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On the asymptotic behaviour of sequences of random variables and of their previsible compensators

by

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INTRODUCTION

This paper is divided into two parts: the first part deals with the comparison or the sets of convergence of two sequences (V_n) and (h_n) of random variables adapted to an increasing family of σ-fields $(\mathcal{F}_n)$ and satisfying the inequality $E(V_{n+1}/\mathcal{F}_n) \leq V_n + h_n$. One of the corollaries of our main theorem of this part is a generalisation of a result of Robbins and Siegmund [*8*]. The second part deals with C-sequences, i. e. sequences of random variables whose previsible predictor do not oscillate. We give a number of conditions for the convergence of such sequences, conditions which include the classical supermartingale convergence theorems. We end by giving simple examples of amarts which are not C-sequences and of C-sequences which are not amarts.

It is known that the convergence theorem for L_1-bounded asymptotic martingales cannot be generalized to the cases of infinite dimensional Banach space valued variables (see [*2*] (*a*) and (*b*)). We hope that our theorem 4 can be generalized in such directions.

NOTATIONS AND CONVENTIONS. — In this paper, $(\Omega, \mathcal{F}, P)$ is a fixed probability space, $(\mathcal{F}_n)_{n \geq 1}$ is a fixed family of increasing σ-algebras contained in $\mathcal{F}$. A sequence (X_n) of random variables will be said to be *adapted* if for each n, X_n is $\mathscr{F}_n$-measurable. Unless otherwise stated, convergence means almost sure (a. s.) convergence to *finite* valued random variables. If $\mathscr{P}$ is a property, $\{ \mathscr{P} \}$ will denote the set

$$\{ \omega : \omega \in \Omega , \qquad \omega \text{ verifies } \mathscr{P} \} .$$

$\uparrow$ (resp. $\downarrow$) indicates « increasing » (resp. decreasing) to. For $A \in \mathscr{F}$, 1_A will denote the characteristic function of A. Finally $\bar{\mathbb{R}}$ will denote the extended real line.

I. SOME RESULTS ON THE CONVERGENCE OF SEQUENCES OF RANDOM VARIABLES

THEOREM 1. — Let $(h_n)_{n \geqq 1}$ and $(V_n)_{n \geqq 1}$ be two adapted sequences of real random variables such that

1) for every n, V_n and h_n are integrable and $E(V_{n+1}/\mathscr{F}_n) \leqq V_n + h_n$

2) $\sup_n E\left[\left(V_n - \sum_{j=1}^{n-1} h_j\right)^-\right] < \infty .$

Then the set on which (V_n) convergences is almost surely equal to the set on which Σh_n convergences.

Proof. — Setting $b_n = \sum_1^n h_j$, $W_n = V_n - b_{n-1}$, it is easily seen that $(W_n)_{n \geqq 2}$ is a supermartingale. The condition 2) then implies that (W_n) convergences a. s. [*6*]. The statement of the theorem then follows immediately.

THEOREM 2. — Let $(h_n)_{n \geqq 1}$ and $(V_n)_{n \geqq 1}$ be two adapted sequences of real random variables such that

1) For every n, h_n and V_n are integrable and $V_n \geqq 0$ a. s.

2) $E(V_{n+1}/\mathscr{F}_n) \leqq V_n + h_n$.

Set
$$B = \left\{ \omega : \sup_n \sum_1^n h_n(\omega) < \infty \right\} .$$

Then on B, the set on which (V_n) converges is a. s. equal to the set on which Σh_j convergences.

Proof. — Setting again $b_n = \sum_1^n h_j$, $1_n^a = 1_{\bigcap_1^n \{b_j < a\}}$ we obtain since (1_n^a) is decreasing for all a :

i) $E(V_{n+1}1_{n+1}^a/\mathscr{F}_n) \leqq E(V_{n+1}1_n^a/\mathscr{F}_n) \leqq (V_n + h_n)1_n^a$

ii) $\sum_1^n h_j(\omega)1_j^a(\omega) = \sum_1^{k(\omega)} h_j(\omega) = \left(\sum_1^{k(\omega)} h_j(\omega)\right)1_{k(\omega)}^a = b_{k(\omega)}1_{k(\omega)}^a$

where $\qquad k(\omega) = \sup \{ i \leqq n : 1_i^a(\omega) = 1 \}$.

Therefore, since $V_n \geqq 0$,

$$\sup_n E\left[\left(V_n 1_n^a - \sum_1^{n-1} h_j 1_n^a\right)^-\right] \leqq \sup_n E\left[\left(\sum_1^{n-1} h_j 1_j^a\right)^-\right] < a$$

and theorem 1, applied to the sequences $(V_n 1_n^a)$ and $(h_n 1_n^a)$, allows us to state that:

$\{ (V_n 1_n^a)$ convergences $\} = \{ \Sigma h_n 1_n^a$ converges $\}$, and therefore using the definition of 1_n^a

$$\bigcap_1^\infty \{ b_n < a \} \cap \{ V_n 1_n^a \text{ converges} \} = \bigcap_1^\infty \{ b_n < a \} \cap \{ \Sigma h_n 1_n^a \text{ converges} \}$$

and the theorem follows by letting a go to $+\infty$.

We now give a few corollaries to theorems 1 and 2.

COROLLARY I.1. — Let (h_n), (V_n) be as in theorem 1. Let (g_n) be an adapted sequences of strictly positive random variables such that:

1) $E[V_{n+1}/\mathscr{F}_n] \leqq g_n V_n + h_n$ for all n

2) $\sup_n E\left[\left(a_{n-1}V_n - \sum_1^{n-1} h_j a_j\right)^-\right] < \infty$, where $a_n = \dfrac{1}{\prod_1^n g_j}$

Then

$$\left\{\left(\frac{1}{a_n}\right) \text{ converges}\right\} \cap \{ (V_n) \text{ converges} \}$$

$$\stackrel{\text{a.s.}}{=} \left\{\left(\frac{1}{a_n}\right) \text{ converges}\right\} \cap \left\{\left(\frac{1}{a_n}\sum_1^n a_j h_j\right) \text{ converges}\right\}$$

Moreover on the set $\left\{\frac{1}{a_n} \to 0\right\}$, the sequence

$$\left(V_n - \frac{1}{a_{n-1}} \sum_{1}^{n-1} h_j a_j\right) \to 0 \qquad \text{a. s.}$$

Proof. — Apply theorem 1 to the sequences $(V_n' = a_{n-1} V_n)$, $(h_n' = a_n h_n)$.

COROLLARY I.2. — Let (X_n) be an adapted sequence of real integrable random variables. If

1) $\sup_n E(X_n^-) < \infty$

2) $\sup_n E\left(\left[\sum_{1}^{n} E(X_{j+1}/\mathscr{F}_j) - X_j\right)\right]^+ < \infty$

then ΣX_n converges a. s. if and only if $\Sigma E(X_{n+1}/\mathscr{F}_n)$ converges a. s.

Proof. — Apply theorem 1 by setting $V_n = \sum_{1}^{n} X_j$, $h_n = E(X_{n+1}/\mathscr{F}_n)$.

COROLLARY I.3. — Let (X_n) be as in corollary I.2. Let (a_n) be a sequence of real numbers tending to ∞. Then:

$$\left|\frac{1}{a_n} \sum_{1}^{n} X_i - \frac{1}{a_n} \sum_{1}^{n} E(X_{i+1}/\mathscr{F}_i)\right| \to 0 \qquad \text{a. s.}$$

In particular, secting $a_n = n$, (X_n) verifies the law of large numbers if and only if the sequence $(E(X_{n+1}/\mathscr{F}_n))$ does.

Proof. — Set $V_n = \frac{1}{a_n} \sum_{1}^{n} X_i$, $h_n = E(X_{n+1}/\mathscr{F}_n)$, $g_n = \frac{a_n}{a_{n+1}}$ and apply corollary I.1.

The following generalises slightly a result of Robbins and Siegmund.

COROLLARY I.4. — Let $(V_n)(\xi_n)(\eta_n)(g_n)$ be adapted sequences. We suppose $V_n \geqq 0$, $\xi_n \geqq 0$, $\eta_n \geqq 0$, $g_n > 0$ and that

$$E[V_{n+1}/\mathscr{F}_n] \leqq g_n V_n + \xi_n - \eta_n .$$

Then the sequences (V_n) and $\left(\sum_1^n \eta_j\right)$ converge almost surely on the set

$$B = \left\{ 0 < \lim_n \prod_1^n g_j < \infty \right\} \cap \left\{ \sum_1^\infty \xi_i < \infty \right\}.$$

Proof. — Setting

$$V'_n = \frac{V_n}{\prod_1^{n-1} g_j}, \qquad \xi'_n = \frac{\xi_n}{\prod_1^n g_j}, \qquad \eta'_n = \frac{\eta_n}{\prod_1^n g_j}$$

we see that

$$E[V'_{n+1}/\mathscr{F}_n] \leqq V'_n + \xi'_n - \eta'_n \leqq V'_n + \xi'_n$$

Moreover, on B, the sequences (V_n) and (V'_n) $\left(\text{resp.} \sum_1^n \xi_k \text{ and } \sum_1^n \xi'_k,\right.$ resp. $\left.\sum_1^n \eta_k \text{ and } \sum_1^n \eta'_k\right)$ have the same set of convergence.

Theorem 2 applied to the sequences (V'_n) and (ξ'_n) implies that (V_n) converges almost surely on B. Set

$$A = \left\{ \sup_n \sum_1^n (\xi'_k - \eta'_k) < \infty \right\}.$$

On A, (V'_n) and $\Sigma(\xi'_k - \eta'_k)$ have the same set of convergence, by theorem 2. Since on B, the series $\Sigma(\xi'_k - \eta'_k)$ converges if and only if $\Sigma\eta_k$ does, the corollary follows.

II. C-SEQUENCES

Before defining C-sequences, we prove a « Doob decomposition theorem ».

THEOREM 3. — Let (V_n) be an adapted sequence of integrable random variables. Then there exists sequences (M_n), $(\tilde{V}_n)$ of random variables such that

1) $V_n = M_n + \tilde{V}_n$
2) $\tilde{V}_1 = 0$ and $\tilde{V}_n$ is $\mathscr{F}_{n-1}$-measurable for every $n \geqq 2$
3) M_n is an $\mathscr{F}_n$-martingale.

This decomposition is unique.

Proof. — Setting $M_1 = V_1$,

$$M_n = \left(V_n - \sum_{1}^{n-1} [E(V_{k+1}/\mathscr{F}_k) - V_k]\right)$$

$$\tilde{V}_n = \sum_{1}^{n-1} [E(V_{k+1}/\mathscr{F}_k) - V_k] \qquad \text{for } n \geqq 2$$

we get the desired decomposition. To prove uniqueness, we note that if $V_n = M'_n + B_n$ is another decomposition verifying 1), 2) and 3), we have

$$\sum_{1}^{n-1} [E(V_{k+1}/\mathscr{F}_k) - V_k] = \sum_{1}^{n-1} [M'_k + B_{k+1} - M'_k - B_k] = B_n - B_1 = B_n$$

$$\text{for } n \geqq 2 .$$

Thus $B = \tilde{V}$ and the uniqueness is proved.

The following terminology and notation is standard.

DEFINITION. — If (V_n) is a sequence verifying the hypotheses of theorem 3, $(\tilde{V}^n)$ will denote the sequence defined by $\tilde{V}_1 = 0$,

$$\tilde{V}_n = \sum_{1}^{n-1} (E(V_{k+1} \mid \mathscr{F}_k) - V_k) \qquad \text{for } n \geqq 2 ;$$

$(\tilde{V}_n)$ is called the *previsible compensator* of (V_n).

DEFINITION. — An adapted sequence of random variables (X_n) is called a *C-sequence* if the V_n's are integrable and if the sequence $(\tilde{V}_n)$ converges in $\bar{\mathbb{R}}$. It is called a *strict* C-sequence if $(\tilde{V}_n)$ converges in $\mathbb{R}$.

Martingales, submartingales, supermartingales, quasi-martingales are C-sequences. Adapted sequences (V_n) satisfying

$$\sum_{1}^{\infty} | E(V_{n+1}/\mathscr{F}_n) - V_n | < \infty \qquad \text{a. s.}$$

are C-sequences but the converse is not true as is seen by the following example.

Let (X_n) be a sequence of independent identically distributed random variables with $E(X_n) = 0$, $0 < E(X_n^2) < \infty$. Then putting $V_n = \frac{X_n}{n}$ it is easy to see that (V_n) a C-sequence but that

$$\sum_1^\infty | E(V_{n+1}/\mathscr{F}_n) - V_n | = \sum_1^\infty \frac{| X_n |}{n} = \infty \qquad \text{a. s.}$$

THEOREM 4. — Let (V_n) be an adapted sequence of integrable random variables such that

1) $\sup_n E(V_n^-) < \infty$

2) $\sup_n E(\tilde{V}_n^+) < \infty$

Then (V_n) converges almost surely if and only if it is a C-sequence.

Proof. — Write $V_n = M_n + \tilde{V}_n$ where (M_n) is a martingale (cf. theorem 3). If (V_n) converges a. s., $\sup_n E(M_n^-) \leqq \sup_n E(\tilde{V}_n^+) + \sup_n E(V_n^-) < \infty$ which implies that (M_n) converges a. s. The same is then true for $(\tilde{V}_n)$. Conversely, suppose $(\tilde{V}_n)$ converges a. s. in $\bar{\mathbb{R}}$. The equalities

$$E(V_n^+) - E(V_n^-) - E(V_1) = E(V_n) - E(V_1) = E(\tilde{V}_{n-1}) = E(\tilde{V}_{n-1}^+) - E(\tilde{V}_{n-1}^-)$$

imply that

$$\sup_n E(\tilde{V}_n^-) \leqq \sup_n [E(\tilde{V}_n^+) + E(V_{n+1}^-) + E(V_1)]$$

and this last term is finite by hypothesis. Using Fatou's lemma, we conclude that $\lim \tilde{V}_n^+$ and $\lim \tilde{V}_n^-$ are finite, i. e. $(\tilde{V}_n)$ converges a. s. (in $\mathbb{R}$). The hypothesis of our theorem allows us now to apply theorem 1 and to conclude that the sequence (V_n) converge a. s.

COROLLARY 4.1. — Let (V_n) be an adapted sequence of integrable random variables. If

1) $\sup_n E(| V_n |) < \infty$

2) there exists a constant k such that

$$\sum_{j=(n-1)k+1}^{nk} E(V_{j+1}/\mathscr{F}_j) \leqq \sum_{j=(n-1)k+1}^{nk} V_j \qquad \text{for all } n = 1, 2, \ldots$$

3) $E(V_{n+1}/\mathscr{F}_n) - V_n$ converges to 0 a. s.

Then (V_n) converges to 0 a. s.

Proof. — We have

$$\sum_{1}^{n} [E(V_{j+1}/\mathscr{F}_j) - V_j] = \sum_{m=1}^{\left[\frac{n}{k}\right]} a_m + \sum_{\left[\frac{n}{k}\right]k+1}^{n} [E(V_{j+1}/\mathscr{F}_j) - V_j]$$

where

$$a_m = \sum_{j=(m-1)k+1}^{mk} [E(V_{j+1}/\mathscr{F}_j) - V_j]$$

condition 2) implies that $a_m \leqq 0$ for all m. Furthermore

$$\left| \sum_{\left[\frac{n}{k}\right]}^{n} [E(V_{j+1}/\mathscr{F}_j) - V_j] \right| \leqq k \max_{\left[\frac{n}{k}\right]k+1 \leqq j \leqq n} | E(V_{j+1}/\mathscr{F}_j) - V_j |$$

This last term converges to 0 a. s. by condition 3). Thus (V_n) is a C-sequence. Since

$$\sup_n E(\tilde{V}_n^+) \leqq \sup_n E\left[\sum_{m=1}^{\left[\frac{n}{k}\right]} a_m\right]^+ + \sup_n E\left[\sum_{\left[\frac{n}{k}\right]k+1}^{nk} (E(V_{j+1}/\mathscr{F}_j) - V_1)\right]^+$$
$$\leqq 2k \sup_n E(|V_n|) < \infty$$

(using the above inequality and condition 1), condition 2) of theorem 4 is satisfied and therefore (V_n) converges a. s.

COROLLARY 4.2. — Let (V_n) be an adapted sequence of integrable random variables. If

1) $\sup_n E(V_n^-) < \infty$

2) $E(V_{n+1}/\mathscr{F}_n) \leqq V_n$ if n is odd
 $E(V_{n+1}/\mathscr{F}_n) \geqq V_n$ if n is even

3) $| E(V_{n+1}/\mathscr{F}_n) - V_n | \downarrow 0$ a. s.

then (V_n) converges a. s.

Proof. — By the convergence theorem for alternating series, (V_n) is a C-sequence. Also we notice that for all n

$$\left| \sum_{1}^{n} (E(V_{k+1}/\mathscr{F}_k) - V_k) \right| \leqq | E(V_2/\mathscr{F}_1) - V_1 |$$

and therefore theorem 4 applies.

We now try to weaken L_1 bounded conditions such that

$$\sup_n E(V_n^-) < \infty \quad \text{or} \quad \sup_n E(|V_n|) < \infty$$

THEOREM 5. — Let (V_n) be an adapted sequence of positive random variables. If (V_n) is a strict C-sequence, then (V_n) converges a. s. Conversely if (V_n) converges a. s., then $(\tilde{V}_n)$ converges a. s. on the set $\{ \sup_n \tilde{V}_n < \infty \}$.

Proof. — Write $E(V_{n+1}/\mathscr{F}_n) = (E(V_{n+1}/\mathscr{F}_n) - V_n)$ and apply theorem 2.

COROLLARY 5.1. — Let (V_n) be an adapted sequence of non negative random variables. Then (V_n) converges a. s. if any one of the following conditions is satisfied

1) $E(V_{n+1}/\mathscr{F}_n) \geqq V_n$ and $\sum_1^\infty (E(V_{n+1}/\mathscr{F}_n) - V_n) < \infty$ a. s.
2) $\sum_1^\infty | E(V_{n+1}/\mathscr{F}_n) - V_n | < \infty$ a. s.
3) For almost all ω, there exist an integer $k(\omega)$ such that
   a) $E(V_{n+1}/\mathscr{F}_n) - V_n$ is alternating when $n \geqq k(\omega)$
   b) $| E(V_{n+1}/\mathscr{F}_n) - V_n | \downarrow 0$ when $n \geqq k(\omega)$.

As an example where this corollary can be used (see [*1*]) take the unit interval with its Borel field and Lebesgue measure and set $V_i = i2^i$ on $\left[0, \frac{1}{2^i}\right]$, 0 elsewhere if i is odd, $\equiv 0$ if i is even.

We now rid ourselves of the hypothesis that the V_n's are positive.

THEOREM 6. — Let (V_n) be an adapted sequence of integrable random variables. Then on the set

$$B = \{ \sup_n \tilde{V}_n^+ < \infty , \sup_n \tilde{V}_n^- < \infty \}$$

(V_n) converges a. s. if and only if $(\tilde{V}_n^+)$ and $(\tilde{V}_n^-)$ converge a. s.

If any two of the four sequences (V_n), (V_n^+), (V_n^-), $(|V_n|)$ are strict C-sequences, then (V_n) converges a. s.

The proof goes very much along the lines of that of Theorem 5.

COROLLARY 6.1. — Let (V_n) be a submartingale. If $(\tilde{V}_n)$ and $(\tilde{V}_n^+)$ converge a. s., then so does (V_n).

Remark. — The conditions in this corollary are weaker than the usual condition $\sup_n E(V_n^+) < \infty$ as can be seen by considering the sequence (V_n) defined on the unit interval by the formula

$$V_n = n2^n 1_{[0,2^{-n}]}$$

COROLLARY 6.2. — Any of the following conditions is sufficient for the almost sure convergence of the martingale (V_n):

i) $\sum_1^\infty [E(|V_{j+1}|/\mathscr{F}_j) - |V_j|] < \infty$ a. s.

ii) $\sum_1^\infty [E(V_{j+1}^+/\mathscr{F}_j) - V_j^+] < \infty$ a. s.

iii) $\sum_1^\infty [E(V_{j+1}^-/\mathscr{F}_j) - V_j^-] < \infty$ a. s.

We now show that asymptotic martingales (« amarts », see [*2*] (*b*) and [*7*]) are not necessarily C-sequences nor are C-sequences necessarily asymptotic martingales. As a matter of fact, the C-sequence defined in the remark following corollary 6.1 is not an asymptotic martingale.

Let (X_n) be a sequence of independent identically distributed random variables such that $|X_n| < 1$. Let (a_n) be a sequence of real numbers diverging to ∞ so slowly that $\sum_1^n \frac{X_i}{a_i}$ does not converge in $\bar{\mathbb{R}}$ (this is possible by the law of iterated logarithm (see [*9*])). Then $\left(V_n = \frac{X_n}{a_n}\right)$ is an asymptotic martingale since V_n converges uniformly to 0. Writing

$$V_n = \sum_{j=1}^{n} \frac{X_j}{a_j} - \sum_{j=1}^{n-1} \frac{X_j}{a_j}$$

and using the uniqueness of the compensator it is seen that (V_n) is not a C-sequence.

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REFERENCES

[1] D. G. Austin, G. A. Edgar, A. Ionescu Tulcea, Pointwise convergence in term of expectation. *Z. Wahrscheinlichkeitstheorie verw. Gebiete*, t. **30**, 1974, p. 17-26.

[2] [a] A. Bellow, On vector valued asymptotic martingales. *Proc. Nat. Acad. Sci. U. S. A.*, t. **73**, n° 6, 1976, p. 1798-1799.
[b] A. Bellow, Several stability properties of the class of asymptotic martingales. *Z. Wahrscheinlichkeitstheorie verw. Gebiete*, t. **37**, 1977, p. 275-290.

[3] A. N. Borodin, Quasi-martingales. Theory of Probability and Appl. 1978-3.

[4] Dan Anbav, An application of a theorem of Robbins and Siegmund. *Annals of Statistics*, t. **4**, n° 5, 1976, p. 1018-1021

[5] G. A. Edgar, L. Sucheston, Amarts, A class of asymptotic martingales (Discrete parameter), *J. Multivariate Anal.*, t. **6**, 1976, p. 193-221.

[6] J. Neveu, a) *Mathematical foundations of the calculus of probability*, Holden Day, San Fransisco, 1965; b) *Martingales discrètes*, Masson, 1972.

[7] M. M. Rao, Quasi-martingales. *Math. Scand.*, t. **24**, 1969, p. 79-92.

[8] H. Robbins, D. Siegmund, A convergence theorem for non negative almost supermartingales and some applications. *Optimization methods in statistics*, A. P. New-York, 1971, p. 233-257.

[9] R. J. Tomkins, A law of the Iterated Logarithm Logarithm for martingales. *Z. Wahrscheinlickeitstheorie verw. Gebiete*, t. **33**, 1975, p. 55-59.

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