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Strong ratio limit theorems for mixing Markov operators

by

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ABSTRACT. — Let P be a Markov operator (positive linear contraction of L_1), which is assumed mixing (i. e., $uP^n \rightarrow 0$ for $u \in L$, with $\int u = 0$). For fixed $f \in L_\infty$, we obtain conditions for the convergence of $\langle uP^n, f \rangle / \langle vP^n, f \rangle$ for every $0 \leq v, u \in L_1$, and for the convergence of $\langle uP^n, g \rangle / \langle vP^n, f \rangle$ for $0 \leq g \leq f$ and every $0 \leq v, u \in L_1$.

1. NOTATIONS

Let (X, Σ, m) be a σ -finite measure space, and let P be a positive linear contraction of $L_1(X, \Sigma, m)$. Its action will be written by uP , while the action of the adjoint on L_∞ will be written as Pf , so that $\langle uP, f \rangle = \langle u, Pf \rangle$.

Identifying L_1 with the space $M(X, \Sigma, m)$ of finite signed measures $\ll m$ (via the Radon-Nikodym theorem), P can be represented as an operator on $M(X, \Sigma, m)$, still denoted by P , so that $d(vP)/dm = uP$ when $u = dv/dm$.

A positive (finite or σ -finite) measure $\lambda \ll m$ is *invariant* if $\lambda P = \lambda$, and many

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limit theorems require (or imply) this assumption. For existence theorems, see [J].

P is called *conservative* if $\sum_{n=0}^{\infty} uP^n$ is 0 or ∞ a. e., for $0 \leq u \in L_1$. It is *ergodic* if $f \in L_{\infty}, Pf = f \Rightarrow f \equiv 1$, (and by the Hahn-Banach theorem, $\int u = 0 \Rightarrow \left\| \frac{1}{N} \sum uP^n \right\| \rightarrow 0$), and *mixing* if uP^n converges weakly to 0 in L_1 , for u with $\int udm = 0$.

2. STRONG RATIO LIMIT THEOREMS

Strong ratio limit theorems for a Markov operator deal with the convergence of the expressions $\langle \eta P^n, g \rangle / \langle \nu P^n, f \rangle$, where $\nu, \eta \ll m$ are probability measures, and $0 \leq f, g \in L_{\infty}$. If P has a finite invariant measure and is ergodic, all these expressions converge if and only if P is mixing. We shall assume now that P is mixing, and irreducible (i. e., if $L_1(A)$ is invariant under P then $m(A) = 0$ or $m(X - A) = 0$). In that case, either P has a finite invariant measure equivalent to m , or P has no finite invariant measures.

LEMMA 2.1. — *If P has no finite invariant measure and uP^n converges weakly, then $\|uP^n\|_1 \rightarrow 0$. This is theorem 5.1 of Krengel and Sucheston [8].*

We first deal with particular cases of the convergence.

THEOREM 2.1. — *Let P be an irreducible mixing Markov operator and let $0 \leq f \in L_{\infty}$. A necessary and sufficient condition for the convergence $\lim \langle \nu P^n, f \rangle / \langle \eta P^n, f \rangle = 1$ for every probability measures $\nu, \eta \ll m$ is the existence of a probability measure $\mu \ll m$ satisfying*

$$(2.1) \quad \limsup \|P^n f / \langle \mu, P^n f \rangle\|_{\infty} < \infty$$

$$(2.2) \quad \liminf_{n \rightarrow \infty} \langle \mu, P^{n+1} f \rangle / \langle \mu, P^n f \rangle \geq 1.$$

Proof. — Let $f_n = P^n f / \langle \mu, P^n f \rangle$. If the convergence holds, $f_n \rightarrow 1$ in the weak-* topology of L_{∞} and (2.1) follows, while (2.2) follows by putting $\nu = \mu P, \eta = \mu$, and necessity is proved.

Sufficiency : If P has a finite invariant measure, mixing implies the convergence, so we may assume there is no finite invariant measure. Fix a

probability measure ν . By lemma 2.1, $\|(v - \mu)P^n\| \rightarrow 0$. For $\varepsilon > 0$ fix k such that $\|(v - \mu)P^k\| < \varepsilon$. By (2.1) $\|f_n\|_\infty \leq M$ for $n > n_0$.

$$\begin{aligned} \limsup_{n \rightarrow \infty} |\langle \nu P^n, f \rangle / \langle \mu P^n, f \rangle - 1| &= \limsup_{n \rightarrow \infty} |\langle (v - \mu)P^k, P^{n-k} f \rangle / \langle \mu P^n, f \rangle| \\ &= \limsup_{n \rightarrow \infty} |\langle (v - \mu)P^k, f_{n-k} \rangle| \{ \langle \mu P^{n-k}, f \rangle / \langle \mu P^n, f \rangle \} \\ &\leq \varepsilon M \limsup_{n \rightarrow \infty} \langle \mu P^{n-k}, f \rangle / \langle \mu P^n, f \rangle \leq \varepsilon M, \end{aligned}$$

the last inequality by (2.2). Let $\varepsilon \rightarrow 0$ to conclude the proof.

THEOREM 2.2. — *Let P be an irreducible mixing Markov operator and let $0 \leq g \leq f \in L_\infty$. Let μ and f satisfy (2.1) and (2.2). If $\langle \mu P^n, g \rangle / \langle \mu P^n, f \rangle$ converges to a nonzero limit α , then for every probability measures $\nu, \eta \ll m$ we have*

$$\lim_{n \rightarrow \infty} \langle \nu, P^n g \rangle / \langle \eta, P^n f \rangle = \alpha.$$

Proof. — We show that μ and g satisfy (2.1) and (2.2).

$$\begin{aligned} \langle \mu P^{n+1}, g \rangle / \langle \mu P^n, g \rangle &= \{ \langle \mu P^{n+1}, g \rangle / \langle \mu P^{n+1}, f \rangle \} \\ &\quad \{ \langle \mu P^{n+1}, f \rangle / \langle \mu P^n, f \rangle \} \{ \langle \mu P^n, f \rangle / \langle \mu P^n, g \rangle \} \rightarrow 1. \end{aligned}$$

If n is large enough,

$$\|P^n g / \langle \mu P^n, g \rangle\|_\infty \leq \|P^n f / \langle \mu P^n, f \rangle\|_\infty \{ \langle \mu P^n, f \rangle / \langle \mu P^n, g \rangle \} \leq M2/\alpha.$$

Apply theorem 2.1 to g .

COROLLARY 2.1. — *Let P be a conservative mixing Markov operator and let $A \in \Sigma$ with $m(A) > 0$. If $\mu \ll m$ is a probability measure such that $\mu P^{n+1}(B) / \mu P^n(A)$ converges for $B \subset A$, and μ satisfies*

$$\limsup \|P^n 1_A / \mu P^n(A)\|_\infty < \infty,$$

then there exists a σ -finite invariant measure $\lambda \sim m$ with $\lambda(A) < \infty$, and there is a sequence of sets $A = A_0 \subset A_1 \subset \dots \subset A_k \uparrow X$ with $\lambda(A_k) < \infty$ such that for $0 \leq f, g \in L_\infty(A_k)$ and every probabilities $\nu, \eta \ll m$ we have

$$(2.3) \quad \lim_{n \rightarrow \infty} \langle \nu P^n, g \rangle / \langle \eta P^n, f \rangle = \int g d\lambda / \int f d\lambda.$$

Proof. — The existence of λ is known ([11], lemma 1.1), and

$$\lim \mu P^{n+1}(B) / \mu P^n(A) = \lambda(B) / \lambda(A).$$

Putting $B = A$ we have (2.2) (with $f = 1_A$) and (2.3) holds for $0 \leq f, g \in L_\infty(A)$ by theorem 2.2. The existence of $\{A_k\}$ follows now from theorem 3.4 of [11].

REMARK 2.1. — The condition of mixing instead of ergodicity in the corol-

lary is not very restrictive: it is satisfied in aperiodic Harris operators [7] [13], and in aperiodic random walks [2]. Point transformations (with $\lambda(X) = \infty$) do not satisfy (2.1) when $\int f d\lambda < \infty$, since (2.1) then implies $\|P^n f\|_\infty \rightarrow 0$.

2. An example of a random walk satisfying the conditions of the corollary was given in [12]. Hence the results are more general than those obtained by Levitan and Smolowitz [10], who assume Harris recurrence (and have longer proofs).

3. If P is conservative and ergodic, then a weaker version of theorem 1 can be proved by the methods of [11]. We need a condition stronger than (2.2) (in a certain sense): For every $v \ll m$ we have

$$\limsup_{n \rightarrow \infty} \langle vP^{n+1}, f \rangle / \langle vP^n, f \rangle = 1.$$

The corresponding version of corollary 2.1 is given in [12].

The next proposition yields a sufficient criterion for corollary 2.1 to hold.

PROPOSITION. — *Let P be ergodic and conservative with σ -finite invariant measure $\lambda \sim m$, and let $0 \leq f \in L_\infty$ satisfy $0 < \int f d\lambda < \infty$. Let $\mu \ll m$ be a probability measure such that $\limsup \|P^n f / \langle \mu P^n, f \rangle\|_\infty < \infty$. If for $0 \leq g \in L_\infty$ with $\int g d\lambda < \infty$ the sequence $P^n g(x) / P^n f(x)$ converges μ -a. e., then $\liminf_{n \rightarrow \infty} \langle \mu P^n, g \rangle / \langle \mu P^n, f \rangle \geq \int g d\lambda / \int f d\lambda$.*

Proof. — By the Chacon-Ornstein theorem

$$\lim_{N \rightarrow \infty} \sum_{n=0}^N P^n g(x) / \sum_{n=0}^N P^n f(x) = \int g d\lambda / \int f d\lambda$$

a. e., and since $\sum_{n=0}^{\infty} P^n f(x) \equiv \infty$, $P^n g(x) / P^n f(x) \rightarrow \int g d\lambda / \int f d\lambda$ μ -a. e. Let S

be the support of μ (i. e., $S = \{d\mu/dm > 0\}$). Fix $\alpha < \int g d\lambda / \int f d\lambda$. For $x \in S$ there is a first $N(x)$ such that for $n \geq N(x)$, $P^n g(x) / P^n f(x) > \alpha$. We have $\{x \in S : N(x) \leq k\} = \bigcap_{n=k}^{\infty} \{x : P^n g(x) - \alpha P^n f(x) > 0\} \cap S$, showing the measurability of $N(x)$ on S . Putting $B_k = \{x \in S : N(x) > k\}$, $\mu(B_k) \downarrow 0$.

Now

$$\begin{aligned} \langle \mu P^n, g \rangle - \alpha \langle \mu P^n, f \rangle &= \int P^n(g - \alpha f) d\mu \geq \int_{B_n} P^n(g - \alpha f) d\mu \geq -\alpha \int_{B_n} P^n f d\mu. \end{aligned}$$

Hence, with $M = \sup_{n \geq n_0} \| P^n f / \langle \mu P^n, f \rangle \|_\infty$, we have

$$\langle \mu P^n, g \rangle / \langle \mu P^n, f \rangle - \alpha \geq -\alpha \int_{B_n} \{ P^n f / \langle \mu P^n, f \rangle \} d\mu \geq -\alpha M \mu(B_n) \rightarrow 0.$$

Hence $\liminf \langle \mu P^n, g \rangle / \langle \mu P^n, f \rangle \geq \alpha$. Now let $\alpha \rightarrow \int g d\lambda / \int f d\lambda$.

COROLLARY 2.2. — *Let P be a conservative mixing Markov operator with σ -finite invariant measure $\lambda \sim m$, and let $0 < \lambda(A) < \infty$. Let μ be a probability measure satisfying (2.1) and (2.2) (with $f = 1_A$). If for every $B \subset A$ $P^n 1_B(x) / P^n 1_A(x)$ converges on a nonnull set, then the results of corollary 2.1 hold.*

Proof. — Fix $B \subset A$. Let μ_0 be a probability supported in

$$\{ x : P^n 1_B(x) / P^n 1_A(x) \text{ converges} \}.$$

By theorem 2.1 μ_0 also satisfies (2.1) and (2.2) (with $f = 1_A$). Applying the proposition to 1_B and to $1_A - 1_B = 1_{A-B}$, with μ_0 , we have

$$\lim \mu_0 P^n(B) / \mu_0 P^n(A) = \lambda(B) / \lambda(A).$$

We now apply theorem 2.2 (with μ_0) to obtain $\mu P^n(B) / \mu P^n(A) \rightarrow \lambda(B) / \lambda(A)$. Since $\mu P^{n+1}(A) / \mu P^n(A) \rightarrow 1$, we can apply corollary 2.1.

Remarks. — 1. If μ and f satisfy (2.1), then a sufficient condition for (2.2) is that $P^{n+1} f(x) / P^n f(x)$ converges μ -a. e., when P is conservative and ergodic.

(Then the limit is 1, since $\sum_{n=0}^{\infty} P^n(x) \equiv \infty$). This is proved as in the proposition, by taking $g = Pf$, and $\alpha < 1$. (We do not require an invariant measure; e. g., $f \equiv 1$).

2. The proof of the proposition is a modification of the proof of the ratio limit theorem (for sums) in [3].

THEOREM 2.3. — *Let P be an irreducible mixing Markov operator and let $\mu \ll m$ be a probability measure.*

(i) *The set of all functions $0 \leq f \in L_\infty$ which satisfy (2.1) and (2.2) is P-invariant and closed under addition.*

(ii) Let f satisfy (2.1) and (2.2). If $0 \leq g \in L_\infty$ satisfies $P^K g \geq \beta f$ and $\sum_{n=0}^N P^n f \geq \alpha g$ for some K, N, α, β positive, then (2.1) and (2.2) are satisfied with g replacing f .

Proof. — (i) P -invariance is trivial. Let $f = f_1 + f_2$ with $0 \leq f_i$ satisfying (2.1) and (2.2).

$$\|P^n f\|_\infty / \langle \mu P^n, f \rangle \leq \|P^n f_1\|_\infty / \langle \mu P^n, f \rangle + \|P^n f_2\|_\infty / \langle \mu P^n, f \rangle \\ \leq \|P^n f_1\| / \langle \mu P^n, f_1 \rangle + \|P^n f_2\| / \langle \mu P^n, f_2 \rangle$$

and (2.1) follows. For $\varepsilon > 0$, if $n \geq n_0(\varepsilon)$,

$$\langle \mu P^{n+1}, f \rangle = \langle \mu P^{n+1}, f_1 \rangle + \langle \mu P^{n+1}, f_2 \rangle \\ \geq (1 - \varepsilon) \langle \mu P^n, f_1 \rangle + (1 - \varepsilon) \langle \mu P^n, f_2 \rangle = (1 - \varepsilon) \langle \mu P^n, f \rangle.$$

(ii) For $n > K$ we have

$$\langle \mu P^n, f \rangle / \langle \mu P^n, g \rangle = \{ \langle \mu P^{n-k}, f \rangle / \langle \mu P^n, g \rangle \} \langle \mu P^n, f \rangle / \langle \mu P^{n-k}, f \rangle \\ \leq \beta^{-1} \langle \mu P^n, f \rangle / \langle \mu P^{n-k}, f \rangle$$

and the right-hand side tends to β^{-1} by theorem 2.1. Hence

$$\langle \mu P^n, f \rangle / \langle \mu P^n, g \rangle \leq M$$

for $n \geq n_1$. For any $k \geq 0$,

$$\alpha \|P^{n-k} g / \langle \mu P^n, g \rangle\| \leq \left\| \sum_{j=0}^N P^{n-k+j} f \right\| / \langle \mu P^n, g \rangle \leq \\ \frac{\langle \mu P^n, f \rangle}{\langle \mu P^n, g \rangle} \sum_{j=0}^N \{ \|P^{n-k+j} f\| / \langle \mu P^{n-k+j}, f \rangle \} \{ \langle \mu P^{n-k+j}, f \rangle / \langle \mu P^n, f \rangle \}.$$

Hence, since the last factors tend to 1, we have

$$\limsup_{n \rightarrow \infty} \|P^{n-k} g / \langle \mu P^n, g \rangle\| \\ \leq \alpha^{-1} M(N+1) \limsup_{i \rightarrow \infty} \|P^i f / \langle \mu P^i, f \rangle\| = M_1.$$

M_1 does not depend on k .

Let $v \ll m$ be a probability. For $\varepsilon > 0$ take k such that $\|(\mu - v)P^k\| < \varepsilon$. Then

$$\limsup | \langle v P^n, g \rangle / \langle \mu P^n, g \rangle - 1 | \\ = \limsup_{n \rightarrow \infty} | \langle (v - \mu)P^k, P^{n-k} g \rangle / \langle \mu P^n, g \rangle | \leq \varepsilon M_1.$$

The theorem shows that if $f = 1_B$ satisfies (2.1) and (2.2), so does 1_A with

$$A = \left\{ x : \sum_{n=0}^N P^n 1_B(x) \geq 1 \right\}$$

DÉFINITIONS. — A function $0 \leq f \in L_\infty$ is *small* if for every $0 \leq g \in L_\infty$

($g \neq 0$) there are K and $\beta > 0$ such that $\sum_{n=0}^K P^n g \geq \beta f$. f is *very small* if for

every $0 \leq g \in L_\infty$ there are K and $\beta > 0$ such that $P^K g \geq \beta f$.

It was proved by Horowitz [4] that if P is conservative and ergodic and has a « small » function $f \neq 0$, then it is Harris. If P is Harris, then there are « very small » functions [5]. Levitan [9] treated ratio limit theorems on sets A which satisfies $\inf_{x,y \in A} q_k(x,y) = \beta > 0$ for some k , where q_k is the kernel bounded by P^k . Clearly 1_A is then « very small ». If we define on $X = \{0, 1, 2, \dots\}$ $p_{ii+1} = 1/2$, $p_{i0} = 1/2$, $p_{ij} = 0$ otherwise, then $P^{j+1} 1_{\{j\}} \geq 1/2^{j+1}$ and 1 is a « very small » function ; but

$$\sup_n \inf_{i,j} p_{ij}^{(n)} = 0,$$

so X itself does not satisfy Levitan's condition.

COROLLARY 2.3. — Let P be an aperiodic Harris operator. It for some « very small » function f and a probability measure $\mu \ll m$ the relations (2.1) and (2.2) hold, then for every « small » function g and probabilities $\eta, \nu \ll m$ we have $\lim \langle \nu P^n, g \rangle / \langle \eta P^n, g \rangle = 1$.

THEOREM 2.4. — Let X be a locally compact σ -compact Abelian group, with Haar measure λ , and let P be an aperiodic random walk on X . If

(i) For some compact set C , $\limsup \{ P^n(0, C) \}^{1/n} = 1$;

(ii) $\limsup \| P^n 1_B / \mu P^n(B) \|_\infty < \infty$ for some relatively compact open set B with $\lambda(\partial B) = 0$, and μ with compact support ;

Then $\lim \langle \nu P^n, g \rangle / \langle \eta P^n, f \rangle = \int g d\lambda / \int f d\lambda$ for $0 \leq f, g \in L_\infty$ with compact supports and any two probabilities $\nu, \eta \ll \lambda$.

Proof. — By Stone [15], condition (i) implies that $P^n 1_B(x) / P^n 1_B(y) \rightarrow 1$ uniformly for x, y in compact sets. We can therefore obtain

$$P^n 1_B(x) / \mu P^n(B) \rightarrow 1$$

uniformly on compact sets. Let $d\mu_1 = \lambda(B)^{-1} 1_B$. Then $\mu_1 P^n(B) / \mu P^n(B) \rightarrow 1$.

Then $\|P^n 1_B\|_\infty / \mu_1 P^n(B) = \{ \|P^n|_B\|_\infty / \mu P^n(B) \} \mu P^n(B) / \mu_1 P^n(B)$ is also bounded. Hence we may and do assume $d\mu/d\lambda = \lambda(B)^{-1} 1_B$.

The adjoint Markov operator is the random walk induced by $p^*(A) = p(-A)$, and condition (i) applies to P^* as well (since we may assume that C is symmetric). Hence by Stone [15] we have

$$P^{*n+r} 1_B(x) / P^{*n} 1_B(y) \rightarrow 1$$

uniformly for x, y in compact sets. Let $0 \leq f, g \in L_\infty$ have compact supports. The above uniform convergence yields easily that

$$(*) \quad \langle \mu P^{n+r}, g \rangle / \langle \mu P^n, f \rangle = \langle g, P^{*n+r} 1_B \rangle / \langle f, P^{*n} 1_B \rangle \rightarrow \int g d\lambda / \int f d\lambda.$$

When $g = f = 1_B$, (2.1) and (2.2) are satisfied.

Let $A = B \cup \{f > 0\} \cup \{g > 0\}$. Then A is relatively compact, and its closure can be included in a relatively compact open set E with $\lambda(\partial E) = 0$ (using Urysohn's lemma).

Since the random walk is aperiodic, for a continuous $0 \leq h \leq 1$ supported in B (e. g., $h(a) = 1$ at a point of B , $h = 0$ outside B) we have $\sum_{n=0}^\infty P^n h(x) > 0$ on \bar{E} , hence the compactness of \bar{E} yields N and α such that

$$\sum_{n=0}^N P^n 1_B \geq \sum_{n=0}^N P^n h \geq \alpha 1_E.$$

By theorem 2.3, we can apply theorem 2.1 (P is mixing by Foguel [2]) to have (2.1) and (2.2) satisfied by μ and $1_{\bar{E}}$. Relation (*) and theorem 2.2 conclude the proof.

Remarks. — 1. We see from the proof that (ii) can be replaced by (ii)' $\limsup \|P^n f / \langle \mu P^n, f \rangle\|_\infty < \infty$ for some $0 \leq f \in L_\infty$ with compact support and μ with $d\mu/d\lambda = \lambda(B)^{-1} 1_B$, where B is as in (ii).

2. The conditions of the theorem are clearly necessary.

Levitan and Smolowitz [10] treated also the case that P is Harris and self-adjoint with respect to the invariant measure λ (i. e., $\langle u, Pf \rangle = \langle f, Pu \rangle$ for $f, u \in L_2(\lambda)$). We obtain results without the Harris assumption. The following lemma was observed by Pruitt [14] for Markov chains, and its arguments are used in [10].

LEMMA 2.2. — Let T be a self-adjoint operator in a Hilbert space H satisfying $\|T\| \leq 1$. If $Tf \neq 0$, then $T^n f \neq 0$ for every n , $\lim \|T^{n+1} f\| / \|T^n f\|$

exists, and does not exceed 1. If $\sum_{n=1}^{\infty} \|T^n f\| = \infty$, then $\|T^{n+1} f\| / \|T^n f\| \rightarrow 1$.

Proof. — The spectrum $\sigma(T)$ is contained in $[-1, 1] = I$. Let f satisfy $Tf \neq 0$. Let $E(\cdot)$ be the resolution of the identity of T . Define, for Borel subsets of $I = [-1, 1]$, $\mu(A) = \langle E(A)f, f \rangle = \|E(A)f\|^2$. By the spectral theorem

$$\begin{aligned} \|T^n f\|^2 &= \langle T^{2n} f, f \rangle = \int_I t^{2n} d\mu = \int_I t^{n+1} t^{n-1} d\mu \\ &\leq \left\{ \int_I t^{2(n+1)} d\mu \right\}^{1/2} \left\{ \int_I t^{2(n-1)} d\mu \right\}^{1/2} = \|T^{n+1} f\| \|T^{n-1} f\|. \end{aligned}$$

Thus $T^n f \neq 0 \Rightarrow T^{n+1} f \neq 0$, and $T^n f \neq 0$ for every n . Furthermore, $\|T^n f\| / \|T^{n-1} f\| \leq \|T^{n+1} f\| / \|T^n f\|$ and the sequence $\|T^{n+1} f\| / \|T^n f\|$ is increasing, and bounded by 1. The last statement is clear.

THEOREM 2.5. — Let P be a conservative mixing Markov operator with σ -finite invariant measure $\lambda \sim m$, and assume that P is self-adjoint. Let $0 \leq f \in L_\infty$ satisfy $0 < \int f d\lambda < \infty$. A necessary and sufficient condition for the convergence $\lim \langle \nu P^n, g \rangle / \langle \eta P^n, f \rangle = \int g d\lambda / \int f d\lambda$ to hold for every $0 \leq g \leq f$ and probabilities $\nu, \eta \ll m$ is $\limsup \|P^{2n} f / \langle P^{2n} f, f \rangle\|_\infty < \infty$.

Proof. — Define μ by $d\mu/d\lambda = \left(\int f d\lambda\right)^{-1} f$. Necessity is clear. $Q = P^2$ is also conservative and mixing. By lemma 2.2, $\|P\|_2 \leq 1$ implies

$$\begin{aligned} \langle \mu Q^{n+1}, f \rangle / \langle \mu Q^n, f \rangle &= \langle P^{2(n+1)} f, f \rangle / \langle P^{2n} f, f \rangle \\ &= \|P^{n+1} f\|_2^2 / \|P^n f\|_2^2 \rightarrow 1, \end{aligned}$$

since,

$$\|f\|_2 \sum_{n=0}^{\infty} \|P^n f\|_2 \geq \sum_{n=0}^{\infty} \|P^n f\|_2^2 = \sum_{n=0}^{\infty} \langle P^n f, P^n f \rangle = \sum_{n=0}^{\infty} \langle f, P^{2n} f \rangle = \infty.$$

Hence f, μ and Q satisfy (2.1) and (2.2) and for every $\nu, \eta \ll m$ we have, by theorem 2.1,

$$\lim \langle \nu P^{2n}, f \rangle / \langle \eta P^{2n}, f \rangle = 1.$$

Substituting νP for ν and ηP for η , we have $\langle \nu P^n, f \rangle / \langle \eta P^n, f \rangle \rightarrow 1$.
If $0 \leq g \leq f$, then

$$\begin{aligned} \langle \mu P^n, g \rangle / \langle \mu P^n, f \rangle \\ = \langle P^n g, f \rangle / \langle P^n f, f \rangle = \langle P^n f, g \rangle / \langle P^n f, f \rangle \rightarrow \int g d\lambda / \int f d\lambda \end{aligned}$$

when $d\nu/d\lambda = \left(\int g d\lambda \right)^{-1} g$. We now apply theorem 2.2.

COROLLARY 2.4. — *Let P be an aperiodic self-adjoint Harris operator. If for some « very small » function f we have $\limsup \| P^{2^n} f / P^{2^n} f, f \rangle \|_\infty < \infty$, then for every « small » functions g, h and probabilities $\nu, \eta \ll m$ we have*

$$\lim \langle \nu P^n, g \rangle / \langle \eta P^n, h \rangle = \int g d\lambda / \int h d\lambda.$$

Proof. — P has a σ -finite invariant measure, and for any « small » function h , $\int h d\lambda < \infty$ (Horowitz [5]).

Now apply theorems 2.3 and 2.5.

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