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A new identity for real Lévy processes

by

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RÉSUMÉ. — Étant donné un processus de Lévy réel $(Z_t)_{t \geqq 0}$ de processus maximal $\overline{Z}_t = \sup \{ Z_s ; s \leqq t \}$, nous étudions les excursions du processus de Markov fort $Z_t - \overline{Z}_t$ en dehors de zéro. Nous montrons que la loi jointe de $((Z - \overline{Z})(\xi -), Z(\xi) - Z(\xi -))$ où ξ désigne le temps de vie de l'excursion a une forme particulièrement simple, et nous en déduisons la transformée de Laplace de $\overline{Z}_T$ où T désigne une variable aléatoire exponentielle indépendante de Z.

1. INTRODUCTION

Let $(Z_t)_{t \geqslant 0}$ be a real Lévy process, with maximum process

$$\overline{Z}_t = \sup \{ Z_s ; s \leqslant t \} .$$

As is well known, $Z_t - \overline{Z}_t$ is a strong Markov process in $(-\infty, 0]$ (see, for example, Bingham [*1*] p. 709); in this paper, we study the excursions of $Z - \overline{Z}$ away from zero. The (excursion) law of most functionals of the excursions of $Z - \overline{Z}$ are not explicitly calculable, but we show that the joint law of $((Z - \overline{Z})(\xi -), Z(\xi) - Z(\xi -))$ has a particularly simple form. Here, ξ is the lifetime of the excursion; informally speaking, the second random variable is the amount by which the excursion overshoots when Z attains a new maximum, and the first measures where the excursion jumped from.

Using this, we can easily prove the identity.

$$(1)\quad \mathrm{E}e^{-\lambda \underline{Z}_T} = \eta\left[\eta + \kappa_+\lambda + \int_{(-\infty,0]} P(\underline{Z}_T \in dy)\int_{-y}^{\infty}\mu(dx)\{1-e^{-\lambda(x+y)}\}\right]^{-1}$$

where T is an exp (η) random variable independent of Z, μ is the Lévy measure of Z, κ_+ is a non-negative real, $\underline{Z}_t \equiv \inf\{Z_s; s \leqslant t\}$ and $\mathrm{Re}\,\lambda \geqslant 0$.

This identity has already been exploited in Rogers [*7*]. The constant κ_+ has the interpretation of the « upward creep » of Z, and in the final section of this paper, we use (1) to deduce a number of results of Millar [*6*], who first considered the problem of deciding when the Lévy process can creep upward.

2. EXCURSIONS OF A LÉVY PROCESS FROM ITS MAXIMUM

Let $(Z_t)_{t\geqslant 0}$ be a real Lévy process with characteristic exponent ψ; for $\mathrm{Re}\, s = 0$,

$$\mathrm{E}e^{sZ_t} = \exp t\psi(s)$$

$$(2)\quad = \exp t\left\{cs + \frac{1}{2}\sigma_0^2 s^2 + \int_{|x|>1}(e^{sx}-1)\mu(dx) + \int_{|x|\leqslant 1}(e^{sx}-1-sx)\mu(dx)\right\}$$

where c, σ_0 are reals, μ a σ-finite measure on $\mathbb{R}$ such that

$$\int(|x|^2 \wedge 1)\mu(dx) < \infty .$$

When $\int(|x| \wedge 1)\mu(dx) < \infty$, we can and shall re-express the characteristic exponent as

$$(3)\qquad \psi(s) = as + \frac{1}{2}\sigma_0^2 s^2 + \int(e^{sx} - 1)\mu(dx) .$$

We define $\overline{Z}_t \equiv \sup\{Z_s; s \leqslant t\}$, and $\underline{Z}_t \equiv \inf\{Z_s; s \leqslant t\}$.

The following is the analogue of a well-known (and pictorially obvious!) result for discrete time random walks (see, for example, Feller [*2*]). It plays a crucial role in the Wiener-Hopf factorization, (see Greenwood-Pitman [*4*]) and in our own development.

DUALITY LEMMA. — Fix $T > 0$. Then the processes $\{Z_t; 0 \leqslant t \leqslant T\}$ and

$$\{Z_t^*; 0 \leqslant t \leqslant T\} \equiv \{Z_{T-} - Z_{(T-t)-}; 0 \leqslant t \leqslant T\}$$

have the same laws.

Proof. — Both processes start at 0, and have cadlag sample paths, so it is enough to show that for bounded continuous f_i,

$$E\prod_{i=1}^{n} f_i(\Delta_i Z) = E\prod_{i=1}^{n} f_i(\Delta_i Z^*)$$

where $\Delta_i Z = Z_{t_i} - Z_{t_{i-1}}$, $0 = t_0 < t_1 < \ldots < t_n = T$. But the jump times of Z are totally inaccessible and at most countable in number, so $P(Z^*_{t_i} = Z^*_{t_i -}$ for all $i) = 1$, and the result follows immediately from the independence and stationarity of the increments of Z.

Remark. — It is obvious that the statement of the duality lemma is true for random times T independent of Z, and it is in this form that we shall principally use it.

The duality lemma allows us easily to deduce other consequences. In particular, we have the following.

COROLLARY 1. — Suppose 0 is regular for $(-\infty, 0)$. Then

i) $P(\text{for some } t,\ Z_t > Z_{t-} = \overline{Z}_{t-}) = 0$;

ii) $P(\text{for some } t,\ Z_t < Z_{t-} = \overline{Z}_{t-}) = 0$.

Proof. — Both parts follow once we can show for each $T > 0$, $\varepsilon > 0$

$$P(\text{for some } 0 \leqslant t \leqslant T,\ |Z_t - Z_{t-}| > \varepsilon,\ Z_{t-} = \overline{Z}_{t-}) = 0.$$

By the duality lemma, this probability is the same as

$$P(\text{for some } 0 \leqslant t \leqslant T,\ |Z^*_t - Z^*_{t-}| > \varepsilon,\ Z^*_t \leqslant Z^*_u \text{ for all } u \geqslant t):$$

since there is a discrete sequence $(\tau_i)_{i \geqslant 0}$ of times when $|Z^*_t - Z^*_{t-}| > \varepsilon$, each a stopping time for the natural filtration of Z^*,

$$\{ Z^*_{\tau_i + t} - Z^*_{\tau_i} ;\ 0 \leqslant t \leqslant T - \tau_i \}$$

has the same law as the initial part of $(Z_t)_{t \geqslant 0}$. But as 0 is a regular for $(-\infty, 0)$ (i. e. $P(\inf\{t ; Z_t < 0\} = 0) = 1$), this implies that almost surely $Z^*_{\tau_i + u} < Z^*_{\tau_i}$ for some $u > 0$, completing the proof.

Remarks. — The first part says that whenever a jump of Z comes which increases $\overline{Z}$, the jump has to have started from below $\overline{Z}$; *the maximum cannot increase by jumps which start from the maximum.* The second part of the Corollary says that *all excursions of* $Z - \overline{Z}$ *from zero will exit zero continuously.* The example of a compound Poisson process with upward drift shows that neither conclusion is valid without assuming 0 regular for $(-\infty, 0)$.

Before stating the first main result, we set up notation. With a graveyard ∂ adjoined to $\mathbb{R}$, we let V be the *space of excursions of* $Z - \overline{Z}$; the elements of V are those maps $\rho : [0, \infty) \to \mathbb{R} \cup \{ \partial \}$ such that for some $0 < \xi \leq \infty$, $\rho(t) = \partial$ for $t > \xi$, ρ is cadlag on $[0, \xi]$, $\rho(t) < 0$ for $0 < t < \xi$. Under the Skorokhod topology V is Polish, and we equip it with its Borel σ-fields. The positive ξ appearing in the definition of ρ is called the *excursion lifetime* of ρ.

It is important to realize that we do *not* insist that $\rho(\xi) \leq 0$; although an excursion of $Z - \overline{Z}$ hits zero at lifetime ξ, it may reach zero by Z jumping across the level of its previous maximum, and we do not want to lose the information about this jump. Indeed, we define maps $q : V \to (-\infty, 0]$, $r : V \to \mathbb{R}$ by

$$q(\rho) \equiv \rho(\xi -)$$

and

$$r(\rho) \equiv \rho(\xi) - \rho(\xi -)$$

which give us respectively the position of the excursion just before the lifetime, and the jump (if any) by which the excursion is terminated.

Let us *assume* 0 *is regular for* $(0, \infty)$ *for* Z, so that 0 is regular for $\{ 0 \}$ for $Z - \overline{Z}$, and there exists a local time $(L_t)_{t \geq 0}$, a continuous increasing additive functional of $Z - \overline{Z}$ whose set of growth points is $\{ t ; Z_t - \overline{Z}_t = 0 \}$ a. s., and whose right continuous inverse A_t is a subordinator. Moreover, there exists a σ-finite measure ν on V, the *excursion measure*, such that $Z - \overline{Z}$ can be decomposed into a Poisson point process with values in V with characteristic measure ν, as Itô [*5*] proved.

Now introduce an exponential random time T, $P(T > t) = e^{-\eta t}$, independent of Z, and kill Z at time T. We define the excursion measure ν^η of the killed process by

$$\nu^\eta(\{ \rho_{t_1} \in A_1, \ldots, \rho_{t_{n-1}} \in A_{n-1}, \rho_{t_n} \geqslant 0, \xi \in dt_n \}) = e^{-\eta t_n} \nu(\{ \rho_{t_1} \in A_1, \ldots, \rho_{t_{n-1}} \in A_{n-1}, \xi \in dt_n \})$$

$$\nu^\eta(\{ \rho_{t_1} \in A_1, \ldots, \rho_{t_n} \in A_n, \xi \in dt_n \}) = \eta e^{-\eta t_n} \nu(\{ \rho_{t_1} \in A_1, \ldots, \rho_{t_n} \in A_n \}) dt_n$$

where $0 < t_1 < \ldots < t_n$, $A_i \in \mathcal{B}((-\infty, 0))$.

Not only is the right continuous inverse A_t of local time L_t a subordinator but, as Fristedt [*3*] shows, the process (A_t, Y_t) is a bivariate subordinator. Here, $Y_t \equiv Z(A_t) = \overline{Z}(A_t)$. We let ϕ denote the bivariate Laplace exponent of (A_t, Y_t);

$$E \exp(-\eta A_t - \lambda Y_t) \equiv \exp\{ -t\phi(\eta, \lambda) \}$$

for $\eta, \lambda > 0$, $t \geq 0$. Fristedt presents an expression for ϕ analogous to the Spitzer-Rogozin identity (see [*3*], Corollary 9.2, and Greenwood-Pitman [*4*]). Here now is the main result.

THEOREM 1. — Suppose 0 is regular for $(0, \infty)$. Then there exists $c_\eta > 0$ such that for $0 < -y < t$,

$$(4) \qquad \begin{aligned} \nu^\eta(\{\rho ; q(\rho) \in dy, r(\rho) = 0\}) &= \eta c_\eta \mathrm{P}(\underline{Z}_\mathrm{T} \in dy) \\ \nu^\eta(\{\rho ; q(\rho) \in dy, r(\rho) \in dt\}) &= \mu(dt) c_\eta \mathrm{P}(\underline{Z}_\mathrm{T} \in dy) \end{aligned}$$

Moreover, the constant c_η satisfies

$$(5) \qquad \eta c_\eta = \phi(\eta, 0).$$

Proof. — Since the excursion law ν is in some sense a limit as $x \uparrow 0$ of the law of Z started at $x < 0$ killed when it enters $\mathbb{R}^+$, it makes sense to investigate the process Z stated at $x < 0$. Let $\tau \equiv \inf\{u > 0 ; Z_u \geq 0\}$, and let $\zeta \equiv \tau \wedge \mathrm{T}$. The key observation is that if $g : \mathbb{R}^2 \to [0, 1]$ is any measurable function (such that $g(0, 0) = 0$), then

$$\mathrm{M}_t \equiv g(Z_\zeta, Z_{\zeta-}) \mathrm{I}_{\{t \geq \zeta\}} - \int_0^{t \wedge \zeta} h(Z_s) ds$$

is a martingale, where for $y < 0$,

$$h(y) \equiv \eta g(y, y) + \int_{-y}^{\infty} \eta(dx) g(x + y, y).$$

Accordingly,

$$\mathrm{E}^x g(Z_\zeta, Z_{\zeta-}) = \mathrm{E}^x \int_0^\zeta h(Z_t) dt = \int_{-\infty}^0 \mathrm{G}_\eta(x, dy) h(y),$$

where

$$\mathrm{G}_\eta(x, y) \equiv \mathrm{E}^x \left[\int_0^\zeta \mathrm{I}_{\{Z_s \leq y\}} ds \right].$$

Hence by choosing g suitably we have immediately that for $x < 0$,

$$(6) \qquad \begin{aligned} \mathrm{P}^x(Z_\zeta = Z_{\zeta-} \in dy) &= \eta \mathrm{G}_\eta(x, dy) && (y < 0) \\ \mathrm{P}^x(Z_{\zeta-} \in dy, Z_\zeta - Z_{\zeta-} \in dt) &= \mu(dt) \mathrm{G}_\eta(x, dy) && (0 < -y < t). \end{aligned}$$

Now according to Itô, under ν^η, the excursion of $Z - \overline{Z}$, once in $(-\infty, 0)$, evolves just like Z killed at time ξ. Specifically,

$$\int_\mathrm{V} \nu^\eta(d\rho) f(\rho_{u+s}, 0 \leq s \leq \xi - u) \mathrm{I}_{\{\xi > u\}} = \int_{-\infty}^0 \nu^\eta(\rho_u \in dx) \mathrm{E}^x f(Z_s ; 0 \leq s \leq \zeta)$$

for any bounded measurable function f on V, and any $u > 0$. Thus from (6) it follows immediately that for $0 < -y < t$,

$$(7) \quad \nu^\eta(\{\xi > u, q(\rho) \in dy, r(\rho) \in dt\}) = \int_{-\infty}^0 \nu^\eta(\rho_u \in dx) \mathrm{G}_\eta(x, dy) \mu(dt)$$

But

$$\int_{-\infty}^{0} \nu^\eta(\rho_u \in dx) G_\eta(x, y) = \int_V \nu^\eta(d\rho) \int_u^\xi I_{\{\rho_s \le y\}} ds ,$$

so letting $u \downarrow 0$, and defining $G_\eta(y)$ by

$$G_\eta(y) \equiv \int_V \nu^\eta(d\rho) \int_0^\xi I_{\{\rho_s \le y\}} ds \qquad (y < 0), \tag{8}$$

(7) and (8) yield immediately

$$\begin{aligned} \nu^\eta(\{ \rho ; q(\rho) \in dy, r(\rho) = 0 \}) &= \eta G_\eta(dy) . \\ \nu^\eta(\{ \rho ; q(\rho) \in dy, r(\rho) \in dt \}) &= \mu(dt) G_\eta(dy) . \end{aligned} \tag{9}$$

All that remains is to identify the measure G_η, which we do with the aid of the duality lemma.

Consider the random variable $H \equiv Z_T - \overline{Z}_T$. When less than zero, it is the value of $\rho(\xi)$ for the first excursion of $Z - \overline{Z}$ which is still in $(-\infty, 0)$ at the lifetime. Thus from the description of the Poisson point process of excursions,

$$P(H \le y \mid H < 0) = G_\eta(y)/G_\eta(0-)$$

However, it is clear from a picture that

$$H \equiv Z_T - \overline{Z}_T = \underline{Z}_T^* ,$$

so, by the duality lemma

$$\begin{aligned} P(H \le y \mid H < 0) &= P(\underline{Z}_T^* \le y \mid \underline{Z}_T^* < 0) \\ &= P(\underline{Z}_T \le y \mid \underline{Z}_T < 0) . \end{aligned}$$

Hence

$$G_\eta(y) = P(\underline{Z}_T \le y) G_\eta(0-)/P(\underline{Z}_T < 0) . \tag{10}$$

Now A_t is a subordinator whose jumps are excursion intervals of $Z - \overline{Z}$; it is also possible that A_t might have some drift $b \ge 0$ (which is positive iff 0 is not regular for $(-\infty, 0)$), so the Laplace exponent $\phi(\eta, 0)$ of A_t is

$$\begin{aligned} \phi(\eta, 0) &= b\eta + \int_V \nu(d\rho)(1 - e^{-\eta\xi}) \\ &\equiv b\eta + \eta G_\eta(0-) \end{aligned} \tag{11}$$

from (8). Hence

$$P(\underline{Z}_T < 0) = \frac{\eta G_\eta(0-)}{b\eta + \eta G_\eta(0-)} = \eta G_\eta(0-)/\phi(\eta, 0) ,$$

which together with (10) identifies $\eta G_\eta(dy)$ as $\phi(\eta, 0)P(\underline{Z}_T \in dy)$, as required.

The arguments just used to prove Theorem 1 also provide us with a means of characterising more fully the bivariate Laplace exponent ϕ.

COROLLARY 2. — Suppose 0 is regular $(0, \infty)$. Then there exists $k \geq 0$ such that for all $\eta, \lambda > 0$,

$$(12) \quad \phi(\eta, \lambda) = \phi(\eta, 0) + k\lambda + \eta^{-1}\phi(\eta, 0)\int_{(-\infty,0]} P(\underline{Z}_T \in dy)\int_{-y}^{\infty} \mu(dx)[1 - e^{-\lambda(x+y)}].$$

Proof. — We begin by supposing also that 0 is regular for $(-\infty, 0)$. By Corollary 1 (*i*), the maximum never increases by jumps which start from the maximum, so the transform of the Lévy measure of (A_t, Y_t) is

$$\int_V \nu(d\rho)(1 - e^{-\eta\xi - \lambda\rho_\xi})$$

and the Laplace exponent differs from this only by drift terms; thus for some $b, k \geq 0$,

$$\begin{aligned}
\phi(\eta, \lambda) &= b\eta + k\lambda + \int_V \nu(d\rho)(1 - e^{-\eta\xi - \lambda\rho_\xi}) \\
&= b\eta + \int_V \nu(d\rho)(1 - e^{-\eta\xi}) + k\lambda + \int_V \nu(d\rho)e^{-\eta\xi}(1 - e^{-\lambda\rho_\xi}) \\
&= \phi(\eta, 0) + k\lambda + \int_V \nu^\eta(d\rho)(1 - e^{-\lambda\rho_\xi^+})
\end{aligned}$$

from (11);

$$= \phi(\eta, 0) + k\lambda + \int_{(-\infty,0)} G_\eta(dy)\int_{-y}^{\infty} \mu(dx)[1 - e^{-\lambda(x+y)}].$$

But $G_\eta(dy) = \eta^{-1}\phi(\eta, 0)P(\underline{Z}_T \in dy)$, establishing (12), at least when 0 is regular for $(-\infty, 0)$. To deduce (12) without this restriction, replace Z_t by $Z_t^{(n)} \equiv Z_t + n^{-1}W_t$, where W_t is a Brownian motion independent of Z; for $Z^{(n)}$, 0 is regular for $(0, \infty)$ and $(-\infty, 0)$. Then (12) is valid for $Z^{(n)}$, and $\underline{Z}_T^{(n)} \to \underline{Z}_T$ a. s., $Z_t^{(n)} \to Z_t$ a. s. for each t. Fristedt's identity ([*3*], Corollary 9.2) states

$$\phi(\eta, \lambda) = \exp\left[\int_0^\infty t^{-1}dt\int_{[0,\infty)} \{e^{-t} - e^{-\eta t - \lambda x}\} P(Z_t \in dx)\right],$$

so $\phi^{(n)}(\eta, \lambda) \to \phi(\eta, \lambda)$ provided $P(Z_t = 0) = 0$ for almost all t. But $\int_0^\infty P(Z_t = 0)dt > 0$ only if Z is a compound Poisson process with zero drift, and this is ruled out by the assumption that 0 is regular for $(0, \infty)$.

Thus the left sides of (12) are converging to the correct limit, as is the first term on the right. We show that the third term on the right of (12) converges to the correct limit, which establishes the result. If $\sigma_0 \neq 0$ (non-zero Gaussian component) or if $\int (|x| \wedge 1)\mu(dx) = +\infty$ (jumps of unbounded variation) then 0 is regular for $(0, \infty)$ and $(-\infty, 0)$; see Rogozin [8]. Thus the Gaussian part of Z must vanish, and its jumps must be of bounded variation. But in that case the map

$$y \mapsto \int_{-y}^{\infty} \mu(dx)[1 - e^{-\lambda(x+y)}]$$

is, for each $\lambda > 0$, bounded and continuous, and $\underline{Z}_T^{(n)} \to \underline{Z}_T$ a. s. and therefore $\underline{Z}_T^{(n)} \to \underline{Z}_T$ in law. Convergence of the third terms on the right of (12) follows immediately.

We exploit this in the next result. The characterization of the law of $\overline{Z}_T$ due to Rogozin (see Rogozin [8] and Greenwood-Pitman [4]) is well-established; we now present another identity which is every bit as useful in practice.

THEOREM 2. — For any Lévy process Z, there exists $\kappa \geq 0$ (depending on η) such that for all λ, Re $\lambda \geq 0$,

$$\text{(13)} \quad Ee^{-\lambda \overline{Z}_T} = \frac{\eta}{\eta + \kappa\lambda + \int_{(-\infty,0]} P(\underline{Z}_T \in dy) \int_{-y}^{\infty} \mu(dx)\{1 - e^{-\lambda(x+y)}\}}$$

Proof. — Suppose firstly that 0 is regular for $(0, \infty)$. Then

$$\text{(14)} \quad Ee^{-\lambda Z_T} = \phi(\eta, 0)/\phi(\eta, \lambda)$$

(See Bingham [1], Fristedt [3], or, for the excursion interpretation, Greenwood-Pitman [4]), so (13) follows from (12) if we write $\kappa \equiv \eta k/\phi(\eta, 0)$. The case when 0 is not regular for $(0, \infty)$ follows from the first by approximating Z_t by $Z_t + n^{-1}W_t$ as before; we leave the details to the reader.

Example. — Suppose Z is spectrally negative (i. e. $\mu(0, \infty) = 0$), and $-Z$ is not a subordinator. Then the identity (13) collapses to the well-known result that $\overline{Z}_T$ has an exponential distribution;

$$Ee^{s\overline{Z}_T} = \eta[\eta - \kappa s]^{-1} \qquad (\text{Re } s = 0)$$

What can we say about the parameter of the distribution? It is easy to show under our assumptions that ψ is well defined in Re $s \geq 0$, and there exists a unique $\sigma > 0$ such that $\psi(\sigma) = \eta$ (see, for example, Bingham [1] §4). Now the classical Wiener-Hopf factorization of Z implies

$$\mathrm{E}e^{s\underline{Z}_T} = \frac{\eta}{\eta - \psi(s)}(\mathrm{E}e^{s\overline{Z}_T})^{-1} = \frac{\eta - \kappa s}{\eta - \psi(s)}.$$

The left side is analytic and bounded in Re $s > 0$, so the right side is also. But the denominator of the right side has a zero at σ, so the numerator must also vanish there; $\kappa = \sigma/\eta$.

Example. — We shall assemble some well-known results to deduce an attractive identity of Silverstein [9] from Theorem 1.

Assume 0 is regular for $(-\infty, 0)$ and $(0, \infty)$. Then the process $Z - \underline{Z}$ has a local time process L^- with right continuous inverse A^- and, as Fristedt [3] shows, $(A_t^-, -Z(A_t^-))$ is a bivariate subordinator, whose Laplace exponent we denote by ϕ^- :

$$\mathrm{E}\exp(-\eta A_t^- + \lambda Z(A_t^-)) = \exp(-t\phi^-(\eta, \lambda)).$$

Silverstein establishes the formula

$$\int_V \nu(d\rho)\left(\int_0^\xi e^{-\eta s + \lambda \rho_s} ds\right) = 1/\phi^-(\eta, \lambda). \tag{15}$$

For us, the left hand side of (15) is

$$\int_V \nu^\eta(d\rho)\int_0^\xi e^{\lambda\rho_s}ds = \int_{(-\infty,0)} G_\eta(dy)e^{\lambda y}$$

by definition of G_η;

$$= \eta^{-1}\phi(\eta, 0)\mathrm{E}e^{\lambda\underline{Z}_T}$$

since

$$\eta G_\eta(dy) = \phi(\eta, 0)P(\underline{Z}_T \in dy)$$

and $P(\underline{Z}_T = 0) = 0$, because 0 is assumed regular for $(-\infty, 0)$.

Further, as in (14)

$$\mathrm{E}e^{\lambda\underline{Z}_T} = \phi^-(\eta, 0)/\phi^-(\eta, \lambda)$$

so that the left side of (15) is

$$\eta^{-1}\phi(\eta, 0)\phi^-(\eta, 0)/\phi^-(\eta, \lambda)$$

and Silverstein's formula is proved, since $\phi(\eta, 0)\phi^-(\eta, 0) = \eta$ (see Bingham [1] p. 713).

Remark. — We could have advanced backwards through the above argument to deduce part of Theorem 1, but it does not seem so natural!

3. CREEPING OF LÉVY PROCESSES

As an application of Theorem 2, we show how to deduce a number of results of Millar [6] on the creeping of a Lévy process. Millar poses the question, « If $T_x \equiv \inf\{t > 0 ; Z_t > x\}$, $x > 0$, when is

$$P(Z(T_x) = x) > 0 ? ».$$

As Kesten observed, this is equivalent to deciding whether the subordinator $Y_t \equiv \overline{Z}(A_t)$ has a positive drift; intuitively, the problem is to deduce whether or not a Lévy process can « creep » across new levels, or whether it must jump over.

Now, by Theorem 2, for Re $s = 0$,

$$\mathrm{E}e^{s\overline{Z}_T} = \eta[\eta - \kappa_+ s + \hat{\gamma}_+(s)]^{-1}, \tag{16}$$

where

$$\begin{aligned} \hat{\gamma}_+(s) &\equiv \int_0^\infty \gamma_+(dx)(1 - e^{sx}) \\ &= \int_{-\infty}^0 P(\underline{Z}_T \in dy) \int_{-y}^\infty \mu(dx)[1 - e^{s(x+y)}]. \end{aligned}$$

This follows from (13); notice that, since γ_+ is a Lévy measure on $(0, \infty)$, $\hat{\gamma}_+(s) = o(|s|)$ as $|s| \to \infty$. Similarly, there exist $\kappa_- \geq 0$, and a Lévy measure γ_- on $(-\infty, 0)$ such that

$$\mathrm{E}e^{s\underline{Z}_T} = \eta[\eta + \kappa_- s + \hat{\gamma}_-(s)]^{-1}. \tag{17}$$

To decide whether or not $\kappa_+ > 0$, we may suppose that μ is supported in $[-1, 1]$; indeed, as Millar shows, by throwing away the big jumps of Z we obtain a Lévy process which agrees with Z up to the time of the first big jump, and the creeping of Z is decided by the behaviour of the process before this time (see Corollary 3.4 of Millar's paper). We shall thus assume that μ is supported in $[-1, 1]$, and that the characteristic exponent of Z is

$$\psi(s) = cs + \frac{1}{2}\sigma_0^2 s^2 + \int_{|x| \leq 1} \mu(dx)(e^{sx} - 1 - sx). \tag{18}$$

Now the classical Wiener-Hopf factorization of Z,

$$\mathrm{E}e^{s\overline{Z}_T}\mathrm{E}e^{s\underline{Z}_T} = \eta(\eta - \psi(s))^{-1},$$

can be immediately re-expressed in terms of (16), (17), and (18); cross-multiplying gives

$$(19)\quad \eta^2 + \eta(\kappa_- - \kappa_+)s - s^2\kappa_+\kappa_- + \eta\,\hat{\gamma}_-(s) + \eta_+\,\hat{\gamma}_+(s) - s\kappa_+\,\hat{\gamma}_-(s) + s\kappa_-\,\hat{\gamma}_+(s) + \hat{\gamma}_+(s)\,\hat{\gamma}_-(s) = \eta^2 - \eta cs - \frac{1}{2}\eta\sigma_0^2 s^2 - \eta\int\mu(dx)(e^{sx} - 1 - sx).$$

We deduce immediately the following result.

COROLLARY 3. — The Lévy process has positive creep in both directions iff it has non-zero Gaussian component. Specifically,

$$\kappa_+\kappa_- = \frac{1}{2}\eta\sigma_0^2.$$

Proof. — Divide either side of (19) by s^2, and let $|s| \to \infty$. This is Theorems 3.3 and 3.5 of Millar [*6*]. Developing (19) further, we have, on integrating by parts,

$$\hat{\gamma}_+(s) = -s\int_0^\infty \bar{\gamma}_+(x)e^{sx}dx = -s\Gamma_+(0) - s^2\int_0^\infty \Gamma_+(x)e^{sx}dx$$

where $\bar{\gamma}_+(x) \equiv \gamma_+((x, \infty))$, and $\Gamma_+(x) \equiv \int_x^\infty \bar{\gamma}_+(u)du$, both vanishing on $(1, \infty)$. Likewise, integrating by parts in the characteristic exponent,

$$\int_0^1 \mu(dx)(e^{sx} - 1 - sx) = s\int_0^1 \bar{\mu}_+(x)(e^{sx} - 1)dx = s^2\int_0^1 \mathrm{M}_+(x)e^{sx}dx$$

where $\bar{\mu}_+(x) \equiv \mu((x, \infty))$, $\mathrm{M}_+(x) = \int_x^\infty \bar{\mu}_+(u)du$.

Returning this to (19), and identifying measures with the same Fourier transforms, yields the fundamental identities

$$\begin{array}{ll} (20)\ \ i) & \\ (20)\ \ ii) & \\ (20)\ \ iii) & \end{array} \left\{ \begin{array}{ll} c = \kappa_+ - \kappa_- + \Gamma_+(0) - \Gamma_-(0) & \\ \eta\mathrm{M}_+(x) = \eta\Gamma_+(x) + \kappa_-\bar{\gamma}_+(x) + g(x) & (x > 0) \\ \eta\mathrm{M}_-(x) = \eta\Gamma_-(x) + \kappa_+\bar{\gamma}_-(x) + g(x) & (x < 0) \end{array} \right.$$

where $g(x) \equiv (\bar{\gamma}_+ * \bar{\gamma}_-)(x)$ is the convolution of $\bar{\gamma}_+$ with $\bar{\gamma}_-$.

These identities are rich in consequences, only a few of which are explored here. For example, if $\sigma_0 = 0$ and $\int |x| \mu(dx) < \infty$, then

$$\int x\mu(dx) = \mathrm{M}_+(0) - \mathrm{M}_-(0),$$

so using the alternative form (3) of ψ,

$$\eta a = \kappa_+(\eta + \bar{\gamma}_-(0)) - \kappa_-(\eta + \bar{\gamma}_+(0)).$$

The following result is now immediate, since at most one of κ_+, κ_- can be non-zero if $\sigma_0 = 0$ (Corollary 3).

COROLLARY 4. — Suppose $\sigma_0 = 0$, $\int (|x| \wedge 1)\mu(dx) < \infty$. Then $\kappa_+ > 0$ iff $a > 0$, and $\kappa_- > 0$ iff $a < 0$.

We turn now to the final case, where $\sigma_0 = 0$, $\int (|x| \wedge 1)\mu(dx) = +\infty$. Here, no complete characterisation is known, but we have the following result.

THEOREM 3. — Suppose $\sigma_0 = 0$.

i) If $\int_0^\infty (|x| \wedge 1)\mu(dx) < \int_{-\infty}^0 (|x| \wedge 1)\mu(dx) = +\infty$,

then $\kappa_+ > 0 = \kappa_-$.

ii) If $\int_0^\infty (|x| \wedge 1)\mu(dx) = \int_{-\infty}^0 (|x| \wedge 1)\mu(dx) = +\infty$,

then

$$\limsup_{x \downarrow 0} \frac{\mathrm{M}_-(-x)}{\mathrm{M}_+(x)} < \infty \Rightarrow \kappa_+ = 0. \tag{21}$$

Proof. — *i*) The monotone limit

$$\begin{aligned}\lim_{x\downarrow 0} g(x) \equiv \lim_{x\downarrow 0} \int_{-\infty}^0 \bar{\gamma}_-(y)\bar{\gamma}_+(x-y)dy \\ = \int_{-\infty}^0 \bar{\gamma}_-(y)\bar{\gamma}_+(-y)dy\end{aligned}$$

exists and is the same as $\lim_{x\uparrow 0} g(x)$. From (20) (*ii*), since $\mathrm{M}_+(0+) < \infty$, it follows that the limit of $g(x)$ is finite; but we suppose $\mathrm{M}_-(0-) = \infty$, which can only happen if $\kappa_+ > 0$. By Corollary 3, since $\sigma_0 = 0$, we deduce $\kappa_- = 0$.

ii) Suppose $\kappa_+ > 0$. We show $\limsup M_-(-x)/M_+(x) = +\infty$. Firstly, $\lim_{x\downarrow 0} \overline{\gamma}_+(x) = \lim_{x\uparrow 0} \overline{\gamma}(x) = +\infty$, for if not, $g(0) < \infty$, and, by (20) (*ii*), $M_+(0+)$ would be finite, contradicting the assumption. Secondly, since $\Gamma_+(0)$, $\Gamma_-(0)$ are finite, (20) (*ii*)-(*iii*) yield

$$\begin{aligned}\liminf_{x\downarrow 0} M_+(x)/M_-(-x) &= \liminf_{x\downarrow 0} g(x)/\{\kappa_+\overline{\gamma}_-(-x) + g(-x)\}\\ &\leq \liminf_{x\downarrow 0} g(x)/\kappa_+\overline{\gamma}_-(-x),\end{aligned}$$

so it is enough to prove $\liminf_{x\downarrow 0} g(x)/\overline{\gamma}_-(-x) = 0$.

If false, then for some $\delta > 0$, for all small enough x

$$g(x) \equiv \int_{-\infty}^{0} \overline{\gamma}_-(u)\overline{\gamma}_+(x-u)du > \delta\overline{\gamma}_-(-x).$$

However,

$$\begin{aligned}&\frac{1}{\overline{\gamma}_-(-x)}\int_{-\infty}^{-x} \overline{\gamma}_-(u)\overline{\gamma}_+(x-u)du\\ &\quad = \int_{-\infty}^{0} \frac{\overline{\gamma}_-(-x+y)}{\overline{\gamma}_-(-x)}\overline{\gamma}_+(2x-y)dy \leq \int_{-\infty}^{0} \frac{\overline{\gamma}_-(-x+y)}{\overline{\gamma}_-(-x)}\overline{\gamma}_+(-y)dy\end{aligned}$$

since $\overline{\gamma}_+$ is decreasing;

$\to 0 \quad$ as $x \downarrow 0$

by dominated convergence. So we may suppose that for all small enough x,

$$\int_{-x}^{0} \overline{\gamma}_-(u)\overline{\gamma}_+(x-u)du \geq \delta\overline{\gamma}_-(-x). \tag{22}$$

Integrate both sides with respect to x; the left side is

$$\begin{aligned}\int_0^{\varepsilon} dx \int_{-x}^{0} \overline{\gamma}_-(u)\overline{\gamma}_+(x-u)du &= \int_{-\varepsilon}^{0} \overline{\gamma}_-(u)du \int_{-u}^{\varepsilon} \overline{\gamma}_+(x-u)dx\\ &= \int_{-\varepsilon}^{0} \overline{\gamma}_-(u)du \int_{-2u}^{\varepsilon-u} \overline{\gamma}_+(t)dt \leq \int_{-\varepsilon}^{0} \overline{\gamma}_-(u)du \int_0^{2\varepsilon} \overline{\gamma}_+(t)dt < \delta \int_{-\varepsilon}^{0} \overline{\gamma}_-(u)du\end{aligned} \tag{23}$$

for ε small enough.

The right side is $\delta \int_{-\varepsilon}^{0} \overline{\gamma}_-(u)du$, which is a contradiction of (22), (23).

Thus $\liminf_{x\downarrow 0} g(x)/\overline{\gamma}_-(-x) = 0$, and the Theorem is proved.

Remarks. — *a*) Obviously the companion to statement (*i*), with the integrability conditions on μ_+, μ_- interchanged, is valid

b) The second part includes Millar's Theorem 3.7.

c) The condition of the second part is sufficient for $\kappa_+ = 0$, but cannot be necessary. Indeed, if it were so, we would have to construct a Lévy measure μ for which

$$0 = \liminf \frac{M_-(-x)}{M_+(x)} < \limsup \frac{M_-(-x)}{M_+(x)} = +\infty$$

and then κ_+ and κ_- would both have to be positive, contracting Corollary 2. The reader will be able to manufacture such M_+, M_- with no difficulty.

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