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Ergodic invariant measures for actions of SL(2, Z)

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ABSTRACT. — We construct uncountably many infinite σ-finite continuous ergodic invariant measures for various actions of the unimodular group SL(2, $\mathbb{Z}$).

RÉSUMÉ. — Nous construisons un nombre non dénombrable de mesures infinies, σ-finies, continues, ergodiques et invariantes pour certaines actions du groupe SL(2, $\mathbb{Z}$).

This note arose from the following question of K. Schmidt (Oberwolfach June 1978). Consider the action of SL(2, $\mathbb{Z}$) on $\mathbb{T}^2$ as automorphisms arising from the linear action on $\mathbb{R}^2$. Does there exist an infinite σ-finite continuous (non-atomic) ergodic invariant measure on $\mathbb{T}^2$ with respect to this action?

We show here, using the simple strategy elicited in the lemma below, that various actions of SL(2, $\mathbb{Z}$) including the one above admit uncountably many infinite σ-finite continuous ergodic invariant measures. A simple proof of the uniqueness of the finite invariant measure in the above case is included, and related problems are discussed.

LEMMA. — Let G be a countable group. Let (X, $\mathscr{B}$) be a Borel G-space. Suppose that there exists a subgroup H, a Borel subset Y of X, and a continuous σ-finite Borel measure λ on Y satisfying the following conditions:

i) H is of infinite (countable) index in G;

ii) Y is H-invariant; λ is an ergodic invariant measure with respect to the action of H on Y obtained by restriction;

iii) for $g \notin H$, $\lambda(gY \cap Y) = 0$.

Then there exists an infinite σ-finite continuous ergodic G-invariant measure μ on (X, $\mathscr{B}$) such that for any Borel subset A of Y, $\mu(A) = \lambda(A)$.

Proof. — Let $\{ g_i \}_i^\infty$ be a set of representatives for the coset space G/H. For any Borel subset E of X define

$$\mu(E) = \sum_{i=1}^{\infty} \lambda(g_i^{-1}E \cap Y).$$

Let A be a Borel subset of Y. Then by *iii*), $\lambda(g_i^{-1}A \cap Y) = 0$ unless $g_i \in H$. Hence $\mu(A) = \lambda(A)$. Clearly μ is a continuous infinite measure on X. Let $\{ A_j \}_1^\infty$ be a sequence of Borel subsets of Y such that $\lambda(A_j)$ is finite for all j and

$$\lambda\left(Y - \bigcup_{j=1}^{\infty} A_j\right) = 0.$$

For $n \in \mathbb{N}$ put

$$E_n = \bigcup_{i=1}^{\infty} \bigcup_{j=1}^{\infty} g_i A_j.$$

Then

$$\mu(E_n) \leqslant \sum_{i,j=1}^{n} \mu(g_i A_j)$$

which is finite since

$$\mu(g_i A_j) = \lambda(A_j).$$

Also

$$\mu\left(X - \bigcup_{n=1}^{\infty} E_n\right) = 0.$$

Hence μ is σ-finite. Next let $E \subset g_j Y$ and $g \in G$. Let $h \in H$ and $i \in \mathbb{N}$ be such that $gg_j = g_i h$. Then

$$\mu(gE) = \mu(gg_j g_j^{-1}E) = \mu(g_i h g_j^{-1}E) = \lambda(hg_j^{-1}E) = \lambda(g_j^{-1}E) = \mu(E).$$

Since

$$\mu\left(X - \bigcup_{j=1}^{\infty} g_j Y\right) = 0$$

the last assertion means that μ is G-invariant. Lastly we show that μ is ergodic with respect to the G-action. Let E be a G-invariant Borel subset of X such that $\mu(E) = 0$. Then there exists $k \in \mathbb{N}$ such that $\mu(E \cap g_k Y) > 0$. Since μ is G-invariant,

$$\mu(E \cap Y) = \lambda(E \cap Y) > 0.$$

Clearly $E \cap Y$ is H-invariant. Since λ is ergodic with respect to the H-action, $\lambda(Y - E) = 0$. Equivalently $\mu(Y - E) = 0$. Hence

$$\mu(X - E) = \mu(\cup g_k Y - E) \leqslant \Sigma\mu(g_k Y - E) = \Sigma\mu(Y - g_k^{-1}E) = 0.$$

Examples.

1. Let $X = \mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$. Consider the linear action of the unimodular group SL(2, $\mathbb{Z}$) on $\mathbb{R}^2$ via the standard basis $\{ e_1, e_2 \}$. The action leaves $\mathbb{Z}^2$ invariant. Hence we obtain an action of the group G = SL(2, $\mathbb{Z}$) on $\mathbb{T}^2$. Let

$$Y = \{ (te_1 + y_0 e_2) + \mathbb{Z}^2/\mathbb{Z}^2 : t \in \mathbb{R} \}$$

where y_0 is a fixed irrational number. Also let

$$H = \left\{ \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} : n \in \mathbb{Z} \right\}$$

It is easy to verify that Y is H-invariant. Observe that Y is a circle on $\mathbb{T}^2$ in the direction of e_1 and that the action of $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ on Y coincides with the irrational rotation through an angle y_0. Hence the Lebesgue measure λ on Y is an ergodic H-invariant measure. Clearly H is a subgroup of infinite index in G. Finally, let $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \notin H$. Then $c \neq 0$. Clearly gY is a circle on X whose tangent direction is always parallel to $ae_1 + ce_2$. Since $c \neq 0$ the circle gY intersects Y only in finitely many points. In particular

$$\lambda(gY \cap Y) = 0$$

whenever $g \in H$. Thus all the requirements of the lemma being satisfied it follows that there exists an infinite σ-finite continuous measure μ which is invariant and ergodic under the action of SL(2, $\mathbb{Z}$) and extends the measure λ. By varying y_0, which was an arbitrary irrational number, we obtain an uncountable family of mutually singular measures satisfying the above properties. Each of these measures is positive (and infinite) on non-empty open subsets of $\mathbb{T}^2$.

2. Let $X = SL(2, \mathbb{R})/\Gamma$ where Γ is a discrete co-compact subgroup of $SL(2, \mathbb{R})$. Let G denote the group $SL(2, \mathbb{Z})$, considered as a subgroup of $SL(2, \mathbb{R})$ and consider the action of G on X (on the left). Let A be a hyperbolic element of $SL(2, \mathbb{Z})$. Then there exists a unique matrix α of trace 0 such that $A = \exp \alpha$. For $t \in \mathbb{R}$ we denote by A^t the element $\exp t\alpha$ in $SL(2, \mathbb{R})$. Let A and B be two hyperbolic elements of $SL(2, \mathbb{Z})$ such that $B = g_0 A^\rho g_0^{-1}$ where ρ is an irrational number. We note that it is possible to find such pairs. For instance let $A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 3 & 2 \\ 1 & 1 \end{pmatrix}$. Let σ and τ denote the unique eigenvalues of A and B respectively such that $\sigma > 1$ and $\tau > 1$. Then there exist $g_1, g_2 \in SL(2, \mathbb{R})$ such that

$$A = g_1 \begin{pmatrix} \sigma & 0 \\ 0 & \sigma^{-1} \end{pmatrix} g_1^{-1} \quad , \quad B = g_2 \begin{pmatrix} \tau & 0 \\ 0 & \tau^{-1} \end{pmatrix} g_2^{-1}.$$

Then $B = gA^\rho g^{-1}$ where $gg_1 = g_2$ and $\rho = \log_\sigma \tau$. It is easy to verify that ρ is irrational. We now show that for A, B as above there exists an analytic circle Y in X such that Y is invariant under the action of either A or B and that the action of the particular element which leaves it invariant is an irrational rotation. Consider the flow on X given by the action of the one-parameter subgroup A^t, $t \in \mathbb{R}$. This may be considered as the geodesic flow on the manifold $K\backslash G/\Gamma$ where K is the subgroup of rotation matrices (Without loss of generality we may assume that Γ does not contain any non-trivial elements of finite order and hence acts properly discontinuously on $K\backslash G$). It is well known that this flow admits closed trajectories (cf. for instance [*1*]). Thus there exists $x \in SL(2, \mathbb{R})$ such that $A^{t_0}x\Gamma = x\Gamma$ for some positive t_0. If t_0 is irrational we put

$$Y = \{ A^t x\Gamma/\Gamma : t \in \mathbb{R} \}.$$

Then Y is a circle invariant under A and the restriction of the action of A is an irrational rotation. If t_0 is rational we put

$$Y = \{ g_0 A^t x\Gamma/\Gamma : t \in \mathbb{R} \}.$$

Then Y is an analytic circle in X and is clearly invariant under the action of the one-parameter group $\{ B^t : t \in \mathbb{R} \}$ where $B = g_0 A^\rho g_0^{-1}$. Observe that $B^{s_0} g_0 x\Gamma = g_0 x\Gamma$ where $s_0 = t_0/\rho$. Since s_0 is irrational this means that $B^r g_0 x\Gamma \neq g_0 x\Gamma$ for any rational number r. For otherwise $g_0 x\Gamma$ would be a fixed point of the flow corresponding to $\{ B^t : t \in \mathbb{R} \}$—which is impossible since Γ is a discrete subgroup of $SL(2, \mathbb{R})$. Thus the restriction of the action of B to Y is an irrational rotation.

Now let Y be as above and $H = \{ g \in SL(2, \mathbb{Z}) : gY = Y \}$. Then clearly H is an infinite cyclic subgroup and has infinite index. The generator of H acts as an irrational rotation of Y. Thus on Y there exist uncountably many mutually singular continuous ergodic measures invariant under the action of H (cf. [3]). Also since for any $g \in SL(2, \mathbb{Z})$, gY is an analytic circle, whenever $g \notin H$, $gY \cap Y$ is finite. Thus if λ is any measure on Y as above then $\lambda(gY \cap Y) = 0$. Hence by the lemma it now follows that the action of SL(2, $\mathbb{Z}$) on $X = SL(2, \mathbb{R})/\Gamma$ admits uncountably many mutually singular, continuous, infinite, σ-finite, ergodic, invariant measures.

3. Let $\mathbb{Q}_p$ be the p-adic field where p is a prime number. The $\mathbb{Q}_p$ vector space of all two by two matrices with entries in $\mathbb{Q}_p$ has a natural product (metric) topology. The closure of SL(2, $\mathbb{Z}$) with respect to this topology is a compact metric group which we denote by X. Since SL(2, $\mathbb{Z}$) is a subgroup of X, it acts on X as a group of Borel transformations, say, on the left. Now let H be any infinite cyclic subgroup of SL(2, $\mathbb{Z}$). Let Y be the closure of H in X. Then Y is a compact group and hence the Haar measure on Y is a finite ergodic H-invariant measure on Y. By [3] there exist uncountably many mutually singular σ-finite continuous measures on Y which are invariant and ergodic under the H-action. Observe that for $g \in SL(2, \mathbb{Z}) - H$, gY and Y are disjoint. Hence by the lemma we deduce that the action of SL(2, $\mathbb{Z}$) on X admits uncountably many infinite σ-finite, continuous, ergodic, invariant measures.

Remark. — The lemma can also be used for other groups. For instance, for the action of SL(n, $\mathbb{Z}$) on $\mathbb{T}^n$ arising from the linear action it is possible to prove existence of measures as in the examples above.

It is interesting to note that though in all the above examples there is an abundance of infinite σ-finite continuous ergodic invariant measures, for each of them there is a unique continuous invariant probability measure. In example 3 this fact is obvious. In example 2 it follows from the unique ergodicity of the horocycle flow (cf. [2]). The following proposition shows that it is also true for example 1.

PROPOSITION. — Consider the action of SL(2, $\mathbb{Z}$) on $\mathbb{T}^2$ arising from the linear action. If σ is a continuous probability measure invariant under this action, then σ is Haar measure on $\mathbb{T}^2$.

Proof. — The action of H (see example 1) is an irrational rotation on each horizontal circle with irrational vertical coordinate. Hence the conditional measure with respect to σ on each of these horizontal circles is the Haar measure. A similar statement is obtained for vertical circles with irrational horizontal coordinates by considering the action of

$$H' = \left\{ \begin{pmatrix} 1 & 0 \\ n & 1 \end{pmatrix} : n \in \mathbb{Z} \right\}.$$

It follows that σ is Haar measure on $\mathbb{T}^2$ unless $\sigma(Z) = 0$, where Z is the set of points of $\mathbb{T}^2$ with at least one rational coordinate. But if $\sigma(Z) \neq 0$, this means that there exists at least one horizontal circle (or vertical circle) C with positive σ-measure. Now let $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $c \neq 0$. Then for any $m \neq n$ the set $g^m C \cap g^n C$ is finite, and by the assumption that σ is continuous, the measure of this intersection is zero. Since σ is invariant, we have $\sigma(\mathbb{T}^2) = \infty$, which is a contradiction.

Comments and questions.

Though our construction produces infinite σ-finite continuous ergodic invariant measures in various examples there are certain situations where it does not seem to be applicable. For instance, consider the action of SL(2, $\mathbb{Z}$) on $\mathbb{T} = \mathbb{R}^2 - (0)/\sim$ (where $\sim$ identifies elements which are scalar multiples) arising from the linear action on $\mathbb{R}^2$. In this case it turns out that all ergodic measures for actions of cyclic subgroups are atomic. Hence our lemma cannot be applied with H as a cyclic subgroup. We note however that in this case existence of infinite σ-finite continuous ergodic measures is known and is proved using hyperfiniteness of the action.

Another situation where the lemma may not be applicable may be visualized as follows. Let us define an action of a countable group on a compact space to be *completely minimal* if the action of each individual element of infinite order is minimal. It would be interesting to know if SL(2, $\mathbb{Z}$) (or equivalently, the free group on two generators) admits a completely minimal action on a compact space with more than one point.

We conclude with the question of whether any action of SL(2, $\mathbb{Z}$) which is recurrent, on an uncountable compact space, will admit uncountably many σ-finite continuous ergodic invariant measures.

REFERENCES

[1] R. Bowen, Periodic orbits of hyperbolic flows. *Amer. J. Math.*, t. **94**, 1972, p. 1-30.
[2] H. Furstenberg, The unique ergodicity of the horocycle flow, Recent Advances in Topological Dynamics. *Lecture Notes in Math.*, t. 318, Springer Verlag, Berlin.
[3] K. Schmidt, Invariant measures on the circle. *Preprint*, Warwick.

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