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Some use of some « symmetries » of some random process

by

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RÉSUMÉ. — Un processus dans lequel des chats sont soumis à des promenades aléatoires uni-dimensionnelles jusqu'à être annihilés en collisions binaires est considéré. Une conjecture de Paul Erdös et Peter Ney est vérifiée, en utilisant l'invariance de certaines « symétries » de la distribution des chats.

ABSTRACT. — A process in which cats undergo one-dimensional independent random walks until being annihilated in binary collisions is considered. A conjecture of Paul Erdös and Peter Ney is verified, using invariance of certain « symmetries » of the cats' distribution.

INTRODUCTION

Suppose there is a cat on each integer point on the line, except O. Then start the following. At each second, each cat jumps, independently of the others, with probability 1/2 to each of the two neighbouring places. If two cats are in danger of a mutual collision (either in mid-air or on some integer), they both disappear just before the collision is to take place (each second cat can be considered an anti-cat).

Set $\equiv \{ \text{O is visited} \}$.

Paul Erdös and Peter Ney have conjectured ([*1*]) that P(A) = 1. Here I prove it.

« Almost surely » is omitted. « Before » and « after » are used in an extended manner ; i. e., X will be said to happen before Y (or : Y after X) even if X happens and Y never does.

First, suppose we have one cat only, initially at some $n \neq 0$. For this case, it is well-known that P(A) = 1. But, just to get the most basic idea that is used here, examine the following very simple proof. Consider the block $0, \ldots, 2n$. One of its end points (0 or $2n$) will be once visited (a run of $2n + 7$ ($2n - 1$ is enough) jumps to the right (say) will take place). There is probability 1/2 that $2n$ will be visited before 0 is. If $2n$ is first visited, then there is probability 1/2 that $4n$ will be visited before 0 is, and so on. So we conclude that

$$P(A^c) = (1/2)^\infty = 0 .$$

PROOF OF THE ERDOS-NEY CONJECTURE (ENc)

In **I** I show that P(A) = 1 for the case in which only the positive integers are initially occupied. In **II** I use this to verify the ENc.

I

Here we consider the case in which only the positive integers are initially occupied.

Sometimes a certain block, $m, \ldots, n$, say, is related as « symmetric ». By this is meant that, according to the information we have collected up to the moment of naming this block « symmetric », every possible history of the block is exactly as probable as the one which is obtained from it by reflection with respect to $(m + n)/2$.

This kind of « symmetricity » depends, of course, on the stage at which we are and on the questions we have posed in order to achieve our information ; but the procedure of « gaining the information » will be explained in detail, and there is no danger of confusion.

Notice that the cats disappear in pairs. If there is a block with an odd number of cats, and if no outsider ever jumps on any of this block's edges, then one of these end-points will be once occupied by a cat initially in the block. If it also happens that this block is « symmetric », then each of the two end-points has a probability of at least 1/2 of being visited not after the other one is.

I aim to prove that here $P(A^c) = 0$, by showing that in the event A^c there are infinitely many stages after which the block $1, \ldots, n_i$ (n_i depends on the stage's number and on the « history », i. e., on that which has taken place in the previous stages) is « symmetric ». If at the stage s_i the « symmetry » of the block $1, \ldots, n_i$ is destroyed without 1 being visited at that stage, which is shown to have a probability smaller than some constant smaller than 1, then there is some $n_{i+1} > n_i$ such that the block $1, \ldots, n_{i+1}$ is « symmetric » after the stage s_i is completed. Notice that if 1 is visited, then there is probability 1/2 that 0 will be visited on the next stage, provided -1 is not occupied.

Suppose that n_i is some odd positive number, and consider the block $1, \ldots, n_i$. There is a probability of at least $P(A^c)$ that a jump $n_i + 1 \to n_i$ will not occur at least until after a jump $n_i \to n_i + 1$ occurs. Suppose that at a certain stage the block $1, \ldots, n_i$ is « symmetric ». Suppose, moreover, that none of the jumps $1 \to 0$, $n_i \to n_i + 1$ has already taken place. Then there is a probability of at least $P(A^c)/2$ that 1 will be once occupied while -1 is still empty, so there is a probability of at least $P(A^c)/4$ that 0 will be visited by a cat initially in the block $1, \ldots, n_i$.

Let

$$A_i \equiv \{ \text{a cat initially in } 1, \ldots, n_i \text{ visits } 0, \text{ and this happens not after a jump } n_i + 1 \to n_i \text{ or } n_i \to n_i + 1 \text{ does} \}.$$

In the case A_i^c, denote by s_i the number of the stage at which something that contradicts A_i has first occurred (i. e., s_i is the number of the first jump which is $n_i \to n_i + 1$ or $n_i + 1 \to n_i$). We are not interested in the case (of probability 0) in which the above definition is not applicable.

Set $n_1 = 1$. If A_i, we are satisfied. If not, stop after stage s_i, and choose n_{i+1} according to the following procedure. Let $B_{i,j,k}$ denote the block $[_{i,j,k}, \ldots,]_{i,j,k}$, where

$$[_{i,j,k} \equiv (k-1)(n_{i,j} + 2s_i + 1) + 1$$

and

$$]_{i,j,k} \equiv [_{i,j,k} + n_{i,j} - 1 .$$

$B_{i,j,k}$ are blocks of $n_{i,j}$ integers, with a separation greater than $2s_i$ between any two of them; so their « histories » up to the stage s_i are independent. Notice that $]_{i,j,k}$ is odd, provided $n_{i,j}$ is.

There is a certain positive probability (greater than $2^{-(n_{i,j}+2s_i)s_i}$, for instance) that up to the stage s_i, and at this stage as well, $B_{i,j,k}(k > 1)$ has always been an exact copy of $B_{i,j,1}$, in the sense that for any $l \in \{ 0, \ldots, n_{i,j}+1 \}$, any jump into $B_{i,j,1}$ (i. e., we are not interested in jumps like $n_{i,j}+1 \to n_{i,j}+2$) was simultaneous with a jump in the same direction from $[_{i,j,k} + l - 1$, and *vice-versa*. There is exactly the same probability for reflection (in the analogous sense; i. e., « the same direction » is to be replaced by « the opposite direction », and « $[_{i,j,k} + l - 1$ » by « $]_{i,j,k} - l + 1$ »). Denote copy by C, reflection by R.

Set $n_{i,1} = n_i$. Examine consecutively $B_{i,1,k}$, $k = 2, 3, \ldots$, until a case of $C \cup R$ is encountered, with $k = k^1$. If the case is R (which has probability 1/2), fix $n_{i+1} =]_{i,1,k^1}$. If not, fix $n_{i,2} =]_{i,1,k^1}$, and examine $B_{i,2,k}$, $k = 2, 3, \ldots$, until finding a first case of $C \cup R$. Go on with this procedure. If you meet a case of R, fix n_{i+1} = the right end-point of the last block you have just examined; and if the case is C, and you have been examining the blocks $B_{i,j,k}$, denote this right end-point by $n_{i,j+1}$, and start examining $B_{i,j+1,k}$, $k = 2, 3, \ldots$ You will have a case of R, as if not you will have infinitely many cases of $C \cup R$, each having a probability of at least 1/2 to be R.

Now we have

$$P(A_{i+1}/A_i^c) \geq P(A^c)/4$$

$$\therefore \quad P(A_{i+1}^c/A_i^c) \leq 1 - P(A^c)/4 .$$

In the case A_i you can define $A_{i+1} \equiv A_i$.

We obtain

$$P(A^c) \leq P\left(\bigcap_i A_i^c\right) \leq [1 - P(A^c)/4]^\infty ;$$

so if $P(A^c) > 0$, then $P(A^c) \leq 0$.

II

Here I show how the ENc is verified, as a consequence of **I**.

If initially only 0 is empty, then we know (by **I**) that at least one of the jumps $1 \to 0$, $-1 \to 0$ will take place. In order that 0 be never visited, every jump $1 \to 0$ has to be accompanied by a simultaneous jump $-1 \to 0$ (and *vice-versa*). If there are infinitely many jumps from 1 towards 0, the probability that every one of them will be accompanied by a jump $-1 \to 0$ (even if -1 is occupied whenever necessary for this to happen) is 0. So it is sufficient to prove the following lemma.

THE FOLLOWING LEMMA. — If initially all the positive integers are occupied, and every cat that jumps on 0 is immediately removed, then there will be infinitely many jumps from 1 towards 0.

Proof. — Suppose a cat initially at some $n > 0$ gets to 0 at stage number s. Denote by n_0 some odd number greater than n, fix $s_0 = s$, and proceed along the line of **I**, assuming the situation after the stage s is the initial one. It follows that 0 will be once visited by a cat initially on the right of n. So, to the right of any cat that visits 0, there is some other that will do so. By **I**, there is at least one; so there are infinitely many, and the ENc is proved.

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