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## **Equivalent-singular dichotomy for quasi-invariant ergodic measures**

by

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**ABSTRACT.** — Let  $E$  be a locally convex Hausdorff space,  $H_1, H_2$  be two linear subspaces of  $E$  and  $\mu_1, \mu_2$  be two probability measures on  $E$ . We prove that if  $\mu_i$  is  $H_i$ -quasi-invariant and  $H_i$ -ergodic ( $i = 1, 2$ ), then  $\mu_1$  and  $\mu_2$  are equivalent or singular. If  $H_1 = H_2$ , the result is well-known. So our interest is in the case  $H_1 \neq H_2$ . This result unifies many known dichotomies such as the dichotomy for product measures on  $\mathbb{R}^\infty$ , Hajek-Feldman's dichotomy for Gaussian measures and Fernique's dichotomy for a product measure and a Gaussian measure.

**RÉSUMÉ.** — Soit  $E$  un espace vectoriel localement convexe séparé. Soient  $H_1, H_2$  deux sous-espaces vectoriels de  $E$ , et  $\mu_1, \mu_2$  deux mesures probabilités sur  $E$ . Nous prouverons que si  $\mu_i$  est  $H_i$ -quasi-invariante et  $H_i$ -ergodique ( $i = 1, 2$ ), alors  $\mu_1$  et  $\mu_2$  sont équivalentes ou singulières. Si  $H_1 = H_2$ , c'est bien connu. Notre intérêt est donc au cas  $H_1 \neq H_2$ . Ce résultat unifie les dichotomies des mesures produits dans  $\mathbb{R}^\infty$ , de Hajek-Feldman pour les mesures gaussiennes, et de Fernique pour la mesure produit et la mesure gaussienne.

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## 1. INTRODUCTION

Let  $\mu, \nu$  be probability measures on a measurable space  $(\Omega, \mathcal{B})$ .  $\mu$  is said to be absolutely continuous with respect to  $\nu$  (denoted by  $\mu < \nu$ ) if  $\nu(A) = 0, A \in \mathcal{B}$  implies that  $\mu(A) = 0$ .  $\mu$  and  $\nu$  are equivalent (denoted by  $\mu \sim \nu$ ) if  $\mu < \nu$  and  $\nu < \mu$ .  $\mu$  and  $\nu$  are singular (denoted by  $\mu \perp \nu$ ) if there exists  $A \in \mathcal{B}$  such that  $\mu(A) = 1$  and  $\nu(A^c) = 1$ .

Let  $H$  be a subset of a locally convex Hausdorff space  $E$  and  $\mu$  be a probability measure on  $C(E, E^*)$  the cylindrical  $\sigma$ -algebra generated by the topological dual  $E^*$ . We set  $\tau_x(y) = y + x, \tau_x(x \in E)$  is  $C(E, E^*)$ -measurable.  $\mu$  is said to be  $H$ -quasi-invariant if it holds  $\tau_x(\mu) \sim \mu$  for every  $x \in H$ , where  $\tau_x(\mu)(A) = \mu(A - x), A \in C(E, E^*)$ . The  $H$ -quasi-invariant measure  $\mu$  is said to be  $H$ -ergodic if  $\mu(A\theta(A - x)) = 0$  for every  $x \in H$ , then  $\mu(A) = 0$  or 1, where  $\theta$  denotes the symmetric difference.

The aim of this paper is to prove the following theorem.

**THEOREM.** — *Let  $E$  be a locally convex Hausdorff space,  $H_1, H_2$  be two linear subspaces of  $E$ ,  $\mu_1, \mu_2$  be two probability measures on  $C(E, E^*)$  and assume that  $\mu_i$  is  $H_i$ -quasi-invariant and  $H_i$ -ergodic ( $i = 1, 2$ ). Then  $\mu_1$  and  $\mu_2$  are equivalent or singular.*

If  $H_1 = H_2$ , the result is well-known, for example see Skorohod [13], § 23. So our interest is in the case  $H_1 \neq H_2$ . This theorem unifies many known dichotomies such as a) dichotomy for product measures on  $\mathbb{R}^\infty$  (this is a special case of the Kakutani dichotomy), b) Hajek-Feldman's dichotomy, c) Fernique's dichotomy, d) dichotomy for symmetric stable measures with discrete Lévy measures, e) dichotomy for the case  $H_1 = H_2$  and f) Kanter's dichotomy, see section 3.

## 2. DICHOTOMY FOR QUASI-INVARIANT ERGODIC MEASURE

Let  $E$  be a locally convex Hausdorff space and  $\mu$  be a probability measure on  $C(E, E^*)$ . We set  $A_\mu = \{x \in E; \tau_x(\mu) \sim \mu\}$ .  $A_\mu$  is an additive subgroup of  $E$  but not necessarily a linear space. Denote by  $A_\mu^0$  the largest linear subspace contained in  $A_\mu$ , that is,  $A_\mu^0 = \{x \in A_\mu; tx \in A_\mu \text{ for every } t \in \mathbb{R}\}$ .

Let  $\{x_i^*\} \subset E^*$  be arbitrary but fixed countable subset. Consider the  $\sigma$ -subalgebra  $\mathcal{A} = C(E, \{x_i^*\}) \subset C(E, E^*)$ , the  $\sigma$ -algebra generated by

$\{x_i^*\}$ . For every  $x \in E$ , the translation  $\tau_x(\tau_x(y) = y + x)$  is  $\mathcal{A} - \mathcal{A}$ -measurable. Let  $\mu_0$  be the restriction of  $\mu$  to  $\mathcal{A}$ . For every  $x \in A_\mu^0$  (that is,  $tx \in A_\mu$  for every  $t \in \mathbb{R}$ ), we can define the one parameter group  $\{U_t^x; t \in \mathbb{R}\}$  of unitary operators on  $L^2(\mathcal{A}, \mu_0)$  by

$$(U_t^x f)(y) = \left[ \frac{d\tau_{tx}(\mu_0)}{d\mu_0}(y) \right]^{1/2} f(y - tx).$$

This one parameter group depends on the sequence  $\{x_i^*\}$ .

LEMMA 1. — Let  $\{x_i^*\}$  be fixed and  $\mathcal{A} = C(E, \{x_i^*\})$  as above. Let  $x \in A_\mu^0$  be also fixed. Then the one parameter group  $\{U_t^x; t \in \mathbb{R}\}$  on  $L^2(\mathcal{A}, \mu_0)$  is strongly continuous, that is, for every  $f \in L^2(\mathcal{A}, \mu_0)$ ,  $t \rightarrow U_t^x f \in L^2(\mathcal{A}, \mu_0)$  is continuous on  $\mathbb{R}$ .

*Proof.* — Remark that  $\mathcal{A}$  has a countable generator  $\{C_i\}$ , that is,  $\{C_i\}$  forms an algebra of subsets and generates  $\mathcal{A}$ . To prove that  $\{U_t^x\}$  is strongly continuous on the separable Hilbert space  $L^2(\mathcal{A}, \mu_0)$ , it is sufficient to show that for every  $f, g \in L^2(\mathcal{A}, \mu_0)$ ,  $t \rightarrow (U_t^x f, g) = \int (U_t^x f)(y)g(y)d\mu_0(y)$  is Lebesgue measurable on  $\mathbb{R}$ , since  $\{U_t^x\}$  is a one parameter unitary group, see von Neumann [10], Hewitt and Ross [5], (22.20).

Consider the transformation  $T_x: \mathbb{R} \times E \rightarrow \mathbb{R} \times E$  given by  $T_x(t, y) = (t, y + tx) = (t, \tau_{tx}(y))$ .  $T_x$  is  $\mathcal{B}(\mathbb{R}) \otimes \mathcal{A}$ -measurable, where  $\mathcal{B}(\mathbb{R})$  is the Borel field of  $\mathbb{R}$ . Moreover, if we put  $\lambda$  be the Lebesgue measure on  $\mathbb{R}$ , then we have  $T_x(\lambda \otimes \mu_0) \sim \lambda \otimes \mu_0$ . In fact for every  $A \in \mathcal{B}(\mathbb{R}) \otimes \mathcal{A}$ , we have  $T_x(\lambda \otimes \mu_0)(A) = \int \tau_{tx}(\mu_0)(A_t)d\lambda(t)$ , where  $A_t = \{y; (t, y) \in A\}$ . So it follows that  $T_x(\lambda \otimes \mu_0)(A) = 0$  if and only if  $\tau_{tx}(\mu_0)(A_t) = 0$  for  $\lambda$ -a. e.  $t$ , and if and only if  $\mu_0(A_t) = 0$  for  $\lambda$ -a. e.  $t$  since  $tx \in A_\mu$ , that is  $\lambda \otimes \mu_0(A) = 0$ .

By the Radon-Nikodym theorem, there is a  $\lambda \otimes \mu_0$ -measurable function  $\alpha(t, y)$  such that  $\lambda \otimes \mu_0(T_x^{-1}A) = \int_A \alpha(t, y)d(\lambda \otimes \mu_0)(t, y)$  for every  $A \in \mathcal{B}(\mathbb{R}) \otimes \mathcal{A}$ .

For every  $B \in \mathcal{B}(\mathbb{R})$  and  $C \in \mathcal{A}$ , we have

$$\int_B \mu_0(\{y; y + tx \in C\})d\lambda(t) = \int_B \int_C \alpha(t, y)d\mu_0(y)d\lambda(t).$$

Hence we have  $\tau_{tx}(\mu_0)(C) = \mu_0(\{y; y + tx \in C\}) = \int_C \alpha(t, y)d\mu_0(y)$   $\lambda$ -a. e. (the null set depends on  $C$ ). Let  $\{C_i\}$  be a countable generator of  $\mathcal{A}$ .

Then we can find a  $\lambda$ -null set  $N \subset \mathbb{R}$  such that for every  $t \in \mathbb{R} \setminus N$ , it holds that  $\tau_{tx}(\mu_0)(C_i) = \int_{C_i} \alpha(t, y) d\mu_0(y)$  for every  $i$ . So we obtain for every  $t \in \mathbb{R} \setminus N$ ,  $\tau_{tx}(\mu_0)(C) = \int_C \alpha(t, y) d\mu_0(y)$  for every  $C \in \mathcal{A}$ , since  $\{C_i\}$  generates  $\mathcal{A}$ . By the uniqueness of the Radon-Nikodym derivative, it follows that for every  $t \in \mathbb{R} \setminus N$ ,  $(d\tau_{tx}(\mu_0)/d\mu_0)(y) = \alpha(t, y)$   $\mu$ -a. e. (the  $\mu$ -null set depends on  $t$ ). Consequently for every  $t \in \mathbb{R} \setminus N$  and for every  $f, g \in L^2(\mathcal{A}, \mu_0)$ ,

$$(U_t^x f, g) = \int_E \alpha(t, y)^{1/2} f(y - tx) g(y) d\mu_0(y),$$

which implies that  $t \rightarrow (U_t^x f, g)$  is  $\lambda$ -measurable since  $\alpha(t, y)$  is  $\lambda \otimes \mu_0$ -measurable.

This proves the Lemma.

LEMMA 2. — Suppose that  $x \in A_\mu^0$ . Then for every  $A \in C(E, E^*)$ , it holds that  $\mu(A \cap (A + tx)) \rightarrow \mu(A)$  as  $t \rightarrow 0$ .

*Proof.* — We show that  $\mu(A\theta(A + tx)) \rightarrow 0$  as  $t \rightarrow 0$ . By the definition of the cylindrical  $\sigma$ -algebra, there exists a sequence  $\{x_i^*\} \subset E^*$  such that  $A \in \mathcal{A} = C(E, \{x_i^*\})$ . We set  $\mu_0 = \mu|_{\mathcal{A}}$  (the restriction) as before. Since  $A + tx \in \mathcal{A}$ , it is sufficient to show that  $\mu_0(A\theta(A + tx)) \rightarrow 0$  as  $t \rightarrow 0$ . We have

$$\begin{aligned} \mu_0(A\theta(A + tx)) &= \int |\chi_A(y) - \chi_A(y - tx)|^2 d\mu_0(y) \\ &\leq 2 \|U_t^x(\chi_A) - \chi_A\|^2 + 2 \int |(d\tau_{tx}(\mu_0)/d\mu_0)(y)^{1/2} \chi_A(y - tx) - \chi_A(y - tx)|^2 d\mu_0(y) \\ &\leq 2 \|U_t^x(\chi_A) - \chi_A\|^2 + 2 \|U_t^x 1 - 1\|^2 \rightarrow 0 \quad \text{as } t \rightarrow 0 \text{ by Lemma 1,} \end{aligned}$$

where  $\chi_A$  is the characteristic function of  $A$  and  $\| \cdot \|$  denotes the  $L^2$ -norm.

This completes the proof.

Now we prove the main dichotomy theorem.

THEOREM. — Let  $E$  be a locally convex Hausdorff space and  $H_1, H_2$  be two linear subspaces of  $E$ . Let  $\mu_1, \mu_2$  be two probability measures on  $C(E, E^*)$  such that  $\mu_i$  is  $H_i$ -quasi-invariant and  $H_i$ -ergodic ( $i = 1, 2$ ). Then  $\mu_1$  and  $\mu_2$  are equivalent or singular.

*Proof.* — Since  $H_i$  is linear, we have  $H_i \subset A_{\mu_i}^0$ , hence  $\mu_i$  is  $A_{\mu_i}^0$ -ergodic. If  $A_{\mu_1}^0 = A_{\mu_2}^0$ , then it is well-known that  $\mu_1 \sim \mu_2$  or  $\mu_1 \perp \mu_2$  holds, since  $\mu_1$  and  $\mu_2$  have the same quasi-invariant subspace  $A_{\mu_1}^0 = A_{\mu_2}^0$ , see for example Skorohod [13], 23. We prove that if  $A_{\mu_1}^0 \neq A_{\mu_2}^0$ , then  $\mu_1$  and  $\mu_2$  are singular.

Suppose that  $A_{\mu_1}^0 \neq A_{\mu_2}^0$  and  $\mu_1$  and  $\mu_2$  are not singular. Without loss of generality we may assume that there exists  $x_0 \in A_{\mu_1}^0 \setminus A_{\mu_2}^0$  (if not, consider  $A_{\mu_2}^0 \setminus A_{\mu_1}^0 \neq \emptyset$ ). Since  $x_0 \notin A_{\mu_2}^0$ , we can find a  $t_0 \in \mathbb{R}$  such that  $t_0 x_0 \notin A_{\mu_2}^0$ . Put  $x = t_0 x_0$ , then we have  $\tau_x(\mu_2) \perp \mu_2$  since both  $\tau_x(\mu_2)$  and  $\mu_2$  are  $A_{\mu_2}^0$ -quasi-invariant and  $A_{\mu_2}^0$ -ergodic by the same subspace  $A_{\mu_2}^0$ .

Since  $\mu_1$  and  $\mu_2$  are not singular, there exists  $A \in C(E, E^*)$  such that  $\mu_1(A) > 0$ ,  $\mu_2(A) > 0$  and

$$(1) \quad \mu_1 \sim \mu_2 \quad \text{on } A.$$

Since  $x \in A_{\mu_1}^0$  and  $A_{\mu_1}^0$  is linear space, we have

$$(2) \quad \mu_1 \sim \tau_{x/n}(\mu_1) \quad \text{for every } n = 1, 2, \dots$$

Consider the translates  $\tau_{x/n}(\mu_2)$ ,  $n = 1, 2, \dots$ . By the  $A_{\mu_2}^0$ -quasi-invariance and  $A_{\mu_2}^0$ -ergodicity,  $\tau_{x/n}(\mu_2) \sim \mu_2$  or  $\tau_{x/n}(\mu_2) \perp \mu_2$  ( $n = 1, 2, \dots$ ). If  $\tau_{x/N}(\mu_2) \sim \mu_2$  for some  $N$ , then it follows that

$$\mu_2 \sim \tau_{x/N}(\mu_2) \sim \tau_{x/N}(\tau_{x/N}(\mu_2)) = \tau_{2x/N}(\mu_2) \sim \dots \sim \tau_x(\mu_2),$$

which contradicts to  $x \notin A_{\mu_2}^0$ . Thus we have

$$(3) \quad \mu_2 \perp \tau_{x/n}(\mu_2) \quad \text{for every } n = 1, 2, \dots$$

By (1), we have

$$(4) \quad \tau_{x/n}(\mu_1) \sim \tau_{x/n}(\mu_2) \quad \text{on } A + \frac{x}{n} \quad \text{for every } n.$$

If we can show that  $\mu_1(A \cap (A + x/m)) > 0$  (by (1) it follows also  $\mu_2(A \cap (A + x/m)) > 0$ ) for some  $m$ , then (2), (3) and (4) imply  $\mu_1 \perp \mu_2$  on  $A \cap (A + x/m)$  which contradicts to (1).

By Lemma 2, we have  $\mu_1(A \cap (A + x/n)) \rightarrow \mu_1(A) > 0$  as  $n \rightarrow \infty$ . So there exists  $m$  such that  $\mu_1(A \cap (A + x/m)) > 0$  as desired.

This completes the proof.

*Remark.* — In the above theorem, if  $\mu_1$  and  $\mu_2$  are Radon probability measures on the Borel field  $\mathcal{B}(E)$ , then the same dichotomy  $\mu_1 \sim \mu_2$  or  $\mu_1 \perp \mu_2$  holds. In fact, for two Radon probability measures  $\mu_1, \mu_2$ ,  $\mu_1 \sim \mu_2$  on  $C(E, E^*)$  implies that  $\mu_1 \sim \mu_2$  on  $\mathcal{B}(E)$ , see Sato and Okazaki [12].

### 3. APPLICATION

We shall point out that many well-known dichotomies are unified by our result. The following dichotomies are known.

a) Let  $\mu, \nu$  be probability measures on  $\mathbb{R}$  equivalent to the Lebesgue measure ( $l \in I$ ) and  $\mu = \otimes \mu_l, \nu = \otimes \nu_l$  be the product measures. By Kakutani's dichotomy [8], it follows that  $\mu \sim \nu$  or  $\mu \perp \nu$ . We remark that Kakutani's dichotomy is more general and it requires no linear structure.

b) Let  $E$  be a locally convex Hausdorff space and  $\mu, \nu$  be two Gaussian measures on  $C(E, E^*)$ . Then Hajek and Feldman [2] [4] proved that either  $\mu \sim \nu$  or  $\mu \perp \nu$  holds.

c) Let  $\mathbb{R}^\infty$  be the countable product of real numbers. Let  $\mu_i$  be a probability measure on  $\mathbb{R}$  equivalent to the Lebesgue measure and  $\mu = \otimes \mu_i$  be the product measure. Let  $\nu$  be arbitrary Gaussian measure on  $\mathbb{R}^\infty$ . Then Fernique [3] proved that either  $\mu \sim \nu$  or  $\mu \perp \nu$  holds, answering the problem of Chatterji and Ramaswamy [1].

d) Let  $\mu, \nu$  be symmetric stable measures with discrete Lévy measure on a separable Banach space  $E$ . Then it holds that  $\mu \sim \nu$  or  $\mu \perp \nu$  (this result was informed in Janssen [7]). In general, a symmetric stable measure  $\mu$  has the characteristic functional of the form  $\exp\left(-\int_S |\langle y, x^* \rangle|^p d\sigma(y)\right)$ , where  $0 < p \leq 2$  and  $\sigma$  (the Lévy measure) is a bounded measure on the unit sphere  $S = \{x; \|x\| = 1\}$ , see Tortrat [14]. If  $\sigma$  is concentrated on a countable subset, then  $\mu$  is said to have discrete Lévy measure.

e) Let  $E$  be a locally convex Hausdorff space and  $H$  be a linear subspace of  $E$ . Let  $\mu, \nu$  be  $H$ -quasi-invariant and  $H$ -ergodic Probability measures on  $C(E, E^*)$ . Then it holds either  $\mu \sim \nu$  or  $\mu \perp \nu$ , see for example Skorohod [13], 23.

f) Let  $G_1, G_2$  be countable additive subgroups of  $\mathbb{R}^\infty$  and  $\mu_1, \mu_2$  be two probability measures such that  $\mu_i$  is  $G_i$ -quasi-invariant and  $G_i$ -ergodic,  $i = 1, 2$ . Suppose also that  $\mu_i$  is  $L(G_i)$ -quasi-invariant where  $L(G_i)$  denotes the linear hull of  $G_i$ ,  $i = 1, 2$ . Then Kanter [9] proved that either  $\mu_1 \sim \mu_2$  or  $\mu_1 \perp \mu_2$  holds.

Our dichotomy is a direct generalization of e) and f). But we have used the dichotomy e) in the proof of our theorem. We observe the dichotomies a), b), c) and d) from our viewpoints, and see that our dichotomy unifies them.

In a),  $\mu$  and  $\nu$  are  $\bigoplus_l \mathbb{R}$ -quasi-invariant and  $\bigoplus_l \mathbb{R}$ -ergodic, where  $\bigoplus_l \mathbb{R}$  is the direct sum of  $\mathbb{R}$ , see Zinn [15]. In the dichotomy b), the Gaussian measures  $\mu$  and  $\nu$  are quasi-invariant and ergodic under the translations of their reproducing kernel Hilbert spaces, for details we refer to Rozanov [11]. In the Fernique's dichotomy c),  $\mu$  (resp.  $\nu$ ) is quasi-invariant

and ergodic under the translation of the direct sum  $\bigoplus_i \mathbb{R}$  (resp. the reproducing kernel Hilbert space of  $\nu$ ). Hence the dichotomies  $a$ ),  $b$ ) and  $c$ ) are derived by our Theorem.

We examine the dichotomy  $d$ ). We shall see that  $\mu$  and  $\nu$  are quasi-invariant and ergodic by suitable linear subspaces. Let  $\{x_i\}$  be the discrete support of the Lévy measure  $\sigma$  of  $\mu$ ,  $a_i = \sigma(\{x_i\}) > 0$  and  $y_i = a_i x_i$ . Then the characteristic functional of  $\mu$  is given by  $\exp(-\sum_{i=1}^{\infty} |\langle y_i, x^* \rangle|^p)$ ,  $x^* \in E^*$ . Let  $\{f_i(\omega)\}$  be an independent identically distributed symmetric  $p$ -stable random variables with the characteristic functional  $\exp(-|t|^p)$ .

Then it follows that  $\int \exp(i \langle \sum_{i=1}^n f_i(\omega) y_i, x^* \rangle) dP(\omega) = \exp(-\sum_{i=1}^n |\langle y_i, x^* \rangle|^p)$

converges to the characteristic functional of  $\mu$ . So by Ito-Nisio's theorem [6],  $\sum_{i=1}^n f_i(\omega) y_i$  converges almost everywhere in  $E$  and the distribution of  $\sum_{i=1}^{\infty} f_i(\omega) y_i$  equals  $\mu$ . Let  $\lambda_i$  be the  $p$ -stable measure on  $\mathbb{R}$  with the characteristic functional  $\exp(-|t|^p)$  and  $\lambda = \otimes_i \lambda_i$  be the product measure on  $\mathbb{R}^{\infty}$ . Set  $T: \mathbb{R}^{\infty} \rightarrow E$  by  $T(t_m) = \sum_{n=1}^{\infty} t_n y_n$ . Then we have  $\lambda(\{(t_m) \in \mathbb{R}^{\infty}; \sum_{n=1}^{\infty} t_n y_n \text{ converges in } E\}) = 1$  and the image  $T(\lambda)$  coincides with  $\mu$ . Since  $\lambda$  is  $\bigoplus_i \mathbb{R}$ -quasi-invariant and  $\bigoplus_i \mathbb{R}$ -ergodic, see Zinn [14], it follows easily that  $\mu$  is also  $T(\bigoplus_i \mathbb{R})$ -quasi-invariant and  $T(\bigoplus_i \mathbb{R})$ -ergodic. Thus the dichotomy  $d$ ) is a special case of our Theorem.

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#### REFERENCES

- [1] S. D. CHATTERJI and S. RAMASWAMY, Mesures gaussiennes et mesures produits. Springer-Verlag, *Lecture Notes in Math.*, t. **920**, 1982, p. 570-580.
- [2] J. FELDMAN, Equivalence and perpendicularity of Gaussian Processes. *Pacific J. Math.*, t. **8**, 1958, p. 699-708.
- [3] X. FERNIQUE, Comparaison de mesures gaussiennes et de mesures produits. *Ann. Inst. H. Poincaré*, t. **20**, 1985, p. 165-175.
- [4] J. HAJEK, On a property of normal distributions of any stochastic processes. *Selected translations in Math. Statist. Prob.*, t. **1**, 1958-1961, p. 245-252.
- [5] E. HEWITT and K. A. ROSS, *Abstract harmonic analysis. Die Grundlehren der Math. Wiss.*, t. **115**, Springer-Verlag, 1963.
- [6] K. ITO and M. NISIO, On the convergence of sums of independent random variables. *Osaka J. Math.*, t. **5**, 1968, p. 35-48.



- [7] A. JANSSEN, *A survey about zero-one laws for probability measures on linear spaces and locally compact groups*. Springer-Verlag, *Lecture Notes in Math.*, t. **1064**, 1984, p. 551-563.
- [8] S. KAKUTANI, On equivalence of infinite product measures. *Ann. of Math.*, t. **49**, 1948, p. 214-224.
- [9] M. KANTER, Equivalence singularity dichotomies for a class of ergodic measures. *Math. Proc. Camb. Phil. Soc.*, t. **81**, 1977, p. 249-252.
- [10] J. VON NEUMANN, Uber einen Satz von Herrn M. H. Stone. *Ann. of Math.*, t. **33**, 1932, p. 567-573.
- [11] JU. A. ROZANOV, *Infinite-dimensional Gaussian distributions*. *Proc. Steklov Inst. Math.*, t. **108**, 1968, A. M. S. (English translation).
- [12] H. SATO and Y. OKAZAKI, Separabilities of a Gaussian Radon measure. *Ann. Inst. H. Poincaré*, t. **11**, 1975, p. 287-298.
- [13] A. V. SKOROHOD, Integration in Hilbert space. *Ergebnisse der Math.*, t. **79**, Springer-Verlag, 1974.
- [14] A. TORTRAT, Lois  $e(\lambda)$  dans les espaces vectoriels et lois stables. *Z. Wahrscheinlichkeitstheorie verw.*, t. **37**, 1976, p. 175-182.
- [15] J. ZINN, Zero-one laws for non-Gaussian measures. *Proc. of A. M. S.*, t. **44**, 1974, p. 179-185.

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