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by

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1. INTRODUCTION

There exist a compact metric space K and two singular Radon probability measures on K which agree on all balls ([*5*], Th. II, [*4*]). Therefore, since K is isometric to a compact subset of the Banach space C(K), we can find two singular Radon probability measures μ and ν on C(K) satisfying the condition

(C_0) *for every* $a \in C(K)$ *there exists a* $\delta > 0$ *such that*

$$\mu(B(a\,;r)) = \nu(B(a\,;r)), \qquad 0 < r < \delta .$$

Here $B(a\,;r)$ denotes the closed ball of centre a and radius r. (Compare [*6*], p. 326, and [*9*].)

The main result of this note shows that two Gaussian Radon measures on C(K) (or any Banach space) coincide whenever the condition (C_0) holds (Theorem 3.1). Moreover, we prove that two Gaussian Radon measures on a Banach space are equal, if they agree on all balls of radius one (Theorem 3.2). The same theorem also gives a positive result for dual Banach spaces, equipped with the weak* topology.

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2. THE REPRODUCING KERNEL HILBERT SPACE OF A GAUSSIAN RADON MEASURE

In this section it will always be assumed that E is a fixed locally convex Hausdorff vector space over $\mathbb{R}$. The class of all (centred) Gaussian Radon measures on E is denoted by $\mathcal{G}(\mathrm{E})$ ($\mathcal{G}_0(\mathrm{E})$). In the following, all non-trivial statements will either be proved or, otherwise, they can be found in e. g. [3].

Let $\mu \in \mathcal{G}(\mathrm{E})$ be fixed and denote by b the barycentre of μ. Set $\mu_0(.) = \mu(. + b)$ and $\mathrm{E}'_2(\mu) =$ the closure of E' in $\mathrm{L}_2(\mu_0)$, respectively. Then for every $\eta \in \mathrm{E}'_2(\mu)$, the measure $\eta\mu_0$ has a barycentre $\Lambda(\eta) \in \mathrm{E}$. The map $\Lambda : \mathrm{E}'_2(\mu) \to \mathrm{E}$ is injective. Its range is denoted by $\mathrm{H}(\mu)$. For brevity we write $\Lambda^{-1}h = \tilde{h}$, $h \in \mathrm{H}(\mu)$. Obviously, the scalar product

$$\langle h, k \rangle_\mu = \int \tilde{h}\tilde{k} d\mu_0, \qquad h, k \in \mathrm{H}(\mu),$$

makes $\mathrm{H}(\mu)$ into a Hilbert space, the so-called reproducing kernel Hilbert space of μ. The closed unit ball $\mathrm{O}(\mu)$ of $\mathrm{H}(\mu)$ is a compact subset of E. Moreover,

$$\max_{\mathrm{O}(\mu)} \xi^2 = \int \xi^2 d\mu_0 , \qquad \xi \in \mathrm{E}' .$$

Observing that

$$\int \exp (i\xi) d\mu_0 = \exp\left(-\frac{1}{2} \int \xi^2 d\mu_0 \right), \qquad \xi \in \mathrm{E}' .$$

we have the following useful

THEOREM 2.1. — *Let μ, $\nu \in \mathcal{G}_0(\mathrm{E})$. Then $\mu = \nu$ if $\mathrm{O}(\mu) = \mathrm{O}(\nu)$.*

Our strategy from now on will be to determine $\mathrm{O}(\mu)$ from measures of sufficiently many « balls ». A weak result in this direction follows from e. g. [3], Th. 10.1. Theorems 2.2 and 2.3 below yield stronger conclusions.

Before proceeding, let us introduce

$$\| a \|_\mu^2 = \begin{cases} \langle a, a \rangle_\mu , & a \in \mathrm{H}(\mu), \\ + \infty , & a \in \mathrm{E} \backslash \mathrm{H}(\mu) . \end{cases}$$

Moreover, in the following, measurable always means Borel measurable.

THEOREM 2.2. — *Let $\mu \in \mathcal{G}_0(\mathrm{E})$ and suppose V is a bounded, symmetric, convex, and measurable subset of E such that $\mu(r\mathrm{V}) > 0$, $r > 0$. Then*

$$(2.1) \qquad \lim_{r \to 0^+} \frac{\mu(a + r\mathrm{V})}{\mu(r\mathrm{V})} = \exp\left(-\frac{1}{2} \| a \|_\mu^2 \right), \qquad a \in \mathrm{E} .$$

In many special cases, the behaviour of $\mu(r\mathrm{V})$ for small $r > 0$ is known. For example, J. Hoffmann-Jørgensen [7] and L. A. Shepp [8] give some very precise estimates when E is a Hilbert space and V the unit ball of E.

To prove Theorem 2.2, we need two lemmas.

LEMMA 2.1. — [3], Cor. 2.1 (Cameron-Martin's formula). *For any* $\mu \in \mathcal{G}_0(\mathrm{E})$

$$\mu(. - h) = \left[\exp\left(\tilde{h} - \frac{1}{2}\| h \|_\mu^2\right)\right]\mu(.), \qquad h \in \mathrm{H}(\mu).$$

LEMMA 2.2. — [2], Cor. 2.1, Th. 6.1. *For any* $\mu \in \mathcal{G}_0(\mathrm{E})$

$$\mu_*(\lambda \mathrm{A} + (1 - \lambda)\mathrm{B}) \geq \mu^\lambda(\mathrm{A})\mu^{1-\lambda}(\mathrm{B}), \qquad 0 < \lambda < 1,$$

for all measurable subsets A *and* B *of* E.

In particular,

$$\mu(a + \mathrm{A}) \leq \mu(\mathrm{A}), \qquad a \in \mathrm{E},$$

whenever A *is symmetric, convex, and measurable subset of* E.

PROOF OF THEOREM 2.2. — Let us first assume that $a \in \mathrm{H}(\mu)$. By the Cameron-Martin formula, we have

$$\mu(a + r\mathrm{V}) = \exp\left(-\frac{1}{2}\| a \|_\mu^2\right)\int_{r\mathrm{V}} \exp(-\tilde{a})d\mu. \tag{2.2}$$

Moreover, the Jensen inequality yields

$$\int_{r\mathrm{V}} \exp(-\tilde{a})d\mu \geq \mu(r\mathrm{V})\exp\left(-(\mu(r\mathrm{V}))^{-1}\int_{r\mathrm{V}} \tilde{a}d\mu\right).$$

Since

$$\int_{r\mathrm{V}} \tilde{a}d\mu = 0,$$

it follows that

$$\varliminf_{r\to 0^+} \frac{\mu(a + r\mathrm{V})}{\mu(r\mathrm{V})} \geq \exp\left(-\frac{1}{2}\| a \|_\mu^2\right).$$

We now prove the estimate

$$\varlimsup_{r\to 0^+} \frac{\mu(a + r\mathrm{V})}{\mu(r\mathrm{V})} \leq \exp\left(-\frac{1}{2}\| a \|_\mu^2\right). \tag{2.3}$$

To this end let $\xi \in \mathrm{E}'$ be fixed and set $h = \Lambda\xi$. Then (2.2) gives

$$\mu(a + r\mathrm{V}) \leq \left[\exp\left(-\frac{1}{2}\| a \|_\mu^2 - \inf_{r\mathrm{V}} \xi\right)\right]\int_{r\mathrm{V}} \exp(\xi - \tilde{a})d\mu.$$

Moreover, the Cameron-Martin formula yields

$$\int_{r\mathrm{V}} \exp(\xi - \tilde{a})d\mu = \mu(a - h + r\mathrm{V})\exp\left(\frac{1}{2}\| h - a \|_\mu^2\right).$$

By applying Lemma 2.2, we have

$$\frac{\mu(a + rV)}{\mu(rV)} \leq \exp\left(- \frac{1}{2} \| a \|_\mu^2 - \inf_{rV} \xi + \frac{1}{2} \| h - a \|_\mu^2 \right),$$

and hence

$$\overline{\lim_{r \to 0^+}} \frac{\mu(a + rV)}{\mu(rV)} \leq \exp\left(- \frac{1}{2} \| a \|_\mu^2 + \frac{1}{2} \| h - a \|_\mu^2 \right).$$

By choosing $\xi \in E'$ close to $\tilde{a}$ in $E'_2(\mu)$, the estimate (2.3) follows at once. This proves (2.1) when $a \in H(\mu)$.

Let now $a \in \operatorname{supp}(\mu) \backslash H(\mu)$. Then, for every $n \in \mathbb{N}$, there exists a $\xi_n \in E'$ such that

$$\xi_n^2(a) > (n + 1) \int \xi_n^2 d\mu$$

and

$$\int \xi_n^2 d\mu = 1,$$

respectively. Set $a_n = \xi_n(a)\Lambda\xi_n$ and note that

$$a_n + rV \supseteq \frac{1}{2}(a + rV) + \frac{1}{2}(2a_n - a + rV).$$

By applying Lemma 2.2, we have

$$\mu^2(a_n + rV) \geq \mu(a + rV)\mu(2a_n - a + rV). \tag{2.4}$$

Furthermore, observing that μ is symmetric, the Cameron-Martin formula yields

$$\mu(2a_n - a + rV) = [\exp(-2 \| a_n \|_\mu^2)] \int_{a + rV} \exp(2\tilde{a}_n) d\mu.$$

Since $\| a_n \|_\mu^2 = \xi_n^2(a)$ and $\tilde{a}_n = \xi_n(a)\xi_n$, respectively, we get

$$\mu(2a_n - a + rV) \geq [\exp(2 \inf_{a + rV} \xi_n(a)(\xi_n - \xi_n(a)))]\mu(a + rV).$$

Using (2.4), it follows that

$$\overline{\lim_{r \to 0^+}} \frac{\mu(a_n + rV)}{\mu(rV)} \geq \overline{\lim_{r \to 0^+}} \frac{\mu(a + rV)}{\mu(rV)}.$$

Here, by the first part of the proof, the left-hand side equals $\exp(-\xi_n^2(a)/2)$. Clearly, this expression converges to zero as n tends to plus infinity. This proves (2.1) when $a \in \operatorname{supp}(\mu) \backslash H(\mu)$. Finally, the case $a \in E \backslash \operatorname{supp}(\mu)$ is trivial. This completes the proof of Theorem 2.2.

We also have

THEOREM 2.3. — *Let $\mu \in \mathcal{G}(\mathrm{E})$ and suppose* V *is a bounded measurable subset of* E *with positive μ-measure. Then*

$$(2.5) \qquad \lim_{t \to +\infty} (\mu(ta + \mathrm{V}))^{1/t^2} = \exp\left(- \frac{1}{2} \| a \|_\mu^2 \right), \qquad a \in \mathrm{E}.$$

Proof. — Without loss of generality it can be assumed that $\mu \in \mathcal{G}_0(\mathrm{E})$. Suppose first that $a \in \mathrm{H}(\mu)$. As in the proof of Theorem 2.2, we have

$$(2.6) \qquad \mu(ta + \mathrm{V}) = \left[\exp\left(- \frac{t^2}{2} \| a \|_\mu^2 \right) \right] \int_{\mathrm{V}} \exp(- t\tilde{a}) d\mu,$$

and

$$\int_{\mathrm{V}} \exp(- t\tilde{a}) d\mu \geq \mu(\mathrm{V}) \exp\left(- t(\mu(\mathrm{V}))^{-1} \int_{\mathrm{V}} \tilde{a} d\mu \right),$$

respectively. Hence

$$\varliminf_{t \to +\infty} (\mu(ta + \mathrm{V}))^{1/t^2} \geq \exp\left(- \frac{1}{2} \| a \|_\mu^2 \right).$$

We now prove the estimate

$$(2.7) \qquad \varlimsup_{t \to +\infty} (\mu(ta + \mathrm{V}))^{1/t^2} \leq \exp\left(- \frac{1}{2} \| a \|_\mu^2 \right).$$

To this end let $\xi \in \mathrm{E}'$ be arbitrary and set $h = \Lambda\xi$. Then, assuming $t > 0$, it follows that

$$\int_{\mathrm{V}} \exp(- t\tilde{a}) d\mu \leq [\exp(- t \inf_{\mathrm{V}} \xi)] \int_{\mathrm{V}} \exp(t(\xi - \tilde{a})) d\mu.$$

Using the trivial estimate

$$\int_{\mathrm{V}} \exp(t(\xi - \tilde{a})) d\mu \leq \exp\left(\frac{t^2}{2} \| h - a \|_\mu^2 \right),$$

the relation (2.6) yields

$$\varlimsup_{t \to +\infty} (\mu(ta + \mathrm{V}))^{1/t^2} \leq \exp\left(- \frac{1}{2} \| a \|_\mu^2 + \| h - a \|_\mu^2 \right).$$

By choosing ξ to close to $\tilde{a}$ in $\mathrm{E}'_2(\mu)$, we get (2.7). This proves (2.5) when $a \in \mathrm{H}(\mu)$.

Let now $a \in \mathrm{E} \backslash \mathrm{H}(\mu)$. Then, for every $n \in \mathbb{N}$, there exists a $\xi_n \in \mathrm{E}'$ so that

$$\xi_n^2(a) \geq (n + 1) \int \xi_n^2 d\mu,$$

and $\xi_n(a) = 1$, respectively. Since

$$ta + \mathrm{V} \subseteq \{ \xi_n \geq t + \inf_{\mathrm{V}} \xi_n \}, \qquad t > 0,$$

it follows that

$$\overline{\lim_{t \to +\infty}} \, (\mu(ta + \mathrm{V}))^{1/t^2} \leq \exp(-(n+1)/2).$$

By letting n tend to plus infinity, we get (2.5) for $a \in \mathrm{E} \backslash \mathrm{H}(\mu)$. This concludes the proof of Theorem 2.3.

3. APPLICATIONS

The results proved in Section 2 apply to any locally convex Hausdorff vector space. In order to be concrete, however, we here restrict ourselves to Banach spaces and dual Banach spaces equipped with the weak* topology respectively.

THEOREM 3.1. — *Let* E *be a Banach space and suppose* $\mu \in \mathcal{G}_0(\mathrm{E})$ *and* $\nu \in \mathcal{G}(\mathrm{E})$. *Moreover, assume there exists a function* $\delta : \mathrm{B}(0\,;1) \to \,]0, +\infty[$ *such that*

$$\mu(\mathrm{B}(a\,;r)) = \nu(\mathrm{B}(a\,;r)), \qquad 0 < r < \delta(a), \qquad \|a\| \leq 1.$$

Then $\mu = \nu$.

Proof. — Let c denote the barycentre of ν and note that

$$\nu_0(\mathrm{B}(-c, r)) = \mu(\mathrm{B}(0\,;r)) > 0, \qquad 0 < r < \delta(0),$$

by Lemma 2.2. Hence $-c \in \operatorname{supp} \nu_0 = \overline{\mathrm{H}(\nu)}$ [3], Cor. 8.2. By choosing $k \in \mathrm{B}(c\,;1) \cap \mathrm{H}(\nu)$, we get

$$\mu(\mathrm{B}(c-k\,;r)) = \nu_0(\mathrm{B}(-k\,;r)), \qquad 0 < r < \delta(c-k).$$

Since $\nu_0(\mathrm{B}(0\,;r)) \geq \nu_0(\mathrm{B}(-c\,;r)) > 0$, $r > 0$, the relation

$$1 \geq \frac{\mu(\mathrm{B}(c-k\,;r))}{\mu(\mathrm{B}(0\,;r))} = \frac{\nu_0(\mathrm{B}(-k\,;r))}{\nu_0(\mathrm{B}(0\,;r))} \cdot \frac{\nu_0(\mathrm{B}(0\,;r))}{\nu_0(\mathrm{B}(-c\,;r))}$$

must be true for all $0 < r < \min(\delta(c-k), \delta(0))$. By letting r tend to zero from the right and using Theorem 2.2, we get $-c \in \mathrm{H}(\nu)$. Moreover,

$$\frac{\mu(\mathrm{B}(a\,;r))}{\mu(\mathrm{B}(0\,;r))} = \frac{\nu_0(\mathrm{B}(a-c\,;r))}{\nu_0(\mathrm{B}(0\,;r))} \cdot \frac{\nu_0(\mathrm{B}(0\,;r))}{\nu_0(\mathrm{B}(-c\,;r))}$$

for every $0 < r < \min(\delta(a), \delta(0))$ and $\|a\| \leq 1$. Another application of Theorem 2.2 therefore yields that $\mathrm{H}(\mu) = \mathrm{H}(\nu)$ and

$$\|a\|_\mu^2 = \|a\|_\nu^2 - 2\langle a, c\rangle_\nu, \qquad a \in \mathrm{H}(\mu), \qquad \|a\| \leq 1.$$

Now choosing $a = tc$ and letting t tend to zero, we have $c = 0$. Moreover, $\|\cdot\|_\mu = \|\cdot\|_\nu$. Theorem 2.1 therefore implies that $\mu = \nu$. This proves Theorem 3.1.

THEOREM 3.2. — *Let* E *either be a Banach space or a dual Banach space equipped with the weak* topology. Moreover, let* $\mu \in \mathcal{G}_0(E) \setminus \{$ Dirac measure at $0 \}$ *and* $\nu \in \mathcal{G}(E)$ *be such that*

$$\mu(B(0\,;1)) > 0 \tag{3.1}$$

and

$$\mu(B(a\,;1)) = \nu(B(a\,;1)), \qquad \|a\| > K,$$

where $K > 0$ *is a fixed constant. Then* $\mu = \nu$.

The condition (3.1) is, of course, automatically fulfilled, if E is a Banach space. Note also that the closed unit ball $B(0\,;1)$ is weak* measurable when E is a dual Banach space.

Proof. — Theorems 2.3 and 2.1 tell us that $\mu = \nu_0$. Let c denote the barycentre of ν. It only remains to be proved that $c = 0$. Suppose to the contrary that $c \neq 0$. Let first $a \in E \setminus \{0\}$ be arbitrary and choose $p = p_a \in \mathbb{N}_+$ such that $p\|a\| \geq \|c\| + 1$. Then

$$\|npa + mc\| > K, \; m = 0, \ldots, n, \qquad n > K.$$

For every $n \in \mathbb{N}$, with $n > K$, we therefore get the following chain of equalities

$$\begin{aligned} \mu(B(npa\,;1)) &= \nu_0(B(npa\,;1)) = \nu(B(npa + c\,;1)) \\ &= \mu(B(npa + c\,;1)) = \ldots = \mu(B(n(pa + c)\,;1)). \end{aligned} \tag{3.2}$$

By assuming that $a \in H(\mu) \setminus \{0\}$ and applying Theorem 2.3, we deduce that $c \in H(\mu)$. In the next step, we set $a = c$ and $p = p_c$ in (3.2) and get, again using Theorem 2.3

$$\|p_c c\|_\mu = \|(p_c + 1)c\|_\mu .$$

Hence $c = 0$, which is a contradiction. This, finally, shows that $\mu = \nu$ and concludes the proof of Theorem 3.2.

REMARK 3.1. — Theorem 3.1 is true for a dual Banach space E, equipped with the weak* topology, if we assume that $\mu(B(0\,;r)) > 0$, $r > 0$. However, under these conditions both μ and ν extend to Gaussian Radon measures on the Banach space E [*1*], Th. VI, 2; 1. The result is thus already contained in Theorem 3.1.

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