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The maximum of a Gaussian process with nonconstant variance

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ABSTRACT. — Let $X(t)$, $t \in [0, 1]^n$, be a real separable Gaussian process with mean 0 and continuous covariance function. Suppose that the variance has a unique maximum at some point τ. Under specified conditions the following relation holds for $u \to \infty$:

$$P(\max_{t \in [0,1]^n} X(t) > u) \sim P(X(\tau) > u).$$

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RÉSUMÉ. — Soit $X(t)$, $t \in [0, 1]^n$, un processus Gaussien réel séparable d'espérance 0 et de fonction de covariance continue. On suppose que la variance a un maximum unique en un point τ. Sous des conditions spécifiques, la relation suivante a lieu asymptotiquement lorsque $u \to \infty$:

$$P(\max_{t \in [0,1]^n} X(t) > u) \sim P(X(\tau) > u).$$

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1. INTRODUCTION AND SUMMARY

Let $X(t)$, $t \in [0, 1]^n$, for fixed $n \geq 1$, be a real separable Gaussian process with mean 0 and continuous covariance function; and put $\sigma^2(t) = EX^2(t)$. Suppose that there is a point τ in $[0, 1]^n$ such that $\sigma^2(t)$ has a unique maximum value at $t = \tau$; and put $\sigma^2 = \sigma^2(\tau)$. The main theorem of this paper is that, under a general set of conditions, the distribution tail of the random variable $\max(X(t) : t \in [0, 1]^n)$ is asymptotically equal to the distribution tail of the single random variable $X(\tau)$:

$$(1.1) \qquad P(\max_{[0,1]^n} X(t) > u) \sim P(X(\tau) > u), \qquad \text{for} \quad u \to \infty .$$

Let $\phi(x)$ be the standard normal density function, and define

$$(1.2) \qquad \Psi(z) = \int_z^\infty \phi(y)dy ;$$

then the right hand member of (1.1) is equal to $\Psi(u/\sigma)$.

The hypothesis of the theorem specifies a relation between the local rates of decay of the functions $\sigma^2 - \sigma^2(t)$ and $E(X(t) - X(\tau))^2$ for $t \to \tau$. It requires that the ratio

$$(1.3) \qquad E(X(t) - X(\tau))^2/(\sigma^2 - \sigma^2(t))$$

tend to 0 at a specified rate as $t \to \tau$. The condition signifies that the values of $X(t)$, for t near τ, tend to be very close to $X(\tau)$, so that the maximum of $X(t)$ over a small neighborhood of τ is approximately equal to $X(\tau)$ itself. Then it is shown that the maximum of $X(t)$ over this neighborhood dominates the maximum over the remaining portion of the parameter set because the variance is largest at τ, and so the random variables $X(t)$ for t in this neighborhood have the largest deviations from their mean 0.

The only other major study of this problem is contained in the work of Piterbarg and Prisjaznjuk in the case $n = 1$ [6]. They assumed

$$\sigma^2 - \sigma^2(t) \sim C\,|\,\tau - t\,|^\beta \quad \text{for} \quad t \to \tau$$

$$E\left(\frac{X(t)}{\sigma(t)} - \frac{X(s)}{\sigma(s)}\right)^2 \sim D\,|\,t - s\,|^\alpha, \quad \text{for} \quad s, t \to \tau ,$$

and the uniform condition

$$E\left(\frac{X(s)}{\sigma(s)} - \frac{X(t)}{\sigma(t)}\right)^2 \leq C'\,|\,s - t\,|^{\alpha'}, \qquad s, t \in [0, 1] ,$$

for some α, α' and β; and they showed that (1.1) holds if $\beta < \alpha$. The current paper represents a generalization of their result, and the extent of the generality is illustrated by the examples in Section 4. We also indicate there that our estimate of the distribution of the supremum is sharper than that which follows from the celebrated result of C. Borell [3].

A more specialized result was recently obtained by the author in the particular case of a Gaussian process with stationary increments [2]. If the incremental variance function $\sigma^2(t) = E(X(t) - X(0))^2$ is continuously differentiable and convex, and $\sigma^{2\prime}(1) > 0$ and $\sigma^2(h)/h \to 0$ for $h \to 0$, then (1.1) holds. This does not follow from our main theorem here because the latter requires an explicit convergence rate for $\sigma^2(h)/h$.

2. STATEMENT OF THE MAIN RESULT

Define the metric $\| s - t \| = \max_i | s_i - t_i |$, where (s_i) and (t_i) are the real components of s and t, respectively, for $s, t \in [0, 1]^n$. The well known theorem of Fernique [4] states: If there is a positive nondecreasing function $q(t)$, $t > 0$, such that

$$\limsup_{\|s-t\|\to 0} \frac{E(X(s) - X(t))^2}{q^2(\| t - s \|)} < \infty , \tag{2.1}$$

and

$$\int_1^\infty q(e^{-y^2})dy < \infty , \tag{2.2}$$

then the sample functions are almost surely continuous. Under this hypothesis, there exists at least one such function $q(t)$ such that (2.1) may be strengthened to

$$\limsup_{\|s-t\|\to 0} \frac{E(X(s) - X(t))^2}{q^2(\| s - t \|)} < 1 . \tag{2.3}$$

Indeed, if for some q, the lim sup in (2.1) has the value $M > 1$, then replace q^2 by $2q^2M$.

Now we describe a local version of Fernique's hypothesis. For the point τ in the domain, we suppose that there is a positive nondecreasing function $g(t)$, $t > 0$, such that (2.2) and (2.3) hold with g in the role of q, and with the lim sup for $\| s - t \| \to 0$ replaced by the lim sup for $s, t \to \tau$:

$$\limsup_{s,t\to\tau} \frac{E(X(s) - X(t))^2}{g^2(\| s - t \|)} < 1 , \tag{2.4}$$

and

$$\int_1^\infty g(e^{-y^2})dy < \infty . \tag{2.5}$$

As in [1], we define

$$Q(h) = q(h) + (2 + \sqrt{2}) \int_1^\infty q(h2^{-y^2})dy, \qquad 0 < h \leq 1 , \tag{2.6}$$

and

$$Q^{-1}(x) = \sup (h : Q(h) \leq x) . \tag{2.7}$$

Similarly, we define

$$G(h) = g(h) + (2 + \sqrt{2}) \int_1^\infty g(h2^{-y^2})dy, \qquad 0 < h \leq 1 , \tag{2.8}$$

and

$$G^{-1}(x) = \sup (h : G(h) \leq x) . \tag{2.9}$$

Note that the functions g and G may depend on τ.

For $h > 0$, and fixed $\tau \in [0, 1]^n$, the set

$$B(h) = \{ t : \| t - \tau \| \leq h/2 \}$$

is a cube of edge h and with center at τ. (In terms of the metric $\| s - t \|$ defined above, it is considered a ball of radius $h/2$.) Put $\sigma^2(t) = EX^2(t)$, and define

$$\bar{\sigma}^2(h) = \max_{t \in [0,1]^n \cap {}^c B(h)} \sigma^2(t) . \tag{2.10}$$

THEOREM 2.1. — *Suppose that there is a function q satisfying (2.1) and (2.2), and a function g satisfying (2.4) and (2.5). If, for every $\varepsilon > 0$,*

$$\lim_{h \to 0} [Q^{-1}(G(h)/\varepsilon)]^{-n} \exp\left\{ -\frac{\varepsilon^2}{2\sigma^4} \left[\frac{\sigma^2 - \bar{\sigma}^2(h)}{G^2(h)} \right] \right\} = 0 , \tag{2.11}$$

then,

$$\lim_{u \to \infty} (\Psi(u/\sigma))^{-1} P(\max_{[0,1]^n} X(t) > u) = 1 . \tag{2.12}$$

3. PROOF OF THEOREM 2.1

We recall the well known inequalities relating ϕ and Ψ in (1.1):

$$\phi(z)(z^{-1} - z^{-3}) \leq \Psi(x) \leq \phi(z)z^{-1}, \qquad \text{for} \quad z > 0 . \tag{3.1}$$

The explicit form of ϕ implies

$$\phi(x+y) = \phi(x)\exp\left(-xy - \frac{1}{2}y^2\right), \tag{3.2}$$

for all x and y.

LEMMA 3.1. — *Under* (2.4) *and* (2.5), *we have*,

$$\lim_{\varepsilon\to 0}\limsup_{u\to\infty}(\Psi(u/\sigma))^{-1}P(\max_{t\in B(G^{-1}(\varepsilon/u))}X(t) > u) \leq 1\,. \tag{3.3}$$

Proof. — We adapt the proof of [1] Theorem 3.1. For small $h > 0$ the function g in (2.4) and (2.5) plays the same role for $X(t)$, $t \in B(h)$, as does the function φ for $X(t)$, $t \in B(h)$ in [1]. Similarly, G in (2.8) takes the role of Q in [1]. Define

$$h = G^{-1}(\varepsilon/u)\,, \tag{3.4}$$

so that $G(h) = \varepsilon/u$; and then define

$$\gamma = (1 + 4n\log 2)^{1/2}, \quad \lambda = (5/2)4^n\sqrt{2\pi}\,.$$

The event $\max_{B(h)}X(t) > u$ is included in the union of the three events,

$$X(\tau) > u - \gamma G(h) = u - \gamma\varepsilon/u\,, \tag{3.5}$$

$$\max_{t\in B(h)}(X(t) - X(\tau)) > u\,, \tag{3.6}$$

and

$$0 \leq X(\tau) \leq u - \gamma\varepsilon/u,\ \max_{t\in B(h)}X(t) > u\,. \tag{3.7}$$

We give individual estimates of the probabilities of the three events above.

According to (3.1) and (3.2) we have

$$\limsup_{u\to\infty}\frac{P(X(\tau) > u - \gamma\varepsilon/u)}{\Psi(u/\sigma)} = \limsup_{u\to\infty}\frac{\Psi\left(\dfrac{u-\gamma\varepsilon/u}{\sigma}\right)}{\Psi(u/\sigma)} = e^{\gamma\varepsilon}\,,$$

so that

$$\lim_{\varepsilon\to 0}\limsup_{u\to\infty}(\Psi(u/\sigma))^{-1}P(X(\tau) > u - \gamma\varepsilon/u) = 1\,. \tag{3.8}$$

According to [1], Lemma 2.1, the probability of the event (3.6) is at most equal to

$$\lambda\Psi(u/G(h)) = \lambda\Psi(u^2/\varepsilon)\,;$$

hence, by (3.1)

$$(3.9)\qquad \limsup_{u\to\infty}\,(\Psi(u/\sigma))^{-1}\mathrm{P}(\max_{t\in \mathrm{B}(\mathrm{G}^{-1}(\varepsilon/u))}\mathrm{X}(t) - \mathrm{X}(\tau) > u) = 0\,,$$

for every $\varepsilon > 0$.

According to the proof of [1], Theorem 3.1, the probability of the event (3.7) is at most equal to

$$\int_0^{u-\gamma\varepsilon/u} \mathrm{P}(\max_{t\in\mathrm{B}(h)}[\mathrm{X}(t) - \mathrm{E}(\mathrm{X}(t)\,|\,\mathrm{X}(\tau))] > u - y)(1/\sigma)\phi(y/\sigma)dy\,.$$

By [1], Lemma 2.2, and by (3.4) above, the integral above is at most equal to

$$\lambda\int_0^{u-\gamma\varepsilon/u}\Psi\left(\frac{u-y}{\varepsilon/u}\right)(1/\sigma)\phi(y/\sigma)dy\,.$$

By the change of variable $z = u(u - y)/\varepsilon$, the expression above is equal to

$$\lambda\varepsilon\int_\gamma^{u^2/\varepsilon}\Psi(z)(1/u\sigma)\phi\left(\frac{u - z\varepsilon/u}{\sigma}\right)dz\,,$$

which, by (3.1) and (3.2), is asymptotically equal to

$$(\lambda\varepsilon/\sigma^2)\Psi(u/\sigma)\int_\gamma^{u^2/\varepsilon}\Psi(z)e^{z\varepsilon/\sigma^2}dz\,.$$

From this estimate for the probability of (3.7), it follows that

$$(3.10)\quad \lim_{\epsilon\to 0}\limsup_{u\to\infty}(\Psi(u/\sigma))^{-1}\mathrm{P}(0\le \mathrm{X}(\tau)\le u-\gamma\varepsilon/u,\ \max_{\mathrm{B}(h)}\mathrm{X}(t)>u)=0.$$

The statement of the lemma now follows from (3.8), (3.9) and (3.10).

LEMMA 3.2. — *Under the hypothesis of Theorem 2.1, for every* $\varepsilon > 0$,

$$(3.11)\qquad \lim_{u\to\infty}(\Psi(u/\sigma))^{-1}\mathrm{P}(\max_{t\in[0,1]^n\cap{}^c\mathrm{B}(\mathrm{G}^{-1}(\varepsilon/u))}\mathrm{X}(t) > u) = 0\,.$$

Proof. — The original Fernique inequality implies

$$\mathrm{P}(\max_{[0,1]^n}\mathrm{X}(t) > u) = 0(\Psi(u/s)),\qquad \text{for}\quad u\to\infty$$

for every $s > \max_t \sigma(t)$. For arbitrary s_1 and s_2 such that $0 < s_1 < s_2 < \sigma$, it follows that

$$\mathrm{P}[\max(\mathrm{X}(t) : t\in[0,1]^n, \sigma(t)\le s_1) > u] = 0(\Psi(u/s_2)) = o(\Psi(u/\sigma)).$$

for $u \to \infty$. Therefore, in order to verify (3.11), it suffices to assume that $\sigma^2(t)$ is bounded away from 0 on $[0, 1]^n$.

As a consequence, we may apply [1], Corollary 3.1 and Theorem 3.2:

$$P(\max_{[0,1]^n \cap {}^c B(h)} X(t) > u) = 0([Q^{-1}(1/u)]^{-n}\Psi(u/\bar{\sigma}(h))),$$

for $u \to \infty$, where h is given by (3.4) and $\bar{\sigma}^2(h)$ by (2.10). Thus the expression under the limit in (3.11) is of the order

$$\frac{\Psi(u/\bar{\sigma}(h))}{\Psi(u/\sigma)[Q^{-1}(1/u)]^n}.$$

By (3.1) and (3.2), this is asymptotically equal to

$$[Q^{-1}(1/u)]^{-n}\exp\left\{ -\frac{u^2}{2}\left(\frac{1}{\bar{\sigma}^2(h)} - \frac{1}{\sigma^2}\right)\right\}$$
$$\leq [Q^{-1}(1/u)]^{-n}\exp\{ -(u^2/2\sigma^4)(\sigma^2 - \bar{\sigma}^2(h))\}.$$

By (3.4) the last expression above is equal to

$$[Q^{-1}(G(h)/\varepsilon)]^{-n}\exp\left\{ -(\varepsilon^2/2\sigma^4)\left(\frac{\sigma^2 - \bar{\sigma}^2(h)}{G^2(h)}\right)\right\},$$

which tends to 0 under the hypothesis of the theorem.

Proof of Theorem 2.1. — Lemmas 3.1 and 3.2 imply that

$$\limsup_{u\to\infty} (\Psi(u/\sigma))^{-1}P(\max_{[0,1]^n} X(t) > u) \leq 1.$$

Since the product under the limit sign above is at least equal to 1, it follows that the limit itself exists, and is equal to 1.

4. APPLICATIONS

If there exists functions q and g satisfying the conditions of Theorem 2.1, then it may also be assumed that these functions satisfy

$$g(t) \leq q(t), \qquad \text{for} \quad 0 < t \leq 1. \tag{4.1}$$

Indeed, put $q^*(t) = \max(g(t), q(t))$; then $q^*(t) \geq q(t)$, and so q^* satisfies (2.1) if q does. Furthermore, q^* satisfies (2.2) if q and g satisfy (2.2) and (2.5), respectively. Finally, we note that $q^*(t) \geq g(t)$ and so (4.1) holds if q^* is used in the place of q.

A commonly used sufficient condition for the continuity of the sample functions on $[0, 1]^n$ is that for some $\delta > 0$,

$$\limsup_{\|s-t\|\to 0} \frac{E(X(s) - X(t))^2}{|\log \| s - t \| |^{-1-\delta}} < \infty. \tag{4.2}$$

This corresponds to the condition that (2.1) holds for the function $q(t) = |\log t|^{-(1+\delta)/2}$. A standard calculation then shows that $Q(h) \sim constant\,|\log h|^{-\delta/2}$, for $h \to 0$. Thus, by inversion, we may take Q^{-1} as

$$Q^{-1}(y) = \exp(-\,y^{-2/\delta}\,constant)\,. \tag{4.3}$$

In this case the sufficient condition (2.11) takes the form

$$\lim_{h\to 0} \exp\left[\frac{constant}{(G(h))^{2/\delta}} - constant\ \frac{\sigma^2 - \bar{\sigma}^2(h)}{(G(h))^2}\right] = 0\,. \tag{4.4}$$

EXAMPLE 4.1. — Suppose that there exists $\alpha > 0$ such that

$$\limsup_{s,t,\to\tau} \frac{E(X(s) - X(t))^2}{\|s - t\|^\alpha} < \infty\,;$$

then (2.4) and (2.5) are satisfied with $g(t) = t^{\alpha/2}$. It follows that $G(h) \sim constant\ h^{\alpha/2}$. Assume, furthermore, that (4.2) holds, and that $\sigma^2(t)$ satisfies

$$\sigma^2 - \sigma^2(\tau + h) \sim constant\ \|h\|^\beta \tag{4.5}$$

for $\|h\| \to 0$. Then the limit in (4.4) is equal to

$$\lim_{h\to 0} \exp(constant\ h^{-\alpha/\delta} - constant\ h^{\beta-\alpha})\,. \tag{4.6}$$

This is equal to 0 if

$$\beta < \alpha(1 - 1/\delta)\,. \tag{4.7}$$

EXAMPLE 4.2. — Suppose, in addition to (4.2), that

$$\limsup_{s,t\to\tau} \frac{E(X(s) - X(t))^2}{|\log\|s - t\||^{-1-\alpha}} < \infty$$

for some $\alpha > 0$, so that we may take $G(h)$ as $|\log h|^{-\alpha/2}$. Then by (4.1) we necessarily have $\delta \le \alpha$. Suppose also that

$$\sigma^2 - \sigma^2(\tau + h) \sim constant\ |\log h|^{-\beta}\,,$$

for some $\beta > 0$. Then the sufficient condition (4.4) takes the form

$$\lim_{h\to 0} \exp(constant\ |\log h|^{\alpha/\delta} - constant\ |\log h|^{\alpha-\beta}) = 0\,. \tag{4.8}$$

This holds under the same condition (4.7).

Fernique [5] has shown that a corollary of Borell's isoperimetric inequality [3] yields, in our notation,

$$P(\max_{[0,1]^n} |X(t) - m| > u) \leq 2\Psi(u/\sigma),$$

where m is the median of max $X(t)$. The result of our Theorem 2.1 is obviously better for large u.

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