ANNALES DE L'I. H. P., SECTION B

CRISTINA GZYL

HENRYK GZYL

Occupation time sets of supports of continuous additive functionals

Annales de l'I. H. P., section B, tome 15, n° 1 (1979), p. 41-50

<http://www.numdam.org/item?id=AIHPB_1979__15_1_41_0>

Ann. Inst. Henri Poincaré, Section B :
Vol. XV, n° 1, 1979, p. 41-50. *Calcul des Probabilités et Statistique.*

Occupation time sets of supports of continuous additive functionals

by

Cristina GZYL and **Henryk GZYL**
Escuela de Fìsica y Matematicas, Facultad de Ciencias,
Universidad Central de Venezuela

ABSTRACT. — It Φ is the support of a continuous additive functional (A_t) of a Markov process (X_t), we use results on the structure of the processes (τ_t) and (X_{τ_t}, τ_t) (where τ_t is the right continuous inverse of A_t) to describe the set $\mathscr{H} = \{ t : X_t \in \Phi \}$.

1. INTRODUCTION

Let $X = (\Omega, \mathscr{F}, \mathscr{F}_t, X_t, \theta_t, P^x)$ be a standard process with state space $(E, \mathscr{E})$. $(A_t)_{t \geq 0}$ a continuous additive functional of X and Φ its fine support (see [*1*] for definitions and notation).

Define $(\tau_t)_{t \geq 0}$ to be the right continuous inverse of (A_t), that is,

$$\tau_t = \inf \{ s : A_s > t \},$$

and consider the random sets

$$\begin{aligned} I &= \{ t : A_{t+\varepsilon} - A_t > 0 \quad \text{for all} \quad \varepsilon > 0 \} \\ J &= \{ t : A_{t+\varepsilon} - A_{t-\varepsilon} > 0 \quad \text{for all} \quad \varepsilon > 0 \} \\ \mathscr{H} &= \{ t : X_t \in \Phi \} \\ Q &= \{ t < \infty : \tau_u = t \quad \text{for some } u \}. \end{aligned}$$

It is well known (see [1], Ch. V) that a. s. $Q = IC\mathscr{H}CJ$. Moreover, $J - I = \{ \tau_{t^-} : \tau_{t^-} \neq \tau_t ; t > 0 \}$ and hence

$$\mathscr{H} = \{ \tau_t ; t \geq 0 \} \cup \{ \tau_{t^-} : \tau_{t^-} \neq \tau_t ; X_{\tau_{t^-}} \in \Phi \}$$

These remarks show that $\mathscr{H}$ is essentially the range of (τ_t) and so it is quite natural to try to describe the set $\mathscr{H}$ in terms of the process $(\tau_t)_{t \geq 0}$.

In the case $\Phi = \{ x_0 \}$, with x_0 regular for itself, the sections of $\mathscr{H}$ can be described as follows (see [7], [8] and [9]).

i) a. s. $P^{x_0}\mathscr{H}$ is bounded or unbounded. We will say that $\mathscr{H}$, or x_0, is a. s. P^{x_0} transient or recurrent ;

ii) a. s. $P^{x_0}\mathscr{H}$ has Lebesgue measure zero or a. s. $P^{x_0}\mathscr{H}$ has positive Lebesgue measure. In the first case one calls $\mathscr{H}$ light and in the second case $\mathscr{H}$ is called heavy ;

iii) We will call $\mathscr{H}$ stable if its complement intersects every finite interval (0, T) in a finite union of intervals, and unstable otherwise. Observe that $\mathscr{H}$ is stable if there are finitely many excursions from $\{ x_0 \}$ in every finite interval. One has that a. s. $P^{x_0}\mathscr{H}$ is stable or a. s. $P^{x_0}\mathscr{H}$ is unstable.

The process $(\tau_t)_{t \geq 0}$ is, in the present case, essentially a subordinator with respect to the law P^{x_0}, and, using the structure of (τ_t) one can give criteria for when is $\mathscr{H}$ going to be transient, recurrent, stable, etc. In fact, in [7] and [8] it is proved that if one considers the exponent

$$S(\theta) = \varepsilon\theta + \int_{(0,\infty)} (1 - e^{-\theta x})\mu(dx)$$

in the Lévy representation of the distribution of (τ_t) $\left(\text{i. e. } e^{-tS(\theta)} = E^{x_0}[e^{-\theta\tau_t}]\right)$ then

i) $\mathscr{H}$ is recurrent $\Leftrightarrow \mu\{ \infty \} = 0$,
ii) $\mathscr{H}$ is heavy $\Leftrightarrow \varepsilon > 0$,
iii) $\mathscr{H}$ is stable $\Leftrightarrow \mu$ is a finite measure.

In the next sections, we intend to use some of the results proved by Cinlar in [2], [3] and [4], and Rolin in [10], about the structure of the processes (τ_t) and (X_{τ_t}, τ_t) in the case Φ is a more general set (Φ the support of (A_t)) to study the set $\mathscr{H}$. Our results will extend those in [7] and [8] for the case $\Phi = \{ x_0 \}$. To be more specific: in section 3 we study the set $\mathscr{H}$ with respect to the measures P^x, $x \in \Phi$, by using the results of Cinlar on Lévy systems for (X_{τ_t}, τ_t), and, in section 4, we describe $\mathscr{H}$ conditional on the paths of the time changed process X_{τ_t}. We begin by stating some preliminary results that will be needed in these sections.

2. PRELIMINARIES

Consider, as in the introduction, a standard process X, a continuous additive functional (A_t) of X, with fine support Φ, and let (τ_t) be the right continuous inverse of (A_t). Let $\Phi_\Delta = \Phi \cup \{ \Delta \}$.

Denote by $\underline{\underline{\Phi}}$ the Borel subsets of Φ, $b\underline{\underline{\Phi}}$ the bounded Borel measurable functions on Φ. $\underline{\underline{R}}_+$ the Borel subsets of R_+, and $b\underline{\underline{R}}_+$ the bounded Borel functions on R^+. $\underline{\underline{\overline{R}}}_+$, $\underline{\underline{\Phi}}_\Delta$, etc. have similar meanings.

The joint process (X_{τ_t}, τ_t) is a Markov additive process (see [2]). We assume Φ to be projective, in which case X_{τ_t} will be a Hunt process (see [1], Ch. V) and so it follows from the results in [3] (*) that there is a Lévy system (H, L) for (X_{τ_t}, τ_t) with H a continuous additive functional of (X_{τ_t}), L a kernel from $\underline{\underline{\Phi}}_\Delta \times \underline{\underline{\overline{R}}}_+$ into $\underline{\underline{\Phi}}$ such that

$$(2.1)\quad E^x \sum_{s \leq t} f(X_{\tau_{s-}}, X_{\tau_s}, \tau_s - \tau_s^-) . 1_{\{X_{\tau_s} \neq X_{\tau_{s-}}\} \cup \{ \tau_s \neq \tau_{s-} \}}$$
$$= E^x \int_0^t dH_s \int_{\Phi_\Delta \times \overline{R}_+} L(X_{\tau_s}, dy, du) f(X_{\tau_s}, y, u)$$

for each f in $b\underline{\underline{\Phi}} \times \underline{\underline{\Phi}}_\Delta \times \underline{\underline{\overline{R}}}_+$.

The process (τ_t) can be decomposed as $\tau_t = \tau_t^c + \tau_t^d$ where τ_t^c (the continuous part of τ_t) is a continuous additive functional of (X_{τ_t}) and τ_t^d is a pure jump increasing additive process (see [2] or [4]).

Let us put $C_t = H_t + \tau_t^c + t$; (C_t) is a strictly increasing continuous additive functional of X_{τ_t}.

It is proved in [4] that if we let $\sigma_t = \inf \{ s : C_s > t \}$, then the process $(\widehat{X}_t, \widehat{\tau}_t) = (X_{\tau_{\sigma_t}}, \tau_{\sigma_t})$ is again a Markov additive process and its Lévy system is such that the corresponding additive functional $\widehat{H}_t$ is equal to $t \wedge \zeta$.

Now, we observe that if one defines $B_t = C_{A_t}$ then we obtain.

(2.2) LEMMA. — *i)* (B_t) is a continuous additive functional of X.

ii) The right continuous inverse of B_t coincides with τ_{σ_t}.

iii) (A_t) and (B_t) have the same support Φ.

(*) See note at the end of the paper.

Proof. — *i*) To prove that (B_t) is adapted see [6] section 2, Lemma 14. The additivity of (B_t) follows from the fact that (C_t) is a continuous additive functional of (X_{τ_t}) and A_t is a stopping time relative to $(\mathscr{F}_{\tau_t})$.

ii) $\inf \{ u : C_{A_u} > s \} = \inf \{ u : A_u > \sigma_s \} = \tau_{\sigma_s}$.

iii) This last assertion follows from the fact that $\sigma_0 \equiv 0$, and so

$$\Phi = \{ x : P^x(\tau_0 = 0) = 1 \} = \{ x : P^x(\tau_{\sigma_0} = 0) = 1 \} = \text{support } (B_t).$$

In view of Lemma (2.2) we will assume that the Lévy system for (X_{τ_t}, τ_t) is such that $H_t = t \wedge \zeta$, so that (2.1) can be rewritten as follows:

$$(2.3) \qquad E^x \sum_{s \leq t} f(X_{\tau_{s-}}, X_{\tau_s}, \tau_s - \tau_{s-}) 1_{\{X_{\tau_{s-}} \neq X_{\tau_s}\} \cup \{\tau_{s-} \neq \tau_s\}}$$
$$= E^x \int_0^t d(s \wedge \zeta) \int_{\Phi_\Delta \times \overline{R}_+} L(X_{\tau_s}, dy, du) f(X_{\tau_s}, y, u).$$

By means of an approximation argument one can get a more general relation than (2.3), namely, one can show that if Z_s is adapted to $(\mathscr{F}_{\tau_t})$ positive and left continuous, then one has for f in $b\underline{\underline{\Phi}} \times \underline{\underline{\Phi}}_\Delta \times \underline{\underline{\overline{R}}}_+$

$$(2.4) \quad E^x \sum_{0 \leq s \leq t} Z_s f(X_{\tau_{s-}}, X_{\tau_s}, \tau_s - \tau_{s-}) 1_{\{X_{\tau_{s-}} \neq X_{\tau_s}\} \cup \{\tau_{s-} \neq \tau_s\}}$$
$$= E^x \int_0^t Z_s d(s \wedge \zeta) \int_{\Phi_\Delta \times \overline{R}_+} L(X_{\tau_s}, dy, du) f(X_{\tau_s}, y, u)$$

Finally, observe that since $t = C_{\sigma_t} = H_{\sigma_t} + \tau^c_{\sigma_t} + \sigma_t$, $\tau^c_{\sigma_t}$ is absolutely continuous with respect to t, so we may also assume that

$$(2.5) \qquad \tau^c_t = \int_0^t a(X_{\tau_s}) ds \text{ where a is positive and } \underline{\underline{\Phi}} \text{ measurable.}$$

3. THE SET $\mathscr{H}$

We will now study the set $\mathscr{H}$ with respect to the laws P^x for $x \in \Phi$. The notations and definitions will be the ones introduced in the preceding sections.

Weight

It follows from the fact that $\{ s : X_s \in \Phi \}$ differs from $\{ s : \Delta\tau_{A_s} = 0 \}$ by countably many points, that the « occupation time » of Φ is related to τ^c_t as follows:

$$(3.1) \qquad \tau^c_t = \int_0^{\tau_t} 1_\Phi(X_s) ds \text{ a. s. } P^x,\ x \in \Phi \text{ (see [10], chap. IV).}$$

From (3.1), one gets that $\mathscr{H}$ is heavy a. s. $P^x \Leftrightarrow \tau^c_\infty$ is positive a. s. P^x.

It is clear that if a. s. P^x, the process spends a positive time in a given subset of Φ, then $\mathscr{H}$ will be heavy a. s. P^x.

By writing τ^c_t in terms of the time changed process X_{τ_s} as in (2.5) namely

$$\tau^c_t = \int_0^t a(X_{\tau_s})ds$$

one gets that $\mathscr{H}$ is light a. s. P^x $\forall x \in \Phi$ if $a \equiv 0$. On the other hand, if we let $D = \{ a > 0 \}$ then it is easy to see that

$$\int_0^\infty a(X_{\tau_s})ds = \int_0^\infty 1_D(X_s)ds$$

(a is defined to be zero outside of Φ), and hence $\mathscr{H}$ will be heavy a. s. P^x for $x \in D$ if D is finely open.

Observe that in the case $\Phi = \{ x_0 \}$ (x_0 regular), for all t, one has $X_{\tau_t} = x_0$, $\tau^c_t = \varepsilon t$, $a(X_{\tau_t}) = \varepsilon$; so it is clear from (3.1) and (2.5) that a. s. $P^{x_0}\mathscr{H}$ is heavy or light, and, $\mathscr{H}$ is heavy $\Leftrightarrow \varepsilon > 0$, which coincides with the results given in [7] and [8].

If Φ is a finite set, $\Phi = \{ x_1, \ldots, x_n \}$, with all the x_i being regular, then, if we let $a(x_i) = \varepsilon_i$, we see that x_i is heavy a. s. $P^{x_i} \Leftrightarrow \varepsilon_i > 0$.

Recurrence

We observe that a. s. P^x $\tau_{A_t^-} = \sup \{ s \leq t : X_s \in \Phi \}$ from which it follows that the last exit from Φ coincides with $\tau_{A_\infty^-}$, i. e. $\tau_{A_\infty^-} = \sup \{ s \geq 0 : X_s \in \Phi \}$.

Thus, if we say that Φ is transient for x if $\mathscr{H}$ is bounded a. s. P^x, and recurrent for x if $\mathscr{H}$ is unbounded a. s. P^x, we get that Φ is transient for x if $\tau_{A_\infty^-} < \infty$ a. s. P^x.

In terms of the Lévy system for (X_{τ_t}, τ_t) one has the following results.

(3.2) PROPOSITION. — For all $x \in \Phi$ the following equality holds

$$E^x[e^{-\tau_{A_\infty^-}}] = E^x \int_0^\infty e^{-s} L(X_s, \Phi_\Delta, \{ \infty \})dA_s$$

Proof.

$$E^x[e^{-\tau_{A_\infty^-}}] = E^x \sum_{s>0} e^{-\tau_{s^-}} 1_{\{\infty\}}(\Delta\tau_s)$$

by (2.4) this last term equals

$$= E^x \int_0^\infty ds \int_{\Phi_\Delta \times \bar{R}^+} L(X_{\tau_s}, dy, du) e^{-\tau_s} 1_{\{\infty\}}(u)$$

$$= E^x \int_0^\infty L(X_{\tau_s}, \Phi_\Delta, \{\infty\}) e^{-\tau_s} ds$$

$$= E^x \int_0^\infty e^{-s} L(X_s, \Phi_\Delta, \{\infty\}) dA_s$$

The last equality follows from a well known time change formula (see [1], Ch. V).

It follows from proposition (3.2) that $\tau_{A_\infty^-} = \infty$ a. s. $P^x \Leftrightarrow L(X_{\tau_s}, \Phi_\Delta, \{\infty\})$ is P^x indistinguishable from 0. Or, equivalently

$$\tau_{A_\infty^-} = \infty \text{ a. s. } P^x \Leftrightarrow L((.), \Phi_\Delta, \{\infty\}) = 0 \text{ A a. s. } P^x$$

Observe that in the case $\Phi = \{x_0\}$ these conditions reduce to the condition for recurrence given in [7] and [8] namely that $\mu\{\infty\} = 0$.

Let us denote by $\bar{X}_t$ the left limit X_{t^-}, then, when Φ is transient for x, one has the following expression for the joint distribution of

$$\tau_{A_\infty^-}, \bar{X}_{\tau_{A_\infty}^-}.$$

(3.3) PROPOSITION. — Let Φ be transient for x, then, if $g \in b\underline{\underline{\Phi}}$ and $b > 0$, one has

$$E^x[g(\bar{X}_{\tau_{A_\infty^-}}), b < \tau_{A_\infty^-}] = E^x \int_b^\infty g(X_s) L(X_s, \Phi_\Delta, \{\infty\}) dA_s \tag{3.4}$$

Proof.

$$E^x[g(\bar{X}_{\tau_{A_\infty^-}}) ; b < \tau_{A_\infty^-}] = E^x \sum_{s>0} g(\bar{X}_{\tau_s^-}) 1_{(b,\infty)}(\tau_{s^-}) 1_{\{\infty\}}(\Delta\tau_s)$$

$$= E^x \int_0^\infty g(X_{\tau_s}) 1_{(b,\infty)}(\tau_s) ds \int_{\Phi_\Delta \times \bar{R}^+} L(X_{\tau_s}, dy, du) 1_{\{\infty\}}(u)$$

$$= E^x \int_0^\infty g(X_{\tau_s}) 1_{(b,\infty)}(\tau_s) L(X_{\tau_s}, \Phi_\Delta, \{\infty\}) ds = E^x \int_b^\infty g(X_s) L(X_s, \Phi_\Delta, \{\infty\}) dA_s$$

(3.5) REMARK. — One may check that proposition (3.3) also holds if $x \in E - \Phi$.

(3.6) REMARK. — By taking $b = 0$ and $g = 1$ in (3.4) one gets that

$$P^x(\tau_{A_\infty^-} > 0) = u_C(x)$$

where

$C_t = \int_0^t L(X_s, \Phi_\Delta, \{\infty\})dA_s$ is a natural potential. Hence, if $\tau_{A_\infty^-} <$ a. s.,

Φ is transient in the usual sense (see [5]).

We observe that the fact that the condition for transience is simpler in this case is due to the fact that Φ is the support of a continuous additive functional.

Moreover, with the notation we just introduced, one can rewrite (3.4) as follows.

$$E^x[g(\bar{X}_{\tau_{A_\infty^-}}) ; \tau_{A_\infty^-} > b] = E^x \int_b^\infty g(X_s)L(X_s, \Phi_\Delta, \{\infty\})dA_s = E^x \int_b^\infty g(X_s)dC_s$$
$$= E^x \left\{ E^x \left[\left(\int_0^\infty g(X_s)dC_s \right) \circ \theta_b \mid \mathscr{F}_b \right] \right\} = \int P_b(x, dy)U_C g(y)$$

where $U_C g(y) = E^y \int g(X_t)dC_t$, which provides another proof of proposition (3.3) in Getoor-Sharpe's [5], plus an explicit representation of the additive functional (C_t) in terms of probabilistic objects.

Stability

It follows from the fact that (A_t) increases when $X_t \in \Phi$ and the definition of (τ_t), that, in order to study the excursions from the set Φ in $[0, t]$, one can examine the jumps of τ_s up to A_t, that is

$$(3.7) \qquad \sum_{s>0} 1_{(0,A_t]}(s)1_{(0,\infty)}(\Delta\tau_s) = \sum_{s>0} 1_{(0,t]}(\tau_s)1_{(0,\infty)}(\Delta\tau_s).$$

Taking expectations in (3.7) and using (2.4) we obtain

$$E^x \sum_{s>0} 1_{(0,t]}(\tau_{s^-} + \Delta\tau_s)1_{(0,\infty)}(\Delta\tau_s) = E^x \int_0^\infty ds \int_0^\infty L(X_{\tau_s}, \Phi_\Delta, du)1_{(0,t]}(\tau_s + u)$$
$$= E^x \int_0^\infty ds 1_{(0,t]}(\tau_s)L(X_{\tau_s}, \Phi_\Delta, (0, t - \tau_s]) = E^x \int_0^{A_t} L(X_{\tau_s}, \Phi_\Delta, (0, t - \tau_s])ds$$
$$= \int_\Phi \int_0^t U(x, dy, du)L(y, \Phi_\Delta, (0, t - u])$$

where $U(x, f, g) = E^x \int_0^\infty f(X_{\tau_t})g(\tau_t)dt$.

This last calculation shows that the excursions from Φ can also be studied in terms of the Lévy system for (X_{τ_t}, τ_t). There are some obvious remarks that we can make, namely (3.7) will be finite for all t if

$L(x, \Phi_\Delta, (0, \infty))$ is bounded for all $x \in \Phi$, and infinite for $t = \infty$ if $L(x, \Phi_\Delta, (0, \infty)) = \infty$ for all $x \in \Phi$. However, since (X_{τ_t}, τ_t) may behave differently for different points in Φ, and $L(x, \Phi, (0, \infty))$ varies with $x \in \Phi$, we will give a definition of stability that takes into account this local behaviour.

For $F, G \in \underline{\underline{\Phi}}$, consider

$$R_t = \sum_s 1_G(X_{\tau_s}-)1_F(X_{\tau_s})1_{(0,t]}(\tau_s)1_{(0,\infty)}(\Delta\tau_s)$$

then

$$(3.8)\quad E^x(R_t) = E^x \int_0^\infty 1_{(0,t]}(\tau_s)1_G(X_{\tau_s})L(X_{\tau_s}, F, (0, t - \tau_s]ds)$$
$$= \int_G \int_0^t U(x, dy, du)L(X_{\tau_s}, F, (0, t - u])$$

DEFINITION. — We will say that $\mathscr{H}$ is stable for (x, F, G) if the right hand side of (3.8) is finite. Otherwise we will say that it is unstable.

Remark. — It is clear that when $\Phi = \{ x_0 \}$ we get the criteria in [7].

4. DESCRIPTION OF $\mathscr{H}$ IN TERMS OF CONDITIONAL PROBABILITIES

We will now briefly discuss the weight, recurrence and stability of $\mathscr{H}$ given the paths of the time changed process (X_{τ_t}).

It is proved in [2] and [10] that if we let $\mathscr{K}$ denote the σ-algebra generated by $(X_{\tau_t})_{t \geq 0}$ completed with respect to the family of measures P^μ (μ a finite measure on Φ), and $\mathscr{L}$ the same σ-algebra but with respect to the process $(X_{\tau_t}, \tau_t)_{t \geq 0}$, then there is a regular version of the conditional probability $P^x[. \mid \mathscr{K}]$ on $\mathscr{L}$, which is independent of $x \in \Phi$. Denote this version by $P^\omega(.)$ when evaluated at $\omega \in \Omega$, and let E^ω denote expectation with respect to P^ω.

The process (τ_t) is a process with independent increments on $(\Omega, \mathscr{L}, P^\omega)$ and, one has the following representation

$$(4.1)\qquad E^\omega[e^{-\alpha\tau_t}] = \exp\left[-\alpha\tau_t^c(\omega) - \int_0^\infty (1 - e^{-\alpha u})\nu_t^\omega(du)\right]$$

where

$$(4.2)\qquad \nu_t^\omega(A) = E^\omega \sum_{s \leq t} 1_A(\Delta\tau_s)$$

for A, a Borel set in R^+ (see [9] and [2] for proof).

Just as in the case $\Phi = \{ x_0 \}$, ν_t enables us to study the recurrence and stability as follows:

Let

$$\widehat{\zeta} = \inf \{ t : X_{\tau_t} = \Delta \}, \text{ then, } A_\infty = \widehat{\zeta}, \text{ from which it follows that}$$

$$P^\omega(\tau_{\widehat{\zeta}-} < \infty) = P^\omega(\Delta\tau_{\widehat{\zeta}} = \infty) = E^\omega[1_{\{\infty\}}(\Delta\tau_{\widehat{\zeta}})] = E^\omega \sum_{0<s\leq\widehat{\zeta}} 1_{\{\infty\}}(\Delta\tau_s) = \nu^\omega_{\widehat{\zeta}}\{\infty\}$$

Hence, $\mathscr{H}$ is transient or recurrent with respect to P^ω according as to $\nu^\omega_{\widehat{\zeta}}\{ \infty \}$ is zero or one.

On the other hand, it follows from (4.2) that

$$\nu^\omega_t(0, \infty)1_{\{t<\widehat{\zeta}\}} = E^\omega\left[\sum_{0<s\leq t} 1_{(0,\infty)}(\Delta\tau_s) \,;\, t < \widehat{\zeta}\right]$$

hence, $\mathscr{H}$ is stable or unstable for P^ω according as to $\nu^\omega_t(0, \infty)$ is finite or infinite for all t.

With regards to the weight of $\mathscr{H}$ one has that a. s. P^ω (for each ω) $\mathscr{H}$ is heavy or light, in fact:

$$P^\omega(\tau^c_\infty > 0) = E^x[1_{\{\tau^c_\infty>0\}} \mid \mathscr{H}] = 1_{\{\tau^c_\infty>0\}}$$

where the last equality follows from the fact that τ^c_t is a continuous additive functional of (X_{τ_t}).

Finally, we observe that in the case $\Phi = \{ x_0 \}$, $P^\omega = P^{x_0}$ for almost all $\omega \in \Omega$ (see [*10*]) and we obtain the criteria in [*7*] and [*8*].

Note. — We wish to thank Prof. B. Maisonneuve for the following remark: In order to apply Cinlar's results on the existence of a Lévy system one has to prove that (τ_t) is quasileft continuous with respect to the family $(\mathscr{F}_{\tau_t})$. Let $D_t = \inf \{ s > t : X_s \in \Phi \}$ and let T_n be an increasing sequence of stopping times of $(\mathscr{F}_{\tau_t})$ with limit T.

Then, $\tau_{T_n^-}$ and τ_{T^-} are stopping times of $(\mathscr{F}_{D_t})$ and $\tau_{T_n^-} \uparrow \tau_{T^-}$. Note now that $\tau_t = D_{\tau_t^-}$ and use the quasileft continuity of the process (D_t) with respect to $(\mathscr{F}_{D_t})$, which is proved in B. Maisonneuve's, *Systèmes régénératifs*, Astérisque, 1974, vol. 15, p. 27.

REFERENCES

[*1*] R. M. BLUMENTHAL and R. K. GETOOR, *Markov Processes and Potential Theory*. Academic Press, New York, 1968.

[2] E. CINLAR, Markov Additive Processes II. *Z. Wahrsch.*, t. **24**, 1972, p. 95-121.
[3] E. CINLAR, Levy Systems of Markov Additive Processes. *Z. Wahrsch.*, t. **31**, 1975, p. 175-185.
[4] E. CINLAR, Entrance exit distributions for Markov Additive Processes. *Stochastic Systems: Modeling, Identification and Optimization I.* Ed. R. Wets, North-Holland Co., 1976.
[5] R. K. GETOOR and M. J. SHARPE, Last Exit Times and Additive Functionals. *The Annals of Probability*, vol. 1, n° 4, 1973, p. 550-569.
[6] N. KAROUI and P. A. MEYER, Les changements de temps en théorie générale des processus. In *Springer Lecture Notes in Mathematics*, t. **581**, 1977, p. 65-78.
[7] J. F. C. KINGMAN, Home comings of Markov Processes. *Adv. Appl. Prob.*, t. **5**, 1973, p. 66-102.
[8] B. MAISONNEUVE, Ensembles régénératifs, temps locaux et subordinateurs. In *Springer Lecture Notes in Mathematics*, t. **191**, 1971, p. 147-169.
[9] P. A. MEYER, Ensembles régénératifs d'après Hoffman Jorgensen. In *Springer Lecture Notes in Mathematics*, t. **124**, 1970, p. 133-150.
[10] J. M. ROLIN, *The Inverse of a Continuous Additive Functional*, Ph. D. Thesis, Univ. of California, San Diego, 1973.

(Manuscrit reçu le 21 septembre 1978).