ANNALES DE L'I. H. P., SECTION B

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Annales de l'I. H. P., section B, tome 12, nº 2 (1976), p. 193-197

<http://www.numdam.org/item?id=AIHPB_1976__12_2_193_0>

Ann. Inst. Henri Poincaré,
Vol. XII, n° 2, 1976, p. 193-197.

Section B :
Calcul des Probabilités et Statistique.

Some use of some « symmetries » of some random process

by

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RÉSUMÉ. — Un processus dans lequel des chats sont soumis à des promenades aléatoires uni-dimensionnelles jusqu'à être annihilés en collisions binaires est considéré. Une conjecture de Paul Erdös et Peter Ney est vérifiée, en utilisant l'invariance de certaines « symétries » de la distribution des chats.

ABSTRACT. — A process in which cats undergo one-dimensional independent random walks until being annihilated in binary collisions is considered. A conjecture of Paul Erdös and Peter Ney is verified, using invariance of certain « symmetries » of the cats' distribution.

INTRODUCTION

Suppose there is a cat on each integer point on the line, except O. Then start the following. At each second, each cat jumps, independently of the others, with probability 1/2 to each of the two neighbouring places. If two cats are in danger of a mutual collision (either in mid-air or on some integer), they both disappear just before the collision is to take place (each second cat can be considered an anti-cat).

Set $\equiv \{ \text{O is visited} \}$.

Paul Erdös and Peter Ney have conjectured ([1]) that $P(A) = 1$. Here I prove it.

« Almost surely » is omitted. « Before » and « after » are used in an extended manner; i. e., X will be said to happen before Y (or : Y after X) even if X happens and Y never does.

First, suppose we have one cat only, initially at some $n \neq 0$. For this case, it is well-known that $P(A) = 1$. But, just to get the most basic idea that is used here, examine the following very simple proof. Consider the block $0, \ldots, 2n$. One of its end points (0 or $2n$) will be once visited (a run of $2n + 7$ ($2n - 1$ is enough) jumps to the right (say) will take place). There is probability $1/2$ that $2n$ will be visited before 0 is. If $2n$ is first visited, then there is probability $1/2$ that $4n$ will be visited before 0 is, and so on. So we conclude that

$$P(A^c) = (1/2)^\infty = 0 .$$

PROOF OF THE ERDOS-NEY CONJECTURE (ENc)

In **I** I show that $P(A) = 1$ for the case in which only the positive integers are initially occupied. In **II** I use this to verify the ENc.

I

Here we consider the case in which only the positive integers are initially occupied.

Sometimes a certain block, $m, \ldots, n$, say, is related as « symmetric ». By this is meant that, according to the information we have collected up to the moment of naming this block « symmetric », every possible history of the block is exactly as probable as the one which is obtained from it by reflection with respect to $(m + n)/2$.

This kind of « symmetricity » depends, of course, on the stage at which we are and on the questions we have posed in order to achieve our information; but the procedure of « gaining the information » will be explained in detail, and there is no danger of confusion.

Notice that the cats disappear in pairs. If there is a block with an odd number of cats, and if no outsider ever jumps on any of this block's edges, then one of these end-points will be once occupied by a cat initially in the block. If it also happens that this block is « symmetric », then each of the two end-points has a probability of at least $1/2$ of being visited not after the other one is.

I aim to prove that here $P(A^c) = 0$, by showing that in the event A^c there are infinitely many stages after which the block $1, \ldots, n_i$ (n_i depends on the stage's number and on the « history », i. e., on that which has taken place in the previous stages) is « symmetric ». If at the stage s_i the « symmetry » of the block $1, \ldots, n_i$ is destroyed without 1 being visited at that stage, which is shown to have a probability smaller than some constant smaller than 1, then there is some $n_{i+1} > n_i$ such that the block $1, \ldots, n_{i+1}$ is « symmetric » after the stage s_i is completed. Notice that if 1 is visited, then there is probability 1/2 that 0 will be visited on the next stage, provided -1 is not occupied.

Suppose that n_i is some odd positive number, and consider the block $1, \ldots, n_i$. There is a probability of at least $P(A^c)$ that a jump $n_i + 1 \to n_i$ will not occur at least until after a jump $n_i \to n_i + 1$ occurs. Suppose that at a certain stage the block $1, \ldots, n_i$ is « symmetric ». Suppose, moreover, that none of the jumps $1 \to 0$, $n_i \to n_i + 1$ has already taken place. Then there is a probability of at least $P(A^c)/2$ that 1 will be once occupied while -1 is still empty, so there is a probability of at least $P(A^c)/4$ that 0 will be visited by a cat initially in the block $1, \ldots, n_i$.

Let

$$A_i \equiv \{ \text{a cat initially in } 1, \ldots, n_i \text{ visits } 0, \text{ and this happens not after a jump } n_i + 1 \to n_i \text{ or } n_i \to n_i + 1 \text{ does} \}.$$

In the case A_i^c, denote by s_i the number of the stage at which something that contradicts A_i has first occurred (i. e., s_i is the number of the first jump which is $n_i \to n_i + 1$ or $n_i + 1 \to n_i$). We are not interested in the case (of probability 0) in which the above definition is not applicable.

Set $n_1 = 1$. If A_i, we are satisfied. If not, stop after stage s_i, and choose n_{i+1} according to the following procedure. Let $B_{i,j,k}$ denote the block $[_{i,j,k}, \ldots,]_{i,j,k}$, where

$$[_{i,j,k} \equiv (k-1)(n_{i,j} + 2s_i + 1) + 1$$

and

$$]_{i,j,k} \equiv [_{i,j,k} + n_{i,j} - 1\,.$$

$B_{i,j,k}$ are blocks of $n_{i,j}$ integers, with a separation greater than $2s_i$ between any two of them; so their « histories » up to the stage s_i are independent. Notice that $]_{i,j,k}$ is odd, provided $n_{i,j}$ is.

There is a certain positive probability (greater than $2^{-(n_{i,j}+2s_i)s_i}$, for instance) that up to the stage s_i, and at this stage as well, $B_{i,j,k} (k > 1)$ has always been an exact copy of $B_{i,j,1}$, in the sense that for any $l \in \{0, \ldots, n_{i,j}+1\}$, any jump into $B_{i,j,1}$ (i. e., we are not interested in jumps like $n_{i,j}+1 \to n_{i,j}+2$) was simultaneous with a jump in the same direction from $[_{i,j,k} + l - 1$, and *vice-versa*. There is exactly the same probability for reflection (in the analogous sense; i. e., « the same direction » is to be replaced by « the opposite direction », and « $[_{i,j,k} + l - 1$ » by « $]_{i,j,k} - l + 1$ »). Denote copy by C, reflection by R.

Set $n_{i,1} = n_i$. Examine consecutively $B_{i,1,k}$, $k = 2, 3, \ldots$, until a case of $C \cup R$ is encountered, with $k = k^1$. If the case is R (which has probability 1/2), fix $n_{i+1} =]_{i,1,k^1}$. If not, fix $n_{i,2} =]_{i,1,k^1}$, and examine $B_{i,2,k}$, $k = 2, 3, \ldots$, until finding a first case of $C \cup R$. Go on with this procedure. If you meet a case of R, fix $n_{i+1} =$ the right end-point of the last block you have just examined; and if the case is C, and you have been examining the blocks $B_{i,j,k}$, denote this right end-point by $n_{i,j+1}$, and start examining $B_{i,j+1,k}$, $k = 2, 3, \ldots$ You will have a case of R, as if not you will have infinitely many cases of $C \cup R$, each having a probability of at least 1/2 to be R.

Now we have

$$P(A_{i+1}/A_i^c) \geq P(A^c)/4$$

$$\therefore \qquad P(A_{i+1}^c/A_i^c) \leq 1 - P(A^c)/4 .$$

In the case A_i you can define $A_{i+1} \equiv A_i$.

We obtain

$$P(A^c) \leq P\left(\bigcap_i A_i^c\right) \leq [1 - P(A^c)/4]^{\infty};$$

so if $P(A^c) > 0$, then $P(A^c) \leq 0$.

II

Here I show how the ENc is verified, as a consequence of **I**.

If initially only 0 is empty, then we know (by **I**) that at least one of the jumps $1 \to 0$, $-1 \to 0$ will take place. In order that 0 be never visited, every jump $1 \to 0$ has to be accompanied by a simultaneous jump $-1 \to 0$ (and *vice-versa*). If there are infinitely many jumps from 1 towards 0, the probability that every one of them will be accompanied by a jump $-1 \to 0$ (even if -1 is occupied whenever necessary for this to happen) is 0. So it is sufficient to prove the following lemma.

THE FOLLOWING LEMMA. — If initially all the positive integers are occupied, and every cat that jumps on 0 is immediately removed, then there will be infinitely many jumps from 1 towards 0.

Proof. — Suppose a cat initially at some $n > 0$ gets to 0 at stage number s. Denote by n_0 some odd number greater than n, fix $s_0 = s$, and proceed along the line of **I**, assuming the situation after the stage s is the initial one. It follows that 0 will be once visited by a cat initially on the right of n. So, to the right of any cat that visits 0, there is some other that will do so. By **I**, there is at least one; so there are infinitely many, and the ENc is proved.

ACKNOWLEDGMENT

David Gilat, Shimeon Friedman, Isaac Meilijson, Jacques Neveu, Nili Orlovski and Gideon Schwarz are thanked for fruitful discussions on the subject.

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(Manuscrit reçu le 24 mai 1976)