

# ANNALES DE L'I. H. P., SECTION B

MARK A. PINSKY

## **On non-euclidean harmonic measure**

*Annales de l'I. H. P., section B*, tome 21, n° 1 (1985), p. 39-46

[http://www.numdam.org/item?id=AIHPB\\_1985\\_\\_21\\_1\\_39\\_0](http://www.numdam.org/item?id=AIHPB_1985__21_1_39_0)

© Gauthier-Villars, 1985, tous droits réservés.

L'accès aux archives de la revue « Annales de l'I. H. P., section B » (<http://www.elsevier.com/locate/anihpb>) implique l'accord avec les conditions générales d'utilisation (<http://www.numdam.org/conditions>). Toute utilisation commerciale ou impression systématique est constitutive d'une infraction pénale. Toute copie ou impression de ce fichier doit contenir la présente mention de copyright.

NUMDAM

Article numérisé dans le cadre du programme  
Numérisation de documents anciens mathématiques

<http://www.numdam.org/>

## On Non-Euclidean Harmonic Measure

by

**Mark A. PINSKY**

Department of Mathematics, Northwestern University,  
Lunt Hall Evanston, Ill 60201, U.S.A.

---

**ABSTRACT.** — Let  $H_\varepsilon$  be the harmonic measure of the  $\varepsilon$ -sphere in a Riemannian manifold. We compute  $\lim_{\varepsilon \downarrow 0} \frac{H_\varepsilon - H_{0+}}{\varepsilon^3}$  in terms of the gradient of the Gaussian curvature.

**RÉSUMÉ.** — Soit  $H_\varepsilon$  la mesure harmonique de la sphère du rayon  $\varepsilon$  d'une variété riemannienne de dimension deux. On exprime  $\lim_{\varepsilon \downarrow 0} (H_\varepsilon - H_{0+})/\varepsilon^3$  en fonction de la dérivée de la courbure gaussienne.

---

### 1. INTRODUCTION

Let  $(M, g)$  be an  $n$ -dimensional Riemannian manifold with Brownian motion process  $\{X_t, t \geq 0\}$ . According to the Onsager-Machlup formula [3], the « most probable path » is that of a classical particle in a conservative force field whose potential energy is one-twelfth the scalar curvature. This suggests that the « most probable path » follows the negative gradient of the scalar curvature. The purpose of this note is to make this precise in terms of the harmonic measure of a small sphere. For technical reasons we restrict the discussion to surfaces. The technique follows the method used to study the mean exit time [1].

## 2. NOTATIONS AND DEFINITIONS

Let  $(M, g)$  be an  $n$ -dimensional Riemannian manifold. We use the following notations:

$\bar{M}_m$  is the tangent space at  $m \in M$ .

$B_m(\varepsilon)$  is the ball of radius  $\varepsilon$  in  $M$  with center at  $m \in M$ .

$\bar{B}_m(\varepsilon)$  is the ball of radius  $\varepsilon$  in  $\bar{M}_m$  with center at  $\hat{0} \in \bar{M}_m$ .

$\exp_m$  is the exponential mapping (which is defined on all of  $\bar{M}_m$  in case  $M$  is complete; otherwise it is a mapping) from  $\bar{B}_m(\varepsilon)$  to  $B_m(\varepsilon)$  for sufficiently small  $\varepsilon > 0$ .

$\Phi_\varepsilon$  is the mapping on functions defined by

$$(\Phi_\varepsilon f)(\exp_m \varepsilon x) = f(x);$$

$\Phi_\varepsilon$  maps from  $C^\infty(\bar{B}_m(1))$  to  $C^\infty(B_m(\varepsilon))$  for sufficiently small  $\varepsilon > 0$ .

$\Delta$  is the Laplace-Beltrami operator of the Riemannian manifold:

$$\Delta f = \frac{1}{\sqrt{g}} \frac{\partial}{\partial x_i} \left( \sqrt{g} g^{ij} \frac{\partial f}{\partial x_j} \right) \quad \text{where} \quad g^{ij} = (g^{-1})^{ij}, \quad g = \det(g_{ij}).$$

The following result, which will be used repeatedly, was proved in [1].

**PROPOSITION 2.0.** — *There exist second order differential operators  $(\Delta_{-2}, \Delta_0, \Delta_1, \dots)$  on  $C^\infty(\bar{M}_m)$  such that for each  $N \geq 0$  and each  $f \in C^\infty(\bar{M}_m)$*

$$(2.1) \quad \Phi_\varepsilon^{-1} \Delta \Phi_\varepsilon f = \varepsilon^{-2} \Delta_{-2} f + \sum_{j=0}^N \varepsilon^j \Delta_j f + o(\varepsilon^{N+1}) \quad (\varepsilon \downarrow 0).$$

$\Delta_j$  maps polynomials of degree  $k$  to polynomials of degree  $k + j$ . In any normal coordinate chart  $(x_1, \dots, x_n)$  we have

$$(2.2) \quad \Delta_{-2} f = \sum_{i=1}^n \frac{\partial^2 f}{\partial x_i^2}$$

$$(2.3) \quad \Delta_0 f = (1/3) \sum_{i,a,j,b=1}^n R_{iajb} x_a x_b \frac{\partial^2 f}{\partial x_i \partial x_j} - (2/3) \sum_{i,a=1}^n \rho_{ia} x_a \frac{\partial f}{\partial x_i}$$

Here  $R_{iajb}$  is the Riemann tensor and  $\rho_{ij} = \sum_{a=1}^n R_{iaja}$  is the Ricci tensor at  $m \in M$ .

Let  $(X_t, P_x)$  be the Brownian motion process with infinitesimal generator  $\Delta$ . For each  $m \in M$  let  $T_\varepsilon$  be the exit time from the geodesic ball  $B_m(\varepsilon)$ .

In case  $n = 2$  we introduce geodesic polar coordinates  $(r, \theta)$  by  $x_1 = r \cos \theta$ ,  $x_2 = r \sin \theta$ . The coordinate formulas for the first three operators become

$$(2.4) \quad \Delta_{-2} = \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2}$$

$$(2.5) \quad \Delta_0 = -\frac{K_0}{3} \left( r \frac{\partial}{\partial r} - \frac{\partial^2}{\partial \theta^2} \right)$$

$$(2.6) \quad \Delta_1 = - (K_1 \cos \theta + K_2 \sin \theta) \left[ \frac{r^2}{4} \frac{\partial}{\partial r} - \frac{r}{6} \frac{\partial^2}{\partial \theta^2} \right] \\ - (K_2 \cos \theta - K_1 \sin \theta) \left[ \frac{r}{12} \frac{\partial}{\partial \theta} \right]$$

where  $K$  is the Gaussian curvature and  $K_0 = K(m)$ ,  $K_1 = \frac{\partial K}{\partial x_1}(m)$ ,  $K_2 = \frac{\partial K}{\partial x_2}(m)$ . This computation is carried out in the Appendix, Sec. 5.

### 3. STATEMENT OF RESULTS

To study the harmonic measure we let  $f \in C^\infty(S_1)$ , a smooth function on the circle. Extend  $f$  to  $\mathbb{R}^2 \setminus \{0\}$  by making  $f$  constant along rays through the origin. Equivalently  $f$  can be thought of as a function on  $\overline{M}_m \setminus \{0\}$ . Then  $\Phi_\varepsilon f$  is a smooth function on  $M \setminus \{m\}$  which is constant along geodesics emanating from  $m$ . The harmonic measure operator is

$$H_\varepsilon f(x) = E_x \{ (\Phi_\varepsilon f)(X_{T_\varepsilon}) \} \quad \varepsilon > 0, \quad x \in B_m(\varepsilon).$$

**THEOREM 3.1.** — *When  $\varepsilon \downarrow 0$  we have*

$$H_\varepsilon f(m) = \int_{-\pi}^{\pi} f(\theta) \left[ 1 - \frac{\varepsilon^3}{32} \langle \nabla K, u_\theta \rangle \right] \omega(d\theta) + O(\varepsilon^4)$$

where  $\omega$  is normalized Lebesgue measure and  $\langle \nabla K, u_\theta \rangle = K_1 \cos \theta + K_2 \sin \theta$ ,  $-\pi \leq \theta \leq \pi$ .

*Proof.* — We follow the perturbation method introduced in [1]. Let  $u_i \in C^\infty(\overline{B}_m(1))$  ( $i = 0, 2, 3$ ) be defined by

$$(3.1) \quad \Delta_{-2} u_0 = 0 \quad u_0|_{\partial \overline{B}_m(1)} = f$$

$$(3.2) \quad \Delta_{-2}u_2 + \Delta_0u_0 = 0 \quad u_2|_{\partial\bar{B}_m(1)} = 0$$

$$(3.3) \quad \Delta_{-2}u_3 + \Delta_1u_0 = 0 \quad u_3|_{\partial\bar{B}_m(1)} = 0$$

The proof depends on the following three lemmas:

$$\text{LEMMA 1. — } u_0(0) = \int_{-\pi}^{\pi} f(\theta)\omega(d\theta).$$

$$\text{LEMMA 2. — } u_2(0) = 0.$$

$$\text{LEMMA 3. — } u_3(0) = -\frac{1}{32} \int_{-\pi}^{\pi} (K_1 \cos \theta + K_2 \sin \theta) f(\theta)\omega(d\theta).$$

*Proof of Lemma 1.* —  $u_0$  is the solution of the classical Dirichlet problem in the unit disc. thus

$$u_0 = \int_{-\pi}^{\pi} \frac{(1-r^2)}{(1+r^2-2r\cos(\theta-\phi))} f(\phi)\omega(d\phi).$$

$$\text{In particular } u_0(0) = \int_{-\pi}^{\pi} f(\phi)\omega(d\phi). \quad \square$$

*Proof of Lemma 2.* — Equivalently we write  $u_0$  as a Fourier series

$$u_0 = A_0 + \sum_{n=1}^{\infty} r^n (A_n \cos n\theta + B_n \sin n\theta).$$

Thus

$$\Delta_0 u_0 = -\frac{K_0}{3} \sum_{n=1}^{\infty} (n^2 + n) r^n (A_n \cos n\theta + B_n \sin n\theta).$$

This is a sum of homogeneous polynomials  $\sum_1^{\infty} h_n$ . By Lemma 6.3 of [1] we have

$$u_2(0) = \sum_1^{\infty} \frac{1}{(n+2)^2} \int_{-\pi}^{\pi} h_n(\theta)\omega(d\theta). \text{ But from the orthogonality relations for}$$

the functions  $(\cos n\theta, \sin n\theta)$ , all of these integrals are zero, hence  $u_2(0) = 0$ .  $\square$

(Note: The above depends on the fact that  $\Delta_0 u_0$  is a harmonic function which vanishes at the origin. This may also be seen from the relation

$$\Delta_{-2}\Delta_0 - \Delta_0\Delta_{-2} = -\frac{2K_0}{3}\Delta_{-2} \text{ applied to } u_0.)$$

*Proof of Lemma 3.* — We have

$$\begin{aligned} \left[ \frac{r^2}{4} \frac{\partial}{\partial r} - \frac{r}{6} \frac{\partial^2}{\partial \theta^2} \right] u_0 &= \sum_1^{\infty} r^{n+1} \left( \frac{n}{4} + \frac{n^2}{6} \right) (A_n \cos n\theta + B_n \sin n\theta) \\ \frac{r}{12} \frac{\partial u_0}{\partial \theta} &= \sum_1^{\infty} r^{n+1} \left( \frac{n}{12} \right) (-A_n \sin n\theta + B_n \cos n\theta), \end{aligned}$$

Referring to the formula (2.6), we see that

$$\begin{aligned} \Delta_1 u_0 &= - \sum_1^{\infty} \left( \frac{n}{4} + \frac{n^2}{6} \right) r^{n+1} (K_1 \cos \theta + K_2 \sin \theta) (A_n \cos n\theta + B_n \sin n\theta) \\ &\quad - \sum_1^{\infty} \left( \frac{n}{12} \right) r^{n+1} (K_2 \cos \theta - K_1 \sin \theta) (-A_n \sin n\theta + B_n \cos n\theta) \end{aligned}$$

a sum of homogeneous polynomials  $\sum_2^{\infty} h_k$ . By Lemma 6.3 of [1] we have

$$u_3(0) = \sum_2^{\infty} \frac{1}{(k+2)^2} \int_{-\pi}^{\pi} h_k(\theta) \omega(d\theta).$$

But from the orthogonality relations of Fourier series all of these integrals are zero except for  $k = 2$ , i. e.  $n = 1$ . This gives

$$\begin{aligned} u_3(0) &= - \left( \frac{1}{4} + \frac{1}{6} \right) \frac{1}{4^2} (K_1 A_1 + K_2 B_2) / 2 \\ &\quad - \left( \frac{1}{12} \right) \frac{1}{4^2} (K_2 B_2 + K_1 A_1) / 2 \\ &= - \frac{1}{32} (K_1 A_1 + K_2 B_2) / 2. \end{aligned}$$

On the other hand  $f(\theta) = A_0 + \sum_1^{\infty} (A_n \cos n\theta + B_n \sin n\theta)$  and thus

$$\int_{-\pi}^{\pi} f(\theta) (K_1 \cos \theta + K_2 \sin \theta) \omega(d\theta) = \frac{1}{2} (K_1 A_1 + K_2 B_2).$$

We have proved that  $u_3(0) = -\frac{1}{32} \int_{-\pi}^{\pi} f(\theta) (K_1 \cos \theta + K_2 \sin \theta) \omega(d\theta)$  as required.  $\square$

To complete the proof of the theorem we appeal to Proposition 2.0 with  $f = u_0 + \varepsilon^2 u_2 + \varepsilon^3 u_3$ ; thus

$$\begin{aligned} \Phi_\varepsilon^{-1} \Delta \Phi_\varepsilon (u_0 + \varepsilon^2 u_2 + \varepsilon^3 u_3) &= \varepsilon^{-2} \Delta_{-2} (u_0 + \varepsilon^2 u_2 + \varepsilon^3 u_3) \\ &\quad + \Delta_0 (u_0 + \varepsilon^2 u_2 + \varepsilon^3 u_3) \\ &\quad + \varepsilon \Delta_1 (u_0 + \varepsilon^2 u_2 + \varepsilon^3 u_3) + 0(\varepsilon^2) \\ &= 0(\varepsilon^2), \end{aligned}$$

where we have used the defining relations (3.1)-(3.3). Hence  $\Delta U = 0(\varepsilon^2)$  where  $U = \Phi_\varepsilon (u_0 + \varepsilon^2 u_2 + \varepsilon^3 u_3)$ . Noting that  $U = \Phi_\varepsilon f$  on  $\partial B_m(\varepsilon)$  we have that for any  $x \in B_m(\varepsilon)$  by Dynkin's formula

$$\begin{aligned} H_\varepsilon f(x) &= E_x U(X_{T_\varepsilon}) \\ &= U(x) + E_x \int_0^{T_\varepsilon} \Delta U(X_s) ds \\ &= U(x) + 0(\varepsilon^4). \end{aligned}$$

Setting  $x = m$  we have  $U(m) = u_0(0) + \varepsilon^2 u_2(0) + \varepsilon^3 u_3(0)$  from which the result follows.  $\square$

#### 4. COMPARISON WITH THE NON-STOCHASTIC MEAN VALUE

The mean-value operator of a Riemannian manifold is defined by

$$M_\varepsilon f(m) = \frac{\int_{S_\varepsilon} (\Phi_\varepsilon f) d\sigma_\varepsilon}{\int_{S_\varepsilon} 1 d\sigma_\varepsilon}$$

where  $\sigma_\varepsilon$  is the surface measure on the  $\varepsilon$ -sphere  $S_\varepsilon$  and  $f$  is a smooth function on the unit sphere. We have the following result.

**THEOREM 4.1.** — *When  $\varepsilon \downarrow 0$*

$$M_\varepsilon f(m) = \int_{-\pi}^{\pi} f(\theta) \left[ 1 - \frac{\varepsilon^3}{12} \langle \nabla K, u_\theta \rangle \right] \omega(d\theta) + 0(\varepsilon^4).$$

*Proof.* — From the method of Gray and Willmore [2], we have

$$M_\varepsilon f = \frac{I_\varepsilon f}{I_\varepsilon 1}$$

where

$$I_\varepsilon f = \int_{-\pi}^{\pi} f(\theta)G(\varepsilon, \theta)d\theta.$$

Using the expansion  $G(\varepsilon, \theta) = \varepsilon - \frac{\varepsilon^3}{6} K_0 - \frac{\varepsilon^4}{12} (K_1 \cos\theta + K_2 \sin\theta) + O(\varepsilon^5)$  we have

$$I_\varepsilon f = \left(\varepsilon - \frac{\varepsilon^3}{6} K_0\right) \int_{-\pi}^{\pi} f(\theta)d\theta - \frac{\varepsilon^4}{12} \int_{-\pi}^{\pi} f(\theta)(K_1 \cos\theta + K_2 \sin\theta)d\theta + O(\varepsilon^5).$$

In particular

$$I_\varepsilon 1 = 2\pi\left(\varepsilon - \frac{\varepsilon^3}{6} K_0\right) + O(\varepsilon^5).$$

Thus

$$\frac{I_\varepsilon f}{I_\varepsilon 1} = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta) \left[ 1 - \frac{\varepsilon^3}{12} (K_1 \cos\theta + K_2 \sin\theta) \right] d\theta + O(\varepsilon^4)$$

which was to be proved.

**COROLLARY.** — *If  $H_\varepsilon f(m) = M_\varepsilon f(m)$  for every  $m \in M$ ,  $\varepsilon > 0$ ,  $f \in C^\infty(S^1)$ , then  $(M, g)$  has constant curvature.*

*Proof.* — We have  $\lim_{\varepsilon \downarrow 0} \frac{M_\varepsilon f - H_\varepsilon f}{\varepsilon^3} = \frac{-5}{96} \int_{-\pi}^{\pi} \langle \nabla K, u_\theta \rangle f(\theta) \omega(d\theta).$

If this is zero for all  $f$ , then  $K_1 = K_2 = 0$ , i. e.  $\nabla K(m) = 0$ , and hence  $K$  is constant.

**5. APPENDIX. COMPUTATION OF  $\Delta_{-2}, \Delta_0, \Delta_1$**

In geodesic polar coordinates we have

$$(5.1) \quad \begin{aligned} (g_{ij}) &= \begin{pmatrix} 1 & 0 \\ 0 & G^2 \end{pmatrix} & G &= G(r, \theta) \\ \Delta &= \frac{1}{G} \frac{\partial}{\partial r} \left( G \frac{\partial}{\partial r} \right) + \frac{1}{G} \frac{\partial}{\partial \theta} \left( \frac{1}{G} \frac{\partial}{\partial \theta} \right) \\ &= \frac{\partial^2}{\partial r^2} + \frac{G_r}{G} \frac{\partial}{\partial r} + \frac{1}{G^2} \frac{\partial^2}{\partial \theta^2} - \frac{G_\theta}{G^3} \frac{\partial}{\partial \theta} \end{aligned}$$

The Jacobi equation connecting the metric with the curvature is

$$(5.2) \quad G_{rr} + KG = 0, \quad G(0^+, \theta) = 0, \quad G_r(0^+, \theta) = 1.$$



In a neighborhood of  $m$ , we can write

$$(5.3) \quad \mathbf{K} = \mathbf{K}_0 + r(\mathbf{K}_1 \cos \theta + \mathbf{K}_2 \sin \theta) + O(r^2) \quad (r \downarrow 0).$$

This yields

$$\mathbf{G} = r - \frac{r^3}{6} \mathbf{K}_0 - \frac{r^4}{12} (\mathbf{K}_1 \cos \theta + \mathbf{K}_2 \sin \theta) + O(r^5) \quad (r \downarrow 0).$$

Performing the indicated operations, we have

$$\frac{\mathbf{G}_r}{\mathbf{G}} = \frac{1}{r} - \frac{r}{3} \mathbf{K}_0 - \frac{r^2}{4} (\mathbf{K}_1 \cos \theta + \mathbf{K}_2 \sin \theta) + O(r^3)$$

$$\frac{1}{\mathbf{G}^2} = \frac{1}{r^2} + \frac{\mathbf{K}_0}{3} + \frac{r}{6} (\mathbf{K}_1 \cos \theta + \mathbf{K}_2 \sin \theta) + O(r^2)$$

$$\frac{\mathbf{G}_\theta}{\mathbf{G}^3} = \frac{r}{12} (\mathbf{K}_1 \sin \theta - \mathbf{K}_2 \cos \theta) + O(r^2).$$

Substituting in (5.1) and equating coefficients of  $r$  yields (2.4)-(2.6).

#### ACKNOWLEDGEMENT

I would like to thank Victor Mizel for some helpful suggestions.

#### REFERENCES

- [1] A. GRAY and M. PINSKY, The mean exit time from a small geodesic ball in a Riemannian manifold, *Bulletin des Sciences Mathématiques*, t. **107**, 1983, p. 345-370.
- [2] A. GRAY and T. J. WILLMORE, Mean-value theorems for Riemannian manifolds, *Proceedings of the Royal Society of Edinburgh*, t. **92 A**, 1982, p. 343-364.
- [3] Y. TAKAHASKI and S. WATANABE, The Onsager-Machlup function of diffusion processes, *Stochastic Integrals*, Springer-Verlag *Lecture Notes in Mathematics*, t. **851**, 1981, p. 433-463.

(Manuscrit reçu le 10 Avril 1984)

(Corrigé le 25 Septembre 1984)