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Section B :
Calcul des Probabilités et Statistique.

Some new Chacon-Edgar-type inequalities for stochastic processes, and characterizations of Vitali-conditions

par

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ABSTRACT. — In this paper we prove that Edgar's main inequality in [5] extends to stochastic processes $(X_i, F_i)_{i\in I}$ where $(F_i)_{i\in I}$ satisfies the Vitali-condition V, when we use the notion essential lim sup in this inequality. We also prove that the inequality is right without V, but then using the notion stochastic lim sup. At the same time, we also generalise some maximal inequalities, proved in [12] and some convergence results in [10].

§ 1. INTRODUCTION, TERMINOLOGY AND NOTATION

The main result in [5] can be stated as follows:

THEOREM 1.1*a* [5]. — Let (Ω, F, P) be a probability space and E a separable dual Banach space. Let (X_n, F_n) be an adapted L^1-bounded sequence of E-valued Bochner integrable random variables. Then, if T denotes the set of bounded stopping times w. r. t. (F_n), we have:

$$\int_\Omega \limsup_{n,m\in N} \| X_n - X_m \| \, dP \leqslant 2 \limsup_{\substack{\sigma,\tau\in T \\ \sigma\leqslant\tau}} \int_\Omega \| E^{F_\sigma} X_\tau - X_\sigma \| \, dP$$

where $E^{F_\sigma}(f)$ denotes the conditional expectation of a function f w. r. t. F_σ, and for every $\sigma \in T$:

$$F_\sigma = \{ A \in F \,\|\, A \cap \{ \sigma = n \} \in F_n, \forall n \in N \}$$

Edgar also showed the failure of this inequality in certain Banach spaces with (RNP), by making the link between theorem 1.1*a* above and the neighborly tree structures in [*7*] and [*8*], by which McCartney and O'Brien prove the existence of a separable Banach space with (RNP) but not isomorphic to a subspace of a separable dual space.

In this paper however, we are interested in a slightly different version if this result, valid in (RNP) spaces:

THEOREM 1.1*b*. — Let E be a Banach space with (RNP) and $(X_n, F_n)_{n=1}^{\infty}$ an amart, which is L^1_E-bounded.

Then:

$$\int_\Omega \limsup_{m,n} \| X_m - X_n \| dP \leqslant 2 \limsup_{\substack{\sigma \leqslant \tau \\ \sigma,\tau \in T}} \int_\Omega \| E^{F_\sigma} X_\tau - X_\sigma \| dP$$

This version follows immediately from Edgar's proof, but is not stated explicitely in this way.

This paper studies the validity of the above inequality for amarts which are L^1-bounded, with values in a (RNP) Banach space, but which are indexed by an arbitrary index set.

Thus stated for amarts $(X_i, F_i)_{i \in I}$, where I is directed is obviously false: Indeed: the inequality trivially implies strong convergence of L^1-bounded martingales (even of uniform amarts, see [*2*]) and this is even false in the case $E = R$. In fact, even for $I = N \times N$ we have an L^1-bounded martingale which is not convergent a. s., due to an example of Cairoli [*3*].

In order to have a. s. convergence of martingales, Krickeberg introduced in [*6*] the Vitali-condition V on a stochastic basis $(F_i)_{i \in I}$. By a stochastic basis $(F_i)_{i \in I}$ we mean a net sub-σ-algebras of F with the property that $i \leqslant j \Rightarrow F_i \subset F_j$ for $i, j \in I$. A family $(A_i)_{i \in I}$ of subsets of Ω we call adapted (to the stochastic basis $(F_i)_{i \in I}$) if $A_i \in F_i$, $\forall i \in I$. Let $(F_i)_{i \in I}$ be a stochastic basis. We say that $(F_i)_{i \in I}$ satisfies the Vitali-condition V, if for every adapted family of sets $(A_i)_{i \in I}$, for every $A \in F_\infty = \sigma\left(\bigcup_i F_i\right)$ (i. e. the σ-algebra generated by $\bigcup_i F_i$) such that $A \subset e \limsup A_i$, for every $\varepsilon > 0$, there exist finitely many indices $i_1, \dots, i_n$ in I and pairwise disjoint sets $B_j \subset A_{i_j} (j = 1, \dots, n)$ such that $B_j \in F_{i_j} (j = 1, \dots, n)$ and such that

$$P\left(A \setminus \bigcup_{j=1}^{n} B_j\right) \leqslant \varepsilon$$

The definition of $e \limsup A_i$ is as follows: let $(f_i)_{i \in I}$ be a family of random variables, taking values in $\overline{\mathbb{R}}$. The essential supremum of $(f_i)_{i\in I}$ is the unique almost surely smallest r. v. $e\sup_i f_i$ such that $\forall j \in I$: $e\sup_i f_i \geqslant f_j$ a. s.

The essential infimum of $(f_i)_{i\in I}$, $e\inf f_i$, is defined by $e\inf f_i = -e\sup(-f_i)$. The essential upper limit of $(f_i)_{i\in I}$ is defined by

$$e\limsup f_i = e\inf_j (e\sup_{i\geqslant j} f_i)$$

and the essential lower limit of $(f_i)_{i\in I}$ by $e\liminf f_i = -e\limsup(-f_i)$.

We call $(f_i)_{i\in I}$ essential convergent if $e\limsup f_i = e\liminf f_i$. For a family $(A_i)_{i\in I}$ as above we define the essential supremum: $e\sup A_i$ by $1_{e\sup_i A_i} = e\sup 1_{A_i}$, and analogously the essential infimum:

$$1_{e\inf A_i} = e\inf 1_{A_i}$$

and the essential upper limit:

$$1_{e\limsup A_i} = e\limsup 1_{A_i}.$$

Krickeberg proved:

THEOREM 1.2 [*6*]. — Let $(X_i, F_i)_{i\in I}$ be an L^1-bounded real martingale. Suppose that $(F_i)_{i\in I}$ satisfies the Vitali-condition V. Then $(X_i)_{i\in I}$ converges essentially.

He left open the sufficiency of V for the almost sure convergence of L^1-bounded martingales. This was solved recently by Millet-Sucheston in [*9*] in the negative. For completeness we mention the interesting result of K. Astbury [*1*] stating that V is necessary and sufficient for the convergence of L^1-bounded real amarts.

By theorem 1.2, we are led to the conjecture of the validity of theorem 1.1*b* for general index set in case we suppose V and use $e\limsup$ in the left hand side of the inequality. This is proved to be true in section 2 of this paper.

Let us look now to a weaker notion than essential convergence: the stochastic convergence (see [*10*]):

Let $(f_i)_{i\in I}$ be a family of r. v. taking values in $\overline{\mathbb{R}}$. The stochastic upper limit of $(f_i)_{i\in I}$, $s\limsup f_i$ is

$$e\inf\{ Y \,\|\, \lim_i P(\{ Y < f_i \}) = 0 \}$$

where Y denotes a r. v.

As usual we define the stochastic lower limit, $s\liminf f_i$, to be

$$s\liminf f_i = -s\limsup(-f_i)$$

We say that $(f_i)_{i\in I}$ converges stochastically if $s\limsup f_i = s\liminf f_i$. If $(A_i)_{i\in I}$ is a family of subsets of Ω we again define $s\limsup A_i$ by:

$$1_{s\limsup A_i} = s\limsup 1_{A_i}$$

Let $(F_i)_{i\in I}$ be a stochastic basis. Denote by T′ the set of ordered stopping times: i. e. a function $\tau : \Omega \to I$ such that $\tau(\Omega)$ is finite and linearly ordered.

The following theorem of Millet-Sucheston ([*12*], proposition 1.3) shows that $(F_i)_{i\in I}$ satisfies always a « stochastic Vitali-condition » (with T′, i. e.: in the strongest way):

THEOREM 1.3 [*12*]. — Let $(A_i)_{i\in I}$ be an adapted family of sets. Then, for every $\varepsilon > 0$ and $i_0 \in I$, there is a $\tau \in T'$, $\tau \geqslant i_0$ such that

$$P(s \limsup_i A_i \Delta A_\tau) \leqslant \varepsilon$$

Here $A_\tau = \bigcup_i [\{ \tau = i \} \cap A_i]$.

We mention also the following result of Millet-Sucheston [*10*] (T denotes the set of all simple (i. e. finitely valued) stopping times w. r. t. the stochastic basis $(F_i)_{i\in I}$):

THEOREM 1.4 [*10*]. — Let $(X_i, F_i)_{i\in I}$ be an L^1-bounded real martingale (or even a subpramart). Then $(X_\tau)_{\tau\in T}$ converges stochastically (Compare this with theorem 1.2).

Guided by this result and theorem 1.3, one is led to the conjecture that theorem 1.1 might be true with lim sup replaced by s lim sup, in case of a general index set, even if V is not satisfied. This is proved in section 3, generalising theorem 1.4 in [*12*].

In theorem 1.4 in [*12*] the first stochastic maximal inequality is proved, and which is also an argument for the conjecture just mentioned.

§ 2. THE CASE OF ESSENTIAL UPPER LIMIT

As in [*5*] we firstly prove two lemmas

LEMMA 2.1. — Let $(X_i, F_i)_{i\in I}$ be a stochastic proces, $X_i \geqslant 0$, $\forall i \in I$. Suppose $(F_i)_{i\in I}$ satisfies V. Then:

$$\int_\Omega e \limsup_i X_i \leqslant \limsup_{\tau\in T} \int_\Omega X_\tau$$

Proof. — We may suppose (X_i) to be uniformly bounded: indeed, suppose the inequality proved for uniformly bounded stochastic processes, an appeal to Lebesgues monotone convergence theorem finishes the proof in the general case.

So let $M \in R^+$ be a uniform bound of (X_i).

For $i_1 \leqslant i_2$ in I, define:

$$X_{i_1 i_2} = e \sup \{ X_i \| i_1 \leqslant i \leqslant i_2 \}$$

where e sup is taken w. r. t. F_{i_2}, i. e.: $X_{i_1 i_2}$ is F_{i_2}-measurable. Now, with $X^* = e \limsup X_i$: for every $\varepsilon > 0$, choose $X_{i_1 i_2}$ such that

$$P\left\{ | X^* - X_{i_1 i_2} | > \frac{\varepsilon}{4} \right\} \leqslant \frac{\varepsilon}{8M} (*)$$

Call
$$A_i = \left\{ | X_i - X_{i_1 i_2} | < \frac{\varepsilon}{4} \right\}$$

for $i \geqslant i_2$. Hence $(A_i)_{\substack{i \in I \\ i \geqslant i_2}}$ is an adapted family. Referring to [*14*] p. 121, VI-1-1 and $(*)$, we have that $P(\Omega \backslash e \limsup A_i) \leqslant \frac{\varepsilon}{8M} (**)$.

Using V, there exist $i_1, \ldots, i_n \in I$ and pairswise disjoint $(B_j)_{j=1}^n$, such that $B_j \subset A_{i_j}$, $B_j \in F_{i_j}$ for every $j = 1, \ldots, n$, and such that $P\left(e \limsup A_i \backslash \bigcup_{j=1}^{n} B_j\right) \leqslant \frac{\varepsilon}{8M}$. So:

$$P\left(\Omega \backslash \bigcup_{j=1}^{n} B_j\right) \leqslant \frac{\varepsilon}{4M}.$$

Call:
$$\tau \begin{cases} = i_j & \text{on } B_j \\ = \bigvee_{j=1}^{n} i_j & \text{on co } \left(\bigcup_{j=1}^{n} B_j\right) \end{cases}$$

Hence $\tau \in T$. Furthermore, since $X_i \geqslant 0$:

$$\int_\Omega X^* \leqslant \int_\Omega | X^* - X_\tau | + \int_\Omega X_\tau$$
$$\leqslant \int_{\{ |X^* - X_\tau| < \varepsilon/2 \}} | X^* - X_\tau | + \int_{\{|X^* - X_\tau| \geqslant \varepsilon/2\}} | X^* - X_\tau | + \int_\Omega X_\tau$$

Since:

$$\left\{ | X^* - X_\tau | \geqslant \frac{\varepsilon}{2} \right\} \subset \left\{ | X^* - X_{i_1 i_2} | \geqslant \frac{\varepsilon}{4} \right\} \cup \left\{ | X_{i_1 i_2} - X_\tau | \geqslant \frac{\varepsilon}{4} \right\}$$

since $B_j \subset A_{ij}$, $\forall j = 1, \ldots, n$, and

$$P\left(\Omega \backslash \bigcup_{j=1}^{n} B_j\right) \leqslant \frac{\varepsilon}{4M}$$

We have that

$$P\left(\left\{ | X_{i_1 i_2} - X_\tau | \geqslant \frac{\varepsilon}{4} \right\}\right) \leqslant \frac{\varepsilon}{4M}$$

and hence $P\left(\left\{|X^* - X_\tau| \leqslant \frac{\varepsilon}{2}\right\}\right) \leqslant \frac{\varepsilon}{2M}$.

So:
$$\int_\Omega X^* \leqslant \frac{\varepsilon}{2} + \frac{\varepsilon}{2M} \cdot M + \int_\Omega X_\tau$$
$$= \varepsilon + \int_\Omega X_\tau$$

Hence:
$$\int_\Omega X^* \leqslant \limsup_{\tau \in T} \int_\Omega X_\tau. \quad \square$$

We note that the next lemma is true for every stochastic basis (even without satisfying V). This is not so for the preceding lemma: in this lemma, V is even necessary (see remark 3.6.3).

LEMMA 2.2. — Let $(X_i, F_i)_{i \in I}$ be a stochastic process, with values in an arbitrary Banach space. Let $\sigma \in T$ arbitrary.

Then:
$$\int_\Omega e \sup_{\substack{i \geqslant \sigma \\ i \in I}} \| E^{F_\sigma} X_i \| \leqslant \sup_{\substack{\tau \geqslant \sigma \\ \tau \in T}} \int_\Omega \| E^{F_\sigma} X_\tau \|$$

Proof. — By [14], p. 121, VI-1-1, there is a sequence $\tau_n \geqslant \sigma$, $\tau_n \in T$ for each $n \in N$ such that
$$e \sup_{\substack{i \geqslant \sigma \\ i \in I}} \| E^{F_\sigma} X_i \| = \sup_{n \in N} \| E^{F_\sigma} X_{\tau_n} \|$$

Using the localization property in T we can assure that there is a sequence τ_n' in T, $\tau_n' \geqslant \sigma$, for each $n \in N$ that
$$\| E^{F_\sigma} X_{\tau_n'} \| \uparrow \sup_{n \in N} \| E^{F_\sigma} X_{\tau_n} \|$$

Hence:
$$\| E^{F_\sigma} X_{\tau_n'} \| \uparrow e \sup_{\substack{i \geqslant \sigma \\ i \in I}} \| E^{F_\sigma} X_i \|$$

Lemma 2.2 is now proved, using the monotone convergence theorem. $\square$

The rest of the proof of our theorem is the same as Edgar's, using the fact that $\forall A \in \bigcup_i F_i : \left(\int_A X_i\right)_{i \in I}$ converges. This yields:

THEOREM 2.3. — Let E be *a* (RNP) Banach space. Let $(X_i, F_i)_{i \in I}$ be an amart with values in E, L^1-bounded, and satisfying V.

Then:
$$\int_\Omega e \limsup_{i,j \in I} \| X_i - X_j \| \leqslant 2 \limsup_{\substack{\tau \geqslant \sigma \\ \sigma,\tau \in T}} \int_\Omega \| E^{F_\sigma} X_\tau - X_\sigma \|$$

Let $(F_i)_{i\in I}$ be a stochastic basis. Denote by T′ the set of the finite stopping times τ for which ther ange is linearly ordered; cf. [10]. An ordering $\leqslant | \leqslant$ on T′ is defined as follows: for $\sigma, \tau \in T'$ we define $\sigma \leqslant | \leqslant \tau$ is $\sigma = \tau$ or there is an $i \in I$ such that $\sigma \leqslant i \leqslant \tau$. For this order, T′, $\leqslant | \leqslant$, is a directed set, filtering to the right.

We say that $(F_i)_{i\in I}$ satisfies the ordered Vitali-condition V′ if for every $A \in F_\infty\left(F_\infty = \sigma\left(\bigcup_i F_i\right)\right)$, every adapted family $(A_i)_{i\in I}$ such that $A \subset e \limsup_i A_i$, and every $\varepsilon > 0$, there exist indices $i_1 \leqslant i_2 \leqslant \ldots \leqslant i_n$, and pairwise disjoint sets $B_j \in F_{i_j}$, $B_j \subset A_{i_j}$, $j = 1, \ldots, n$, such that

$$P\left(A \setminus \bigcup_{j=1}^{n} B_j\right) \leqslant \varepsilon$$

LEMMA 2.4. — Let $(X_i, F_i)_{i\in I}$ be a stochastic process, $X_i \geqslant 0$, $\forall i \in I$. Suppose $(F_i)_{i\in I}$ satisfies V′. Then

$$\int_\Omega e \limsup_i X_i \leqslant \limsup_{\tau\in T'} \int_\Omega X_\tau$$

Proof. — This proof is exactly the same as the proof of lemma 2.1, now using V′; the stopping time τ constructed there is indeed in T′ in this case. □

LEMMA 2.5. — Let $(X_i, F_i)_{i\in I}$ be a stochastic process, with values in an arbitrary Banach space. Suppose that $(F_i)_{i\in I}$ satisfies V′. Let $\sigma \in T'$.

Then:

$$\int_\Omega e \sup_{\substack{i\geqslant\sigma \\ i\in I}} \| E^{F_\sigma} X_i \| \leqslant \sup_{\substack{\tau \geqslant | \geqslant \sigma \\ \tau\in T'}} \int_\Omega \| E^{F_\sigma} X_\tau \|$$

Proof. — This proof is done in the same way as the proof of lemma 2.1, now using sets

$$A_i = \{ \, | \, \| E^{F_\sigma} X_i \| - e \sup_{\substack{j\geqslant\sigma \\ j\in I}} \| E^{F_\sigma} X_j \| \, | \leqslant \varepsilon \}.$$

We have now:

THEOREM 2.6. — Let E be a (RNP) Banach space. Let $(X_i, F_i)_{i\in I}$ be an amart with values in E, L^1-bounded, and satisfying V′.

Then:

$$\int_\Omega e \limsup_{i,j\in I} \| X_i - X_j \| \leqslant 2 \limsup_{\substack{\tau \geqslant | \geqslant \sigma \\ \sigma,\tau\in T'}} \int_\Omega \| E^{F_\sigma} X_\tau - X_\sigma \|$$

REMARK 2.7. — 1. Lemma 2.1 is a generalization of (*1*) ⇒ (*8*) in theorem 3.1 in Millet-Sucheston [*12*].

2. We can also derive (*1*) ⇒ (*7*) in this theorem from our lemma 2.1:

THEOREM 2.7.1 [*12*]. — For each positive stochastic process $(X_i, F_i)_{i\in I}$ such that $(F_i)_{i\in I}$ satisfies V we have, $\forall\lambda > 0$:

$$P[\{ e \limsup_i X_i \geqslant \lambda \}] \leqslant \frac{1}{\lambda} \limsup_{\tau\in T} \int X_\tau$$

Proof. — Denote $X^* = e \limsup X_i$. By an application of Fubini, we have:

$$\int_\Omega X^* = \int_0^\infty P(\{ X^* \geqslant \xi \})d\xi$$

Now

$$\begin{aligned} &\int_0^\infty P(\{ X^* \geqslant \xi \})d\xi \\ &\geqslant \int_0^\lambda P(\{ X^* \geqslant \xi \})d\xi \qquad \forall\lambda > 0 \\ &\geqslant \int_0^\lambda P(\{ X^* \geqslant \lambda \})d\xi \\ &= \lambda P(\{ X^* \geqslant \lambda \}) \end{aligned}$$

So by lemma 2.1:

$$\lambda P(\{ X^* \geqslant \lambda \}) \leqslant \limsup_{\tau\in T} \int_\Omega X_\tau,$$

or: $\forall\lambda > 0$:

$$P(\{ X^* \geqslant \lambda \}) \leqslant \frac{1}{\lambda} \limsup_{\tau\in T} \int_\Omega X_\tau. \quad \square$$

In the same way we can derive (*1*) ⇒ (*8*) in theorem 4.1 in [*12*], from our lemma 2.4.

§ 3. THE CASE OF STOCHASTIC UPPER LIMIT

As mentioned in the introduction, we shall now prove the analogue of theorem 2.3, with $e \limsup_{i,j}$ replaced by $s \limsup_{i.j}$ in the left hand side of the inequality, without supposing V.

LEMMA 3.1. — Let $(X_i, F_i)_{i\in I}$ be a positive stochastic process. Then

$$\int_\Omega s \limsup_i X_i \leqslant \limsup_{\tau\in T'} \int_\Omega X_\tau \leqslant \limsup_{\tau\in T} \int_\Omega X_\tau$$

Proof. — Since the last inequality is obvious, we only have to prove the first one. Fix $\varepsilon > 0$ arbitrarily. As in lemma 2.1, we can suppose the (X_i) to be uniformly bounded, say by M.

Define

$$\tilde{X} = s \limsup_i X_i \in L^1$$

Put, $\forall i \in I$

$$A_i = \left\{ | X_i - E^{F_i}\tilde{X} | < \frac{\varepsilon}{4} \right\}$$

So $(A_i)_{i\in I}$ is adapted. Furthermore since $\lim_{i\in I} E^{F_i}\tilde{X} = \tilde{X}$ in L^1-sense, we see :

$$\tilde{A} = : s \limsup_i A_i = \Omega$$

We use theorem 1.3 to yield a $\tau \in T'$ with $P(\tilde{A}\Delta A_\tau) \leqslant \frac{\varepsilon}{4M}$

So:

$$P(\Omega \backslash A_\tau) \leqslant \frac{\varepsilon}{4M}.$$

Hence:

$$\int_\Omega \tilde{X} \leqslant \int_\Omega | \tilde{X} - E^{F_\tau}\tilde{X} | + \int_{\left\{ |E^{F_\tau}\tilde{X} - X_\tau| < \frac{\varepsilon}{4} \right\}} | E^{F_\tau}\tilde{X} - X_\tau |$$
$$+ \int_{\left\{ |E^{F_\tau}\tilde{X} - X_\tau| \geqslant \frac{\varepsilon}{4} \right\}} | E^{F_\tau}\tilde{X} - X_\tau | + \int_\Omega X_\tau$$
$$\leqslant \frac{\varepsilon}{2} + \frac{\varepsilon}{4} + \frac{\varepsilon}{4M} . M + \int_\Omega X_\tau$$
$$= \varepsilon + \int_\Omega X_\tau. \quad \square$$

Lemma 3.1 and 2.2 give now (see the proof of [*5*]):

THEOREM 3.2. — Let E be a (RNP) Banach space. Let $(X_i, F_i)_{i\in I}$ be an L^1-bounded amart, with values in E. Then:

$$\int_\Omega s \limsup_{i,j\in I} \| X_i - X_j \| \leqslant 2 \limsup_{\substack{\sigma \leqslant \tau \\ \sigma,\tau \in T}} \int_\Omega \| E^{F_\sigma}X_\tau - X_\sigma \|$$

We note the following corollary of lemma 3.1 analogous to theorem 2.7.1, proving that lemma 3.1 is a generalization of theorem 1.4 in [*12*].

COROLLARY 3.3 [*12*]. — For any positive stochastic process we have, $\forall \lambda > 0$:

$$P[\{ s \limsup_{i\in I} X_i \geqslant \lambda \}] \leqslant \frac{1}{\lambda} \limsup_{\tau\in T'} \int_\Omega X_\tau$$

Proof. — The same as the proof of theorem 2.7.1, now using

$$\tilde{X} = s \limsup_{i \in I} X_i$$

and lemma 3.1. □

REMARKS 3.4. — 1. Of course, in theorem 2.3, V is also necessary for the inequality to be satisfied: indeed: All one-dimensional amarts which are L^1-bounded are included in this theorem. By the inequality we have convergence. By K. Astbury's result [*1*], we have V.

2. An analogous remark is true for theorem 2.6 w. r. t. V′, now using [*10*] (theorem 7.4).

3. Also in lemma 2.1, V is necessary: indeed: Once we have the inequality in lemma 2.1, we can prove theorem 2.3 without V. We now refer to our first remark 3.4.1. From this we have:

COROLLARY 3.4.3. — There exists an L^1-bounded, real valued positive stochastic process $(X_i, F_i)_{i \in I}$ such that

$$\int s \limsup X_i \leqslant \limsup_{\tau \in T} \int X_\tau < \int e \limsup X_i .$$

4. From the preceding remark and from [*1*] we derive:

THEOREM 3.4.4. — Let $(F_i)_{i \in I}$ be a stochastic basis. The following properties are equivalent:

i) For every amart $(X_i, F_i)_{i \in I}$:

$$\int_\Omega e \limsup |X_i| \leqslant \limsup_{\tau \in T} \int_\Omega |X_\tau|$$

ii) Every L^1-bounded amart $(X_i, F_i)_{i \in I}$ converges essentially.

Proof. — *ii*) ⇒ *i*) follows immediately by [*1*] (saying that V is equivalent with *ii*)), and by lemma 2.1.

i) ⇒ *ii*) Let $(X_i, F_i)_{i \in I}$ be an L^1-bounded amart. By [*1*]: (X_i^+) and (X_i^-) are positive amarts. By *i*), we can again prove the inequality in theorem 2.3 without V, proving that (X_i^+) and (X_i^-) converge essentially. Hence so does (X_i). □

This theorem 3.4.4, should be compared with his martingale-analogue in theorem 2.5 in [*13*].

COROLLARY 3.5. — If $(X_i, F_i)_{i \in I}$ is an E-valued L^1-bounded uniform amart, where E is a (RNP) Banach space, then $(X_i)_{i \in I}$ converges stochastically to an integrable function. If $(F_i)_{i \in I}$ satisfies V, then (X_i) converges essentially to this function.

This follows immediately from theorems 3.2 and 2.3, and extends theorem 12.4 in [*10*] and [*2*].

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