0^T (P_t - P_{t+s}) f dt ds \right\|_\infty \\
&\quad \leqslant \left\| \int_0^T \phi(s) \int_0^s P_t f dt ds \right\|_\infty + \left\| \int_0^T \phi(s) \int_T^{T+s} P_t f dt ds \right\|_\infty \\
&\quad + \left\| \int_T^\infty \phi(s) \int_0^T P_t f dt ds \right\|_\infty + \left\| \int_T^\infty \phi(s) \int_s^{T+s} P_t f dt ds \right\|_\infty \\
&\quad \leqslant 4 \| f \|_\infty \int_0^\infty s\phi(s) ds
\end{aligned}$$

THEOREM 2.2. — The Markov process $\{ P_t \}$ is conservative if and only if its convex combination Q is conservative.

Proof. — If Q is not conservative then there exist a function $f \geqslant 0$ such that $Qf \leqslant f$ and $Qf \neq f$ (see [7]). Debote $0 \leqslant g = f - Qf$ by lemma 2.1

$$\left\| \int_0^\infty P_t g dt \right\|_\infty < \infty,$$

hence $\{ P_t \}$ is not conservative.

On the other hand if $\{ P_t \}$ is not conservative then there exists a function $f \geqslant 0$ such that $\int_0^\infty P_t f dt < \infty$ (If $\{ P_t \}$ is not conservative then by theorem 2.1 of [11] P_{t_0} is not conservative, for each $t_0 > 0$, and hence there exists a function $h \geqslant 0$ with $P_{t_0}h \leqslant h$ and $P_{t_0}h \neq h$ take $f = h - P_{t_0}h$ and then $\int_0^\infty P_t f dt < \infty$). Denote $g = \int_0^\infty P_t f dt$

$$Qg = \int_0^\infty \phi(s) P_s \int_0^\infty P_t f dt ds = \int_0^\infty \phi(s) \int_s^\infty P_t f dt ds \leqslant g$$

and $Qg \neq g$. So Q is not conservative.

Remark. — An analogous theorem for a single Markov operator is given in [8] theorem 1.1.

DÉFINITION 2.2. — A conservative Markov process $\{ P_t \}$ is said to be *ergodic* if $Qf = f$, $f \in L_\infty(m)$ $f =$ const. when Q is any convex combination.

LEMMA 2.3. — A conservative Markov process $\{ P_t \}$ is ergodic if and only if $0 \neq f \geqslant 0$ $\int_0^\infty P_t f dt = \infty$ and hence the definition of ergodicity does not depend on the choice of the convex combination.

Proof. — If for each $0 \neq f \geqslant 0$ we have $\int_0^\infty \phi(t) P_t f dt > 0$. So, Q is ergodic. On the other hand if there exist sets A and B such that $\int_0^\infty P_t 1_A dt = 0$ on B, then $Q^n 1_A = 0$ on B for each n, because

$$Q^n 1_A = \int_0^\infty \phi * \phi * \ldots * \phi P_t 1_A dt = 0$$

(convolution n times) on B (see [5]) and Q is not ergodic.

Remark. — In [5] is also proved that μ is an invariant measure under $\{ P_t \}$ if and only if $\mu Q = \mu$.

3. ON QUASI-COMPACT SEMI-GROUPS

DÉFINITION 3.1. — Let $\{ P_t \}$ be an ergodic and conservative Markov process, let $Q = \int_0^\infty \phi(t) P_t dt$ be a convex combination, $\{ P_t \}$ is said to be quasi-compact if Q is a quasi-compact operator.

THEOREM 3.1. — Let $\{ P_t \}$ be an ergodic and conservative Markov process, then the following are equivalent:

(*a*) $\{ P_t \}$ is quasi-compact.

(*b*) For every set B there exists $\alpha = \alpha(B) > 0$ and $T = T(B)$ such that

$$\int_0^T P_t 1_B dt \geqslant \alpha.$$

(*c*) There exists a finite invariant measure μ and for every function f with $\int f d\mu = 0$ we have

$$\left\| \frac{1}{T} \int_0^T P_t f dt \right\|_\infty \underset{T \to \infty}{\longrightarrow} 0.$$

(*d*) There exists a finite invariant measure and let E be the projection $Ef = \int f d\mu$ then $\left\| \frac{1}{T} \int_0^T P_t dt - E \right\|_\infty \underset{T \to 0}{\longrightarrow} 0$ in the operator norm.

Proof

(*d*) $\Rightarrow$ (*c*) trivial.

(*c*) $\Rightarrow$ (*b*) also obvious.

(*b*) $\Rightarrow$ (*a*) For every set B there exists $\alpha = \alpha(B)$ and $T = T(B)$ such that

$$\int_0^T P_t 1_B dt \geqslant \alpha$$

and hence

$$Q1_B = \int_0^\infty \phi(t) P_t 1_B dt \geqslant \phi(T) \int_0^T P_t 1_B dt \geqslant \alpha\phi(T)$$

and by theorem 4.1 of [*10*] Q is quasi-compact. (*a*) $\Rightarrow$ (*d*) Let Q be quasi-compact, denote $L_\infty^0 = \left\{ f \mid \int f d\mu = 0 \right\}$ (by theorem 4.1 of [*10*] there exists a finite invariant measure $\mu = \mu Q$) and $(I - Q)L_\infty^0 = L_\infty^0$ and hence for every function f there exists a function $g \in L_\infty^0$ such that

$$g - Qg = f - \int f d\mu. \ .$$

Hence by lemma 2.1

$$\left\| \frac{1}{T} \int_0^T P_t\left(f - \int f d\mu \right) dt \right\|_\infty = \left\| \frac{1}{T} \int_0^T P_t (I - Q) g dt \right\|_\infty$$
$$\leqslant \frac{4}{T} \| g \|_\infty \int_0^\infty t\phi(t) dt \leqslant \frac{4C}{T} \int_0^\infty t\phi(t) dt$$

where C is the norm of the operator $(I - Q)^{-1}$ acting on L_∞^0. Thus

$$\lim_{T\to\infty} \sup_{\|f\|_\infty \leq 1} \left\| \frac{1}{T}\int_0^T P_t f dt - \int f d\mu \right\|_\infty = 0$$

COROLLARY 1. — The definition 3.1 does not depend on the choice of the convex combination.

Remark. — In [2] is proved that $U^1 = \int_0^\infty e^t P_t dt$ is quasi-compact if and only if $\lambda U^\lambda = \lambda \int_0^\infty e^{-\lambda t} P_t dt$ is for each λ, this is a special case of this corollary.

COROLLARY 2. — Let $\{ P_t \}$ be an ergodic and conservative Markov process and P_{t_0} is a quasi-compact operator for some t_0 then the process is quasi-compact.

Proof. — By theorem 4.1 of [*10*], for every function $f \in L_\infty$ there exists a function $g \in L_\infty$ with $\int g d\mu = 0$, where μ is the invariant measure for P_{t_0}, such that $f - \int f d\mu = g - P_{t_0} g$. Hence

$$\left\| \frac{1}{T}\int_0^T P_t\left(f - \int f d\mu\right) dt \right\|_\infty = \left\| \frac{1}{T}\int_0^T P_t(g - P_{t_0} g) dt \right\|_\infty$$
$$= \left\| \frac{1}{T}\int_0^{t_0} P_t g dt - \frac{1}{T}\int_T^{T+t_0} P_t g dt \right\|_\infty \leq \frac{2t_0 \| g \|_\infty}{T} \xrightarrow[T\to\infty]{} 0$$

and by theorem 3.1 the process is quasi-compact.

Remark. — The converse is not true, for example if $\{ P_t \}$ is the semi-group of rotations on the circle then it is easy to see that the process is quasi-compact but each P_t is not.

THEOREM 3.2. — Let $\{ P_t \}$ be an ergodic and conservative Markov process and there exists *no* pure charge (a finite additive measure which does not dominate any measure) ν such that $\nu P_t = \nu$ for each t then the process is quasi-compact.

Proof. — By the Fixed Point Theorem there exists a positive functional λ on L_∞ such that $\lambda P = \lambda$. λ, as a functional on L_∞, can be written uniquely as a sum $\lambda = \mu + \nu$ where μ is a measure and ν a pure charge. It is clear that $\mu P_t \geqslant \mu$ and by the conservativity of P_t, $\mu P_t = \mu$ for each t, and by the ergodicity μ is a unique finite invariant measure. Define the space $L = \text{spn}\{(I - P_t)L_\infty \mid 0 < t < \infty\}$. The orthogonal compliment of L is $L^\perp = \{\nu \in L_\infty^* \mid \nu P = \nu, \forall t\}$ and by the conditions of the theorem we have that $L^\perp$ is the one dimensional space $\{\alpha\mu\}$. So, by the Hahn-Banach Theorem if $\int f d\mu = 0$ then $f \in L$ and hence for each $\varepsilon > 0$ there exist functions $f_1, f_2, \ldots, f_j \in L_\infty$ and real numbers $t_1, t_2, \ldots, t_j$ such that

$$\| (f_1 - P_{t_1} f_1) + (f_2 - P_{t_2} f_2) + \ldots + (f_j - P_{t_j} f_j) - f \|_\infty \leqslant \varepsilon$$

Thus,

$$\left\| \frac{1}{T} \int_0^T P_t f dt \right\|_\infty \leqslant \left\| \frac{1}{T} \int_0^T P_t(f_1 - P_{t_1} f_1) dt \right\|_\infty + \ldots$$
$$+ \left\| \frac{1}{T} \int_0^T P_t(f_j - P_{t_j} f_j) dt \right\|_\infty$$
$$+ \left\| \frac{1}{T} \int_0^T P_t[(f_1 - P_{t_1} f_1) + \ldots + (f_j - P_{t_j} f_j) - f] dt \right\|_\infty$$

the last element of the sum is less then ε and for each $1 \leqslant i \leqslant j$ we have

$$\left\| \frac{1}{T} \int_0^T P_t(f_i - P_{t_i} f_i) dt \right\|_\infty \leqslant \left\| \frac{1}{T} \int_0^{t_i} P_t f_i dt \right\|_\infty + \left\| \frac{1}{T} \int_T^{T+t_i} P_t f_i dt \right\|_\infty$$
$$\leqslant \frac{2t_i \| f_i \|}{T} \underset{T \to \infty}{\longrightarrow} 0$$

and hence

$$\left\| \frac{1}{T} \int_0^T P_t f dt \right\|_\infty \underset{T \to \infty}{\longrightarrow} 0$$

and by theorem 3.1 the process is quasi-compact.

4. HARRIS PROCESSES

A single Markov operator P is said to be a Harris operator if there exist an integral operator K, $Kf(x) = \int k(x, y) f(y) m(dy)$ and an integer n such that $0 < K \leqslant P^n$ (for details see [6], Chapter V). Let P be a Markov operator and A a set, define $P_A = I_A \sum_{n=0}^{\infty} (PI_A)^n PI_A$ where I_A is the operator $I_A f(x) = \begin{cases} f(x) & x \in A \\ 0 & x \notin A \end{cases}$ in [6] is shown that P_A is a Markov operator on (A, Σ_A, mI_A).

DÉFINITION 4.1. — Let $\{ P_t \}$ a Markov process and Q a convex combination of it, $\{ P_t \}$ is said to be a Harris process if Q is a Harris operator.

Since Q is a Harris operator it has a unique σ-finite invariant measure μ (see [6], Chapter VI).

THEOREM 4.1. — Let $\{ P_t \}$ be an ergodic and conservative Markov process then the following are equivalent:

(a) $\{ P_t \}$ is a Harris process.

(b) There exists a set A such that for every set $B \subset A$ there exist $T = T(B)$ and $0 < \alpha = \alpha(B)$ such that $\int_0^T P_t 1_B dt \geqslant \alpha 1_A$.

(c) There exist a set A and a constant C such that if supp $f \subset A$ and $\int f d\mu = 0$ then $\left\| \int_0^T P_t f dt \right\|_\infty \leqslant C \| f \|_\infty$.

Proof. — (b) $\Rightarrow$ (a) Let Q be the convex combination $Qf = \int_0^\infty \phi(t) P_t f dt$, let $B \subset A$ be a set, there exist $T = T(B)$ and $\alpha = \alpha(T)$ such that

$$\int_0^T P_t 1_B dt \geqslant \alpha 1_A$$

and hence

$$Q1_B = \int_0^\infty \phi(t) P_t 1_B dt \geqslant \phi(T) \int_0^T P_t 1_B dt \geqslant \alpha \phi(T) 1_A$$

and by theorem 3.4 of [10] Q is a Harris operator.

(c) $\Rightarrow$ (b) Assume that there exist a set A with $\mu(A) < \infty$ and a constant C such that is supp $f \subset A$ and $\int f d\mu = 0$ then

$$\left\| \int_0^T P_t f dt \right\|_\infty \leqslant C \| f \|_\infty = K.$$

Let $E \subset A$, take $f = 1_A - \frac{\mu(A)}{\mu(E)} 1_E$, then $\int f d\mu = 0$ and supp $f \subset A$, and hence we have

$$\left\| \int_0^T P_t \left(1_A - \frac{\mu(A)}{\mu(E)} 1_E \right) dt \right\|_\infty \leqslant K$$

where K is a constant independent on T. By the conservativity and Egorov's Theorem there exists a set $B \subset A$ such that $\int_0^N P_t 1_A dt \underset{N\to\infty}{\longrightarrow} \infty$ uniformly on B. Hence there exists an integer N such that $\int_0^N P_t 1_A dt \geqslant 2K1_B$. Therefore

$$2K1_B \leqslant \int_0^N P_t 1_A dt \leqslant K + \frac{\mu(A)}{\mu(E)} \int_0^N P_t 1_E dt$$

or

$$\int_0^N P_t 1_E dt \geqslant \frac{\mu(E)}{\mu(A)} . K1_B .$$

So, for every set $E \subset B$ there exist an integer $N = N(E)$ and a positive number $\alpha = \alpha(E)$ such that $\int_0^N P_t 1_E dt \geqslant \alpha 1_B$.

(*a*) $\Rightarrow$ (*c*) Q is a Harris operator. By theorem 5.2 of [*10*] there exists a set A such that Q_A is quasi-compact. By theorem 4.1 of [*10*] we have that for each $f \in L_\infty$ with supp $f \subset A$ and $\int f d\mu = 0$ there exist $g \in L_\infty$ with supp $g \subset A$ and $\int g d\mu = 0$ such that $(I_A - Q_A)g = f$ and $\| g \|_\infty \leqslant C \| f \|_\infty$, where C is a constant independent on *f*.

By the calculations of [*3*] we have

$$(I_A - Q_A)g = (I - Q)\sum_{n=0}^{\infty} (I_A Q)^n I_A g \quad \text{where} \quad \left\| \sum_{n=0}^{\infty} (I_A Q)^n I_A g \right\|_\infty \leqslant \| g \|_\infty$$

By lemma 2.1 we have

$$\left\| \int_0^T P_t f dt \right\|_\infty = \left\| \int_0^T P_t (I - Q) \sum_{n=0}^{\infty} (I_A Q)^n I_A g dt \right\|_\infty$$
$$\leqslant 4 \| g \|_\infty \int_0^\infty t\phi(t) dt \leqslant 4C \| f \|_\infty \int_0^\infty t\phi(t) dt.$$

COROLLARY . — The definition 3.1 does not depend on the choice of the convex combination.

Remark. — Theorem 4.1 is a generalization of some theorems of [*1*], [*4*] and [*12*].

5. ON σ-FINITE INVARIANT MEASURES

THEOREM 5.1. — A necessary and sufficient condition for the existence of a σ-finite invariant measure μ for the conservative and ergodic Markov process $\{ P_t \}$ which is finite on the set A is that for each $0 \leqslant f \in L_\infty$ with supp $f \subset A$ we have:

$$\overline{\lim_{T \to \infty}} \frac{\int_0^T P_t f dt}{\int_0^T P_t 1_A dt} \not\equiv 0$$

Proof. — If a σ-finite invariant measure exists then by the ratio limit theorem (see [*11*]) $\lim_{T \to \infty} \frac{\int_0^T P_t f dt}{\int_0^T P_t 1_A dt}$ exists and is different from zero on a set of positive measure. Hence the condition is necessary. Let us prove that the condition is sufficient. Let Q be the convex combination

$$Qf = \int_0^\infty e^{-t} P_t f dt.$$

By lemma 1.1 of [*1*] μ is a σ-finite invariant measure for $\{ P_t \}$ if and only if it is an invariant measure for Q, so, it is sufficient to show that there exists a σ-finite invariant measure for Q, which is finite on the set A. It is known (see for example [*6*], Chapter VI, theorem C) that there exists such a measure for Q if and only if there exists a finite invariant measure for Q_A. It is also known (see for example [*9*] lemma 1) that if there exists no finite invariant measure for the Markov operator P, then the space $\overline{(I - P)L_\infty}$ contains positive functions. Hence if there exists no σ-finite invariant measure for Q which is finite on A then there exists $f \geqslant 0$ with supp $f \subset A$ such that for each $\varepsilon > 0$ there exists $g \in L_\infty$ with supp $g \subset A$ such that $| f - g + Q_A g | \leqslant \varepsilon 1_A$ and we have:

$$\left| \frac{\int_0^T P_t f dt}{\int_0^T P_t 1_A dt} \right| \leqslant \left| \frac{\int_0^T P_t (I_A - Q_A) g dt}{\int_0^T P_t 1_A dt} \right| + \left| \frac{\int_0^T P_t (f - g + Q_A g) dt}{\int_0^T P_t 1_A dt} \right|$$

The second element of the sum in the left-hand side of the inequality is less than ε, while for the numerator of the first element we have by the calculations of [3]

$$(I_A - Q_A)g = (I - Q)\sum_{n=0}^{\infty}(I_A Q)^n I_A g$$

where

$$\left\| \sum_{n=0}^{\infty}(I_A Q)^n I_A g \right\|_{\infty} \leqslant \| g \|_{\infty}$$

and by lemma 2.1 we have

$$\left\| \int_0^T P_t(I_A - Q_A)g dt \right\|_{\infty} = \left\| \int_0^T P_t(I - Q)\sum_{n=0}^{\infty}(I_A Q)^n I_A g dt \right\|_{\infty} \leqslant 4 \| g \|_{\infty}.$$

So

$$\lim_{T\to\infty} \left| \frac{\int_0^T P_t(I_A - Q_A)g dt}{\int_0^T P_t 1_A dt} \right| \leqslant \lim_{T\to\infty} \frac{4 \| g \|_{\infty}}{\int_0^T P_t 1_A dt} \equiv 0$$

and hence

$$\lim_{T\to\infty} \frac{\int_0^T P_t f dt}{\int_0^T P_t 1_A dt} \equiv 0$$

and the theorem is proved.

Remark. — The theorem of [9] is the analogous theorem for a single Markov operator.

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