ANNALES DE L'I. H. P., SECTION B

R. ÉMILION

Additive and superadditive local theorems

Annales de l'I. H. P., section B, tome 22, nº 1 (1986), p. 19-36
<http://www.numdam.org/item?id=AIHPB_1986__22_1_19_0>

Ann. Inst. Henri Poincaré,
Vol. 22, n° 1, 1986, p. 19-36. *Probabilités et Statistiques*

Additive and superadditive local theorems

by

R. ÉMILION

Université Paris VI, Laboratoire de Probabilités, U. A., C. N. R. S. n° 224,
4, place Jussieu, Tour 56, 3^{e} étage, 75230 Paris Cedex 05

ABSTRACT. — We first prove a local ergodic superadditive theorem for positive contractions on L_1 (*). As an application of this result we obtain the local additive theorem for nonpositive L_1-contractions. Finally we prove the local ergodic theorem for n-parameter semigroups of nonpositive L_1-contractions.

These two last results give an answer to questions raised by U. Krengel ([*16*], p. 169).

RÉSUMÉ. — Nous prouvons d'abord un théorème suradditif pour des contractions positives de L_1 (*). Nous appliquons ce résultat pour obtenir un théorème additif pour des contractions non positives de L_1. Nous prouvons enfin le théorème ergodique local pour des semi-groupes à n paramètres de contractions non positives de L_1.

Ces deux derniers résultats répondent à des questions posées par U. Krengel ([*16*], p. 169).

(*) Part of a conference given at Marseille, 6 July 1982, « Journées de théorie ergodique ». C. I. R. M. Luminy.

(*) Partie d'une conférence donnée à Marseille, 6 juillet 1982, « Journées de théorie ergodique ». C. I. R. M. Luminy.

1. INTRODUCTION AND NOTATIONS

Let L_1 denote the usual space of equivalence classes of *complex*-valued integrable functions on a σ-finite measure space $(X, \mathscr{F}, \mu)$.

L_1^+ denotes the set of positive real-valued functions of L_1.

1.1 CONTINUOUS SEMIGROUPS.

Let $n \geqq 1$ be a fixed integer and let $T = (T_t)_{t \in (\mathbb{R}^+ - \{0\})^n}$ be a strongly continuous semigroup of linear contractions on L_1 (Note that we have not assumed the continuity of T at 0).

T is said *positive* if $T_t(L_1^+) \subset L_1^+$ for all t, and T is said *markovian* (resp. *submarkovian*) if T_t is positive and $\int T_t f \, d\mu = \int f \, d\mu$ for all $f \in L_1$ (resp. T_t is a positive contraction on L_1).

1.2 THE INITIALLY CONSERVATIVE AND DISSIPATIVE PARTS.

If $n = 1$ and T is submarkovian, then let $f \in L_1^+$, $f > 0$ μ-a. e. and $C = \left\{ \int_0^1 T_s f \, ds > 0 \right\}$.

The set C is independent of the choice of f and is called the *initially conservative* part of T. $D = X \backslash C$ is called the *initially dissipative* part of T.

$L_1(C, \mathscr{F}, \mu) = L_1(C)$ is invariant under T and the restriction of T to $L_1(C)$ can be completed continuously at $t = 0$. We therefore can define

$$R_0 = \text{strong} - \lim_{t \to 0} T_{t|L_1(C)} .$$

The results mentioned above are due to M. Akcoglu-R. V. Chacon [*1*]. They also hold for $n \geqq 1$ (M. Akcoglu-A. Del Junco [*2*]) and for bounded semigroups (the author [*9*]).

1.3 ADDITIVE AND SUPERADDITIVE PROCESSES.

If $n = 1$ and if T is as in Section 1.1, then a family $\{ F_t, t > 0 \}$, $F_t \in L_1$, is called an *additive process* with respect to T if

$$F_{t+s} = F_t + T_t F_s \qquad \text{for any} \quad t, s > 0 .$$

The process will be said *bounded* if $\sup_{0<t<1} \frac{\| F_t \|}{t} < +\infty$.

If T is submarkovian, then a family $\{ F_t, t > 0 \}$, $F_t \in L_1^+$, is called a *positive superadditive process* with respect to T if

$$F_{t+s} \geqq F_t + T_t F_s \qquad \text{for any} \quad t, s > 0 .$$

1.4 THE RESULTS of this paper concern local ergodic convergence of processes, i. e. the a. e. convergence of F_t/t as $t \to 0^+$ through any countable set.

In Section 2 we will study the case of superadditive processes for positive operators and in Section 3 the case of additive processes for nonpositive operators.

The last Section contains the results for n-parameter semigroups and also a continuous version of an inequality of A. Brunel [*5*].

2. SUPERADDITIVE LOCAL THEOREMS

We assume here that $n = 1$ and that T is submarkovian.

2.1 THEOREM. (*M. Akcoglu, U. Krengel* [*3*], *Section 4*).

If T is markovian and if $\{ F_t, t \geqq 0 \}$ is a positive superadditive process with respect to T, then

$$\lim_{t \to 0^+} \frac{F_t}{t} \text{ exists a. e. on } X .$$

In the submarkovian case the existence of the limit a. e. on X may fail as one can see with the trivial example $T_t = 0$.

In [*10*], B. Hachem and the author have shown that a supplementary condition yields the a. e. convergence on X.

In fact the a. e. convergence always holds on the conservative part:

2.2 THEOREM (D. Feyel [*12*]).

If T is submarkovian and if $\{ F_t, t > 0 \}$ is a positive superadditive process with respect to T, then

$$\lim_{t \to 0^+} \frac{F_t}{t} \text{ exists a. e. on } C .$$

D. Feyel's theorem more generally holds for superabelian processes and therefore for superadditive processes, by the use of a tauberian theorem.

In this Section we will show that a refinement of our previous arguments in [10] yields the a. e. convergence on C, without using the Tauberian Theory.

The following theorem, the main result of this Section, slightly improves the two theorems mentioned above, since the superadditivity is only assumed on the set of positive dyadic rationals B, where $B = \{ k2^{-i}, k, i = 1, 2, 3... \}$. We also recall that a positive operator on L_1 can be extended to $M^+(X) = \{ f : X \to \mathbb{R}^+ \cup \{ +\infty \} \mid f \text{ measurable} \}$.

2.3 THEOREM.

Let $F = (F_t)_{t>0}$ *be a family such that*

(2.4) $$F_t \in M^+(X) \quad \textit{for all } t > 0 .$$

(2.5) $$F_{t+s} \geqq F_t + T_t F_s \quad \textit{for all } s, t \in B .$$

(2.6) $$F_t \geqq F_s \quad \textit{for all } t, s > 0,\ t \geqq s .$$

Then we have

(2.7) $\lim_{t \to 0^+} F_{t/t}$ *exists a. e. on* X *if* $F_t \in L_1^+$ *and if* T *is markovian*

(2.8) $\lim_{t \to 0^+} \frac{F_t}{t}$ *exists a. e. on* C *if* $F_t \in M^+$ *and if* T *is markovian*

(2.9) $\lim_{t \to 0^+} \frac{F_t}{t}$ *exists a. e. on* C *if* $F_t \in M^+$ *and if* T *submarkovian*.

Note that condition 2.6 and the condition of positivity can be dropped:

2.10 COROLLARY.

Let $F = (F_t)_{t \in B}$ *be a family such that*

- $F_t \in M^+$ *for all* $t \in B$
- $F_{t+s} \geqq F_t + T_t F_s$ *for all* $t, s \in B$.

Then the conclusion of Theorem 2.1 holds if the limits are taken as $t \to 0^+$ *through* B.

(Apply the Theorem 2.3 to $F = (F_t)_{t>0}$ where $F_t = \sup_{\substack{0 < s \leqq t \\ s \in B}} F_s$).

2.11 COROLLARY.

Let $(F_t)_{t \in B}$ *be a family such that*

- $F_t : X \to \mathbb{R} \cup \{ -\infty, +\infty \}$ is a measurable function for all $t \in B$
- $F_t^- \in L_1^+$ for all $t \in B$
- $\sup_{\substack{0 < t < 1 \\ t \in B}} \frac{\| F_t^- \|}{t} < +\infty$
- $F_{t+s} \geqq F_t + T_t F_s$ for all $t, s \in B$.

Then the conclusion of Theorem 2.3 holds if the limits are taken as $t \to 0^+$ *through* B.

Proof of Corollary 2.11. — We have $F_{t+s}^- \leqq F_t^- + T_t F_s^-$ and thus there exists an additive process $(H_t)_{t \in B}$ such that $F_t^- \leqq H_t$ for all $t \in B$. $(F_t + H_t)$ then verifies the hypothesis of Corollary 2.10. Since $\lim_{t \to 0^+} \frac{H_t}{t}$ exists and since $F_t = (F_t + H_t) - H_t$, $\lim_{t \to 0^+} \frac{F_t}{t}$ exists a. e. This is a standard argument ([3] [12]).

Proof of Theorem 2.3.

I. *Proof of 2.7.* — This is a refinement of the arguments of M. Akcoglu-U. Krengel ([3], Section 4) together with a combinatorial lemma (lemma 1 in the paper of M. Akcoglu-L. Sucheston [4]).

By (2.6) we can define $\int_0^t F_s \, ds$ as the strong limit of the Riemann sums.

Let $0 < t < t_0$, t and $t \in B : t = k2^{-i} \, t_0 = l2^{-i}$.

By (2.5) the sequence $(F_{j2^{-i}})_{j \in \mathbb{N}}$ is superadditive with respect to the positive operator $T_{2^{-i}}$. Hence the combinatorial lemma yields

$$\left(1 - \frac{t}{t_0}\right) F_t \leqq (I - T_t) \frac{2^{-i}}{t_0} \sum_{j=1}^{l} F_{j2^{-i}} + 2^{-i} \sum_{j=1}^{k} T_{j2^{-i}} \frac{1}{t_0} F_{t_0}.$$

As $i \to +\infty$ we obtain

$$\left(1 - \frac{t}{t_0}\right) F_t \leqq (I - T_t) \frac{1}{t_0} \int_0^{t_0} F_s ds + \int_0^t T_s\left(\frac{1}{t_0} F_{t_0}\right) ds \tag{2.12}$$

for all $t, t_0 \in B$ $0 < t < t_0$.

Since T is markovian 2.12 implies

$$\text{for } \; 0 < t < t_0 \;\; t, t_0 \in B : \left(1 - \frac{t}{t_0}\right) \int \frac{F_t}{t} d\mu \leqq \frac{1}{t_0} \int F_{t_0} d\mu < +\infty . \tag{2.13}$$

Again since T is markovian (2.5) and (2.13) imply

$$\gamma = \lim_{\substack{t \to 0^+ \\ t \in B}} \frac{1}{t} \int F_t d\mu \quad \text{exists and is finite.} \tag{2.14}$$

By (2.5), (2.14) and lemma 4.1 in [3] we can find a positive additive process, say $(G_t)_{t>0}$, such that $G_t \leqq F_t$ for all $\underline{t \in B}$ and $\int \frac{G_t}{t} d\mu = \gamma$.

Since (G_t) is continuous ($\gamma < +\infty$) and F is increasing (2.6), we have

$G_t \leqq F_t$ *for all* $t > 0$. Moreover, since $(G_t)_{t>0}$ is a positive additive process such that $\gamma = \int \frac{G_t}{t} < +\infty$, $\lim_{t\to 0^+} \frac{G_t}{t}$ exists and is finite a. e. on X. Therefore, if we write $F_t = (F_t - G_t) + G_t$, we see that it suffices to prove (2.7) when $\gamma = 0$.

In this case, let $\varepsilon > 0$ and $t_0 \in B$ such that $\frac{1}{t_0}\int F_{t_0} d\mu < \varepsilon$.

Put $H_t = (I - T_t)\frac{1}{t_0}\int_0^{t_0} F_s ds + \int_0^t T_s\left(\frac{1}{t_0}F_{t_0}\right)ds$ for all $t > 0$.

The strong continuity of T at $t > 0$ shows that strong $-\lim_{u\to t} H_u = H_t$ for any $t > 0$.

Moreover, (2.12) shows that $H_t \geqq 0$ for all $t \in B$, $0 < t < t_0$, and hence $H_t \geqq 0$ *for all* $t0 < t < t_0$. Since H_t is an additive process, this implies that $H_t \geqq 0$ for all $t > 0$.

Since F is increasing (2.6) and since H is continuous, (2.12) shows that $\left(1 - \frac{t}{t_0}\right)F_t \leqq H_t$ *for all* t, $0 < t < t_0$.

Now, $\frac{1}{t}\int H_t d\mu = \frac{1}{t_0}\int F_{t_0} d\mu \leqq \varepsilon < +\infty$ and H_t is a positive additive process, so $\lim_{t\to 0^+} \frac{1}{t} H_t$ exists a. e. on X.

Consequently

$$\int \overline{\lim_{t\to 0^+}} \frac{F_t}{t} d\mu \leq \int \overline{\lim_{t\to 0^+}} \frac{H_t}{t} d\mu \leqq \lim_{t\to 0^+} \frac{1}{t}\int H_t d\mu \leqq \varepsilon .$$

ε being arbitrary, $\lim_{t\to 0^+} \frac{F_t}{t} = 0$ a. e. on X when $\gamma = 0$.

Proof of 2.8. — We use here an argument of D. Feyel [*12*] to prove that 2.7 implies 2.8. Let $h \in L_1^+(C)$ such that $h < \overline{\lim_{t\to 0^+}} \frac{F_t}{t}$ a. e. on a measurable set $Y \subset C$.

The process $G_t = \inf\left(F_t, \int_0^t T_s h ds\right)$ verifies (2.4), (2.5), (2.6) and $G_t \in L_1^+$; by (2.7) $\lim_{t\to 0^+} t^{-1} G_t$ exists a. e. on X.

But, by the local theorem of U. Krengel we have

$$\lim_{t\to 0^+} \frac{1}{t}\int_0^t T_s h ds = R_0 h = h \ [15] .$$

Hence $h \leq \underline{\lim_{t\to 0^+}} \frac{F_t}{t}$ and $\lim_{t\to 0^+} \frac{1}{t}\frac{F_t}{t}$ exists a. e. on C.

Proof of 2.9. — For details see R. Emilion-B. Hachem [*10*].

As in the proof of 2.8, it suffices to prove 2.9 when $F_t \in L_1^+$. Let $\omega \notin X$ and $\overline{X} = X \cup \{\omega\}$. Let $\overline{\mu}$ be the measure on $\overline{X}$ defined by $\overline{\mu}|_X = \mu$ and $\overline{\mu}\{\omega\} = 1$. Then, if $\overline{f} \in L_1(\overline{X})$ and if $f = \overline{f}|_X$,

$$\overline{T}_t(\overline{f}) = (T_t f)1_X + \left(\int f d\mu - \int T_t f d\mu + \overline{f}(\omega)\right)1_{\{\omega\}}$$

defines a markovian semigroup on $L_1(\overline{X}, \overline{\mu})$.

As in [*10*], let us define a family $(\beta_t)_{t>0}$, $\beta_t \in \mathbb{R}^+ \cup \{+\infty\}$ such that

$$\beta_t = \sup_{\substack{0<s<t \\ s \in B}} \beta_s$$

and

$$\beta_s = \lim_{i \to +\infty} \uparrow \beta_s^i$$

where $s = k2^{-i} \in B$ and $\beta_s^i = \sum_{j=0}^{k-1} \int (F_{j2^{-i}} - T_{2^{-i}} F_{j2^{-i}}) d\mu$.

(Note that β_t can be infinite).

In ([*10*] 4.1 and 4.2) it is proved that $\beta_{t+s} \geqq \beta_t + \beta_s + \int (F_t - T_s F_t) d\mu$ for all $t, s \in B$.

Therefore the process $\overline{F}_t = F_t 1_X + \beta_t 1_{\{\omega\}}$, $t > 0$, verifies

- $\overline{F}_t \in M^+(\overline{X})$
- $\overline{F}_{t+s} \geqq \overline{F}_t + \overline{T}_t \overline{F}_s$ for all $t, s \in B$.
- $\overline{F}_t \geqq \overline{F}_s$ for all $t \geqq s > 0$.

Hence, by (2.8) $\lim_{t \to 0^+} \frac{\overline{F}_t}{t}$ exists a. e. on $\overline{C} = C \cup \{\omega\}$, the initially conservative part of $\overline{T}$.

Since $F_t = 1_X \overline{F}_t$ and since $C \subset X$, we have $\lim_{t \to 0^+} \frac{F_t}{t}$ exists a. e. on C. This completes the proof.

Remarks.

- Since strong-$\lim_{t \to 0^+} T_{t|L_1(C)} = R_0$ is a conditionnal expectation operator ([*3*], p. 208), there is no loss in generality in assuming that $R_0 = I$ [*12*].
- (2.7) and (2.8) improve the superadditive theorem in [*3*] and [*12*], since the condition 2.5 only holds on B.
- (2.9) improve the superadditive theorem in [*10*]. Also note that the extra condition given in [*10*] is not necessary for the a. e. convergence to hold on X: take $T_t = 0$ and $F_t = tf$, $f \in L_1^+$.
- As in [*10*] we deduce the result for the submarkovian case from the markovian one.

2.14 L_p-SUPERADDITIVE PROCESSES $(1 \leqq p \leqq \infty)$.

The L_p-local additive theorems were proved by using the L_1-ones (R. Sato [*21*], M. Lin [*18*]). The same holds for superadditive processes (the author [*8*]). Hence, since the present results on L_1 improve those of [*10*], they also improve the L_p-results of [*8*]. We omit the statements.

3. AN ADDITIVE LOCAL THEOREM

In this section we prove the following result (Theorem 3.3), the assumption of continuity of a nonpositive-L_1-contractions semigroup at 0 can be dropped in the local ergodic theorems of C. Kipnis, Y. Kubokawa ([*14*] [*17*]) and D. Feyel [*11*]. This generalizes the positive operators case (M. A. Akcoglu, R. V. Chacon, U. Krengel [*11*] [*3*]) and answer a question raised by U. Krengel ([*16*], p. 169).

3.1 THE SEMIGROUP.

We consider a semigroup $T = (T_t)_{t>0}$ of L_1-contractions which are not necessarily positive. It is assumed that T is strongly measurable and thus strongly continuous at every $t > 0$, but T need not be continuous at 0. $\hat{T} = (\hat{T}_t)_{t>0}$ will denote the modulus semigroup of T (see C. Kipnis [*14*], p. 372). It is important to mention that we consider here *complex* L_1-spaces.

3.2 THE RESULTS OF THIS SECTION.

Theorem 3.3 generalizes the local additive theorem of D. Feyel [*11*] which holds if the resolvent of $\hat{T}$ exists (hence $\hat{T}$ must be continuous at every $t > 0$) and is proper, that is $X = \hat{C}$ the initially conservative part of $\hat{T}$. These conditions imply that $\hat{T}$ is continuous at 0 ([*1*] or [*9*]) and thus T is necessarily continuous at 0 (see 3.6 below). Our result can be proved by using D. Feyel's theorem [*11*] but as [*11*] depends on a rather nontrivial dilation theorem, we prove the additive theorem for semigroups verifying 3.1 as an interesting application of the local superadditive theorem for positive contractions ([*12*] or 2.9). We also prove that we may assume that T_0 exists and that $T_0 = I$ the identity operator. This generalizes the positive operator case [*2*] and the proof depends on a decomposition of the space due to M. A. Akcoglu, A. Brunel [*0*]. The last part of this section contains some generalizations of the results obtained in the positive operators case and also a very simple proof of a local theorem of R. Sato [*19*].

3.3 THEOREM.

Let T *be as in Section 3.1 and let* $F = (F_t)_{t>0}$ *be a locally bounded additive process with respect to* T. *Then* $\lim_{t \to 0^+} \frac{F_t}{t}$ *exists a. e. on* X.

Proof. — We divide the proof into four parts.

Part I. — The continuity of the modulus semigroup.

If $h : \mathbb{R}^+ - \{0\} \to \mathbb{R}$ is any right-continuous function then for any $a \in \mathbb{R}$ the set $\{ t > 0 \mid h(t) > a \}$ is a countable union of intervals, hence h is measurable. Now, for any $f \in L_1, f$ being real-valued, the map $t \to \hat{T}_t f$ is right-continuous at every point $t > 0$ (C. Kipnis [14], p. 374). Since $\hat{T}$ maps real valued functions to real functions, the above argument shows that $t \to \int (\hat{T}_t f) g d\mu$ is measurable for any real-valued $g \in L_\infty$, and moreover the right-continuity implies that the set $\{ \hat{T}_t f, t > 0 \}$ is separable. Consequently the map $t \to \hat{T}_t f$ is measurable (see [7], III.6.11) for any real-valued f and thus for any complex-valued $f \in L_1$, and since the $\hat{T}_t$ are contractions, $t \to \hat{T}_t$ is strongly continuous at every point $t > 0$ (see the proof of VII.1.3 in [7], p. 616).

Therefore we may define the initially conservative and dissipative parts of $\hat{T} : \hat{C} = \{ f_0 > 0 \}$ and $\hat{D} = X \backslash \hat{C}$, where $f_0 = \int_0^1 \hat{T}_s f ds$ for any fixed $f \in L_1^+, f > 0$ a. e. (Akcoglu, Chacon [1]).

Since $1_D \hat{T}_t = 0$ and since $| T_t | \leqq \hat{T}_t$, we can consider the restrictions of $\hat{T}$ and T to $L_1(\hat{C})$ and we put

$$\hat{R}_0 = \text{strong-}\lim_{t \to 0^+} \hat{T}_{t|L_1(\hat{C})} \qquad \text{(see [1] or [9])}$$

and

$$R_0 = \text{strong-}\lim_{t \to 0^+} T_{t|L_1(\hat{C})} \qquad \text{(see 3.6 below)}.$$

Also, note that the additivity relation $F_{t+s} = F_t + T_t F_s$ together with $\sup_{0<t<1} \| F_t/t \|_1 < +\infty$ imply that F is continuous, $F_t \in L_1(\hat{C})$ and $F_t = R_0 F_t$.

Part II. — The reduction to the superadditive case.

The inequality $| F_{t+s} | \leqq | F_t | + \hat{T}_t | F_s |$ and lemma 4.1 in [3] imply that there exists an additive positive process with respect to $\hat{T}$ such that $\sup_{0<t<1} \| H_t/t \| < +\infty$ and such that $| F_t | \leqq H_t$ for any dyadic rational $t > 0$ and hence for any $t > 0$ because F and H are continuous. By the main result of [3] we know that $\lim_{t\to 0^+} H_t/t$ exists a. e. and is finite, and since H is increasing we also have $q\text{-}\sup_{0<t<1} H_t/t < +\infty$ a. e. and thus $q\text{-}\sup_{0<t<1} | F_t/t | < +\infty$ a. e. (q-sup means that the sup is taken over Q^+).

Admit for a moment the following point : (E).

(E) Let f_n be a sequence of complex valued functions which are measurable with respect to a measure space $(X, \mathscr{B}, \mu)$. If $\sup_n | f_n | < +\infty$ a. e. then there exists a complex-valued $\mathscr{B}$-measurable function such that $\liminf_{n\to+\infty} | f_n - f | = 0$ a. e.

Then we will prove the theorem 3.3 as follows:

Let t_n be a sequence of rational numbers such that $t_n > 0$ and $t_n \to 0^+$.

Let $f_n = F_{t_n}/t_n$ and apply (E) to f_n in order to obtain a function f. Then $f \in L_1(\hat{C})$ since $| f | \leqq \lim_{t\to 0^+} H_t/t$. Suppose that $R_0 f = f$.

The process $\left(| F_t - \int_0^t T_s f\, ds | \right)_{t>0}$ is subadditive with respect to $\hat{T}$. Let $(I_t)_{t>0}$ be a bounded additive process with respect to $\hat{T}$ such that $\left| F_t - \int_0^t T_s f\, ds \right| \leqq I_t$ for all $t > 0$. Then, by (2.9) or [12]

$$\lim_{t\to 0^+} \left(\frac{I_t}{t} - \left| \frac{F_t}{t} - \left(\frac{1}{t}\right) \int_0^t T_s f\, ds \right| \right)$$

exists a. e. on C and hence a. e. on X. Therefore $\lim_{t\to 0^+} \left| \frac{F_t}{t} - \frac{1}{t} \int_0^t T_s f\, ds \right|$ exists a. e. on X.

It then suffices to apply the local theorem of Kipnis, Kubokawa to obtain 3.3:

$$\begin{aligned}
\limsup_{t\to 0^+} | F_{t/t} - R_0 f | &= \limsup_{t\to 0^+} \left| F_{t/t} - \left(\frac{1}{t}\right) \int_0^t T_s f\, ds \right| \\
&= \liminf_{t\to 0^+} \left| F_{t/t} - (1/_t) \int_0^t T_s f\, ds \right| \\
&= \liminf_{t\to 0^+} | F_{t/t} - R_0 f | \\
&= \liminf_{t\to 0^+} | F_{t/t} - f | \\
&\leqq \liminf_{n\to+\infty} | f_n - f | = 0 \quad \textit{a. e.} \text{ on } X.
\end{aligned}$$

Hence $\lim_{t\to 0^+} F_{t/t} = R_0 f = f$ a. e. on X.

Part III. — The identification of the limit operator.

To prove that $R_0 f = f$ and more generally that we may assume that $R_0 = I$, we use a decomposition of the space due to Akcoglu, Brunel [*10*].

Clearly $\hat{R}_0$ is conservative because $\hat{R}_0 f_0 = f_0$ and, after a change of measure, $\hat{R}_0$ is a conditional expectation operator with respect to a sub-σ-algebra of $\hat{C}$, say $\mathcal{B}$ ([*13*], p. 208).

Let $\Gamma_0 \subset \hat{C}$ be the union of the supports of all R_0^*-invariant functions of $L_\infty(\hat{C})$, and let $\Delta_0 = \hat{C} \backslash \Gamma_0$.

Let $g \in L_\infty(\hat{C})$ and put $h = \int_0^t T_s^* g ds$. For any $u \in L_1(\hat{C})$, we have

$$\langle R_0^*(1_{\hat{C}} h), u \rangle = \langle 1_{\hat{C}} h, R_0 u \rangle = \langle h, 1_{\hat{C}} R_0 u \rangle = \langle h, R_0 u \rangle = \langle R_0^* h, 1_{\hat{C}} u \rangle = \langle h, 1_{\hat{C}} u \rangle = \langle 1_{\hat{C}} h, u \rangle .$$

Hence $R_0^*(1_{\hat{C}} h) = 1_{\hat{C}} h$ and $1_{\Delta_0} h = 0$. Furthermore

$$0 = \langle u, 1_{\Delta_0} h \rangle = \left\langle \int_0^t T_s(1_{\Delta_0} u) ds, g \right\rangle \qquad \text{for any} \quad g \in L_\infty(\hat{C}),$$

implies that $\int_0^t T_s(1_{\Delta_0} u) ds = 0$. Since $R_0 = \text{strong-}\lim_{s \to 0^+} T_{s|L_1(\hat{C})}$, it follows that $R_0 = \text{strong-}\lim_{t \to 0^+} (1/_t) \int_0^t T_{s|L_1(\hat{C})} ds$ and consequently $R_0(1_{\Delta_0} u) = 0$ if $u \in L_1(\hat{C})$.

Recalling that $F_t \in L_1(\hat{C})$, and that $|R_0| = \hat{R}_0$ ([*14*]) we then get $F_t = R_0 F_t = R_0(1_{\Gamma_0} F_t) = \bar{\varepsilon} \hat{R}_0(\varepsilon 1_{\Gamma_0} F_t)$ where $\varepsilon \in L_\infty(\Gamma_0)$ and $|\varepsilon| = 1$ a. e. (by [*0*]).

Thus $\varepsilon F_t = \hat{R}_0(\varepsilon 1_{\Gamma_0} F_t)$ is $\mathcal{B}$-measurable and by (E), εf is also $\mathcal{B}$-measurable, that is $\varepsilon f = \hat{R}_0(\varepsilon f) = \varepsilon R_0(1_{\Gamma_0} f) = \varepsilon R_0(f)$. Consequently $f = R_0 f$ a. e. on Γ_0. But the relation $F_t = \bar{\varepsilon} \hat{R}_0(\varepsilon 1_{\Gamma_0} F_t)$ shows that $F_t \in L_1(\Gamma_0)$, Hence $f \in L_1(\Gamma_0)$ and since Γ_0 is invariant we have $f = R_0 f$ a. e.

Part IV. — A selection lemma.

Finally it remains to prove (E) in part II. This is independent of the other arguments.

If the f_n are real-valued it suffices to take $f = \liminf_n f_n$.

If the f_n are complex-valued this is less obvious.

We identify the complex field with $\mathbb{R}^2$.

First suppose that $|f_n| \leq 1$ a. e.

Let $\bar{A}_k(x)$ denote the closure of the set $\{ f_n(x) \mid n \geq k \}$. Let $A(x) = \bigcap_{k \geq 0} \bar{A}_k(x)$. so that $A(x)$ is a compact set (since we have assumed that $|f_n| \leq 1$ a. e.).

Let $r(x) = \liminf_n \operatorname{Re}(f_n(x)) = \inf\{\operatorname{Re}(z) \mid z \in A(x)\}$. where $\operatorname{Re}(z)$ denotes the real part of the complex number z. Then r is $\mathscr{B}$-measurable.

Let $s(x) = \inf\{y \mid (r(x), y) \in A(x)\}$ and finally let $f(x) = (r(x), s(x))$. We will show that s is $\mathscr{B}$-measurable.

Let $\alpha \in \mathbb{R}$. Let $g_n^\alpha(x) = r(x)$ if $\operatorname{Im} f_n(x) \geqq \alpha$ and $g_n^\alpha(x) = 2$ if $\operatorname{Im} f_n(x) < \alpha$. Since r is measurable, so is g_n^α, and since $r(x) < 2$ a. e. we have

$$\{x \in X \mid s(x) < \alpha\} = \{x \mid \liminf_n g_n^\alpha = 2\},$$

which shows that s is $\mathscr{B}$-measurable.

f is then $\mathscr{B}$-measurable and $f(x) \in A(x)$ implies that there exists a subsequence $n_j(x) \to \infty$ such that $f_{n_j(x)} \to f(x)$, that is $\liminf_n |f_n - f| = 0$ a. e.

(E) is then proved whenever $|f_n| \leqq 1$ a. e. If we only have $\sup_n |f_n| < +\infty$ a. e. then let $g_n = f_n/(1 + |f_n|)$. The above argument shows that there exists a $\mathscr{B}$-measurable function g such that for a. e. x there exists a sequence $n_j(x)$ such that $g_{n_j(x)}(x)$ converges to $g(x)$. This implies that $|g_{n_j(x)}(x)|$ converges to $|g(x)|$ and thus $|f_{n_j(x)}(x)|$ converges to $|g(x)|/(1 - |g(x)|)$ (note that $\sup_n |f_n| < +\infty$ a. e. implies that $|g| \neq 1$ a. e.).

Finally $f_{n_j(x)}(x)$ converges to $g(x)(1 + (|g(x)|/(1 - |g(x)|)) = f(x)$. So, f is $\mathscr{B}$-measurable and verifies $\liminf_n |f_n = f| = 0$ a. e.

This completes the proof of (E) and therefore of theorem 3.3.

The following points generalize the results obtained in the positive operators case. (3.8) was proved by R. Sato [*19*] in the particular case $M = 1$, we give here a very simple proof of this result.

3.4 THEOREM.

Let T *he as in Section 3.1. Then*

3.5 $\hat{T}$ *is strongly continuous at every point* $t > 0$.

3.6 $T|_{L_1(\hat{C})}$ *is strongly continuous at 0.*

3.7 T *is continuous at 0 if and only if* $\hat{T}$ *is continuous at 0.*

3.8 *If there exists* $M > 0$ *such that* $\|T_t f\|_\infty \leqq M \|f\|_\infty$ *for any* $f \in L_1 \cap L_\infty$, *and* $0 < t < 1$, *then* T *is continuous at 0.*

Proof. — 3.5 is proved above and 3.7 implies 3.6 as $\hat{T}|_{L_1(\hat{C})}$ is continuous at 0. The « only if » part of 3.7 is known ([*14*] [*17*]). Conversely suppose that $\hat{T}$ is continuous at 0.

Then, consider the vector space $H = \{f \in L_1 \mid \lim_{t \to 0^+} T_t f \text{ exists}\}$ which is closed and also weakly closed. For any $f \in L_1$, the inequality $|T_f f| \leqq \hat{T}_t |f|$

implies that there exists $f^* \in L_1$ and a sequence $t_n \to 0^+$ such that $f^* = \underset{t_n \to 0^+}{w\text{-lim}}\, T_{t_n} f$. Since $T_{t_n} f \in H$ we have $f^* \in H$. This implies that $f \in H$ as $T_t f^* = T_t f$ for any $t > 0$. Hence $H = L_1$ (See also the proof of lemma 1 in [*19*]).

To prove 3.8 it suffices to show that $1_A \in H$ for any A such that $\mu(A) < +\infty$. This is easy: $|T_t 1_A| \leqq M$ implies that there exists $f^* \in L_1$ and $t_n \to 0^+$ such that $f^* = \underset{t \to 0^+}{w\text{-lim}}\, T_{t_n} 1_A$; it then suffices to complete the proof as above.

Remark. — If μ is finite we can replace in 3.8 $\| \ \|_\infty$ by $\| \ \|_p$ $(1 < p < \infty)$ (see the proof of 2.1 in [*9*]).

4. AN n-PARAMETER LOCAL ERGODIC THEOREM

In this last Section we prove an inequality for n-parameter semigroups (Theorem 4.3 below) in order to obtain the following local ergodic theorem. We consider complex L_1 spaces.

4.1 THEOREM.

Let $T = \{ T_t \in (\mathbb{R}^+)^n \}$ *be a strongly continuous n-parameter semigroup of* L_1*-nonpositive contractions. Then, for any* $f \in L_1$

$$\lim_{r \to 0^+} r^{-n} \int_0^r \dots \int_0^r T_{(t_1,\dots,t_n)} f \, dt_1 \dots dt_n = T_0 f \quad a.\,e.\ on\ X.$$

This theoiem answers a question also raised by U. Krengel ([*16*], p. 169) and it improves previous local theorems obtained in the following cases for semigroups which are continuous at zero:

. T positive and $n = 1$ (U. Krengel [*15*])
. T positive and $n \geqq 1$ (T. R. Terrell [*21*])
. T nonpositive and $n = 1$ (C. Kipnis [*14*], Y. Kubokawa [*17*])
. T nonpositive, $n \geqq 1$ and T_t L_∞-contraction (T. R. Terrell [*21*])
. T nonpositive, $n \geqq 1$ and T_t L_1-isometry (S. A. McGrath [*33*]).

Our result is proved by using the above basic theorem of U. Krengel stated in dimension one for positive operators.

4.2 REDUCTION OF THE DIMENSION.

To reduce the dimension we first show that Dunford, Schwartz [*7*] and C. Kipnis [*14*] techniques yield the following continuous version of an inequality of A. Brunel [*5*].

4.3 THEOREM.

Let $T = \{ T_{(t_1,\dots,t_n)}, t_i > 0, i = 1, \dots, n \}$ *be a strongly continuous semi-groups of positive* L_1*-contractions. Then there exists a one-parameter strongly continuous semigroup of positive* L_1*-contractions, say* $(U_t)_{t>0}$, *such that*

$$\forall r > 0, \forall f \in L_1, r^{-n} \int_0^r \dots \int_0^r | T_{(t_1,\dots,t_n)} | \, |f| \, dt_1 \dots dt_n \leqq c_n \bar{r}^{-1} \int_0^{\bar{r}} (U_t |f|) dt$$

where $\bar{r} = r^{2-k}$ *if* n *is such that* $2^{k-1} < n \leqq 2^k$, *and* c_n *is a constant which is independent of* T *and* f.

Moreover, if T *is continuous at zero, then so is* U *and* $U_0 = | T_0 |$ *the linear modulus of* T_0.

The continuous case also implies the discrete one:

4.4 COROLLARY.

Let $T_1, \dots, T_n$ be commuting L_1-contractions, then there exists two positive L_1-contractions U and A such that

$$\forall r \in \mathbb{N}, \quad r \geqq 1, \quad \forall f \in L_1,$$

$$r^{-n} \sum_{i_1=0}^{r-1} \dots \sum_{i_n=0}^{r-1} | T_1^{i_1} \dots T_n^{i_n} | \, |f| \leqq d_n \bar{r}^{-1} \sum_{i=0}^{\bar{r}-1} U^i(A |f|)$$

where $\bar{r}$ is the integer part of $r^{2^{-k}} + 2$ if $2^{k-1} < n \leqq 2^k$, and d_n is a constant which is independent of the T_i and of f.

Remarks.

. 4.3 yields several simplifications in the proofs of Dunford, Schwartz.

. 4.4 is a weaker form of A. Brunel's inequality [*5*] but it yields the discrete pointwise theorems of [*5*].

Proof of Theorem 4.3. — We follow the technique introduced by Dunford, Schwartz ([*7*], p. 700-702) and C. Kipnis ([*14*], p. 372-374). Hence we omit the details.

We may assume that $n = 2^k$, $k \in \{ 1, 2, 3 \dots \}$.

Let $h : (\mathbb{R}^+ - \{ 0 \})^n \to \mathbb{R}$. We will say that h is right-continuous if $\lim_{v \to 0^+} h_{u+v} = h_u$, where $v = (v_1, \dots, v_n) \to 0^+$ means that $v_i \to 0^+$ for $i = 1, \dots, n$. For such an h, it is easily seen that for any $a \in \mathbb{R}$, and any bounded interval I of $\mathbb{R}^n$, the set $\{ t \in (\mathbb{R}^+ - \{ 0 \})^n / h(t) > a \} \cap I$ is a countable union of intervals of $\mathbb{R}^n$. Hence h is measurable.

Now, if $P_u = | T_u |$ is the linear modulus of the contraction T_u (see Chacon, Krengel [*6*]) then $P = (P_u)$ is a sub-semigroup (i. e. $P_{u+v} \leqq P_u P_v$) and

$\text{strong-}\lim_{v \to 0^+} P_{u+v} = P_u$. Hence, as in the proof of 3.3, we see that P is strongly measurable.

Therefore if $p = \frac{n}{2}$ and $\alpha_x(t) = 1_{\{\mathbb{R}^+ - \{0\}\}} \frac{x}{2} (\pi^2 t)^{-3/2} e^{-x^2/4t}$ for any $x > 0$ and $t \in \mathbb{R}$, we can define the p-dimensional sub-semigroup of Dunford, Schwartz:

$$S_{(x_1, \ldots, x_p)} f = \int_0^\infty \ldots \int_0^\infty \alpha_{x_1}(t_1) \alpha_{x_1}(t_2) \ldots \alpha_{x_p}(t_{n-1}) \alpha_{x_p}(t_n) P_{(t_1, \ldots, t_n)} f \, dt_1 \ldots dt_n$$

for any $f \in L_1$.

It is then easily seen that $S = (S_{(x_1, \ldots, x_p)})$ is strongly continuous at every point $x = (x_1, \ldots, x_p)$, $x_i > 0$ (see e. g. [*21*], p. 271).

Hence, since $n = 2^k$, Dunford, Schwartz inductive construction yields a one-dimensional strongly continuous sub-semigroup of positive contractions, say $(A_t)_{t>0}$, such that

$$r^{-n} \int_0^r \ldots \int_0^r P_{(t_1, \ldots, t_n)} | f | \, dt_1 \ldots dt_n \leqq c_n \bar{r}^{-1} \int_0^{\bar{r}} A_t | f | \, dt$$

for any $f \in L_1$ and $r > 0$.

Finally, following C. Kipnis we put

$$U_t = \sup_{(t_i)} A_{t_1} \ldots A_{t_n - t_{n-1}}, \qquad \text{where} \quad (t_i)$$

denote any finite partition of $[0, t]$ such that $0 < t_1 < \ldots < t_n = t$. $(U_t)_{t>0}$ is a semigroup which verifies $A_t \leqq U_t$ and the proof of 3.3 shows that (U_t) is strongly continuous at every $t > 0$.

Of course, we have the first assertion of the Theorem 4.3:

$$r^{-n} \int_0^r \ldots \int_0^r | T_{(t_1, \ldots, t_n)} | \, | f | \, dt_1 \ldots dt_n \leqq c_n \bar{r}^{-1} \int_0^{\bar{r}} U_t | f | \, dt .$$

It T is strongly continuous at zero, then $\text{strong-}\lim_{u \to 0^+} | T_u | = | T_0 |$ by the inequality $| T_u | = | T_u T_0 | \leqq | T_u | \, | T_0 |$ together with the fact that T_u is a contraction for all u.

It is then easily seen that the reduced sub-semigroups are continuous at zero (see e. g. [*21*], p. 271):

$$\text{strong-}\lim_{x_i \to 0^+} S_{(x_1, \ldots, x_p)} = | T_0 | .$$

And finally, the one-dimensional semigroup $(U_t)_{t>0}$ also verifies

$$\text{strong-}\lim_{t \to 0^+} U_t = | T_0 | .$$

This completes the proof of Theorem 4.3.

Proof of Corollary 4.4. — Let $P(i_1, \ldots, i_n) = | T_1^{i_1} \ldots T_n^{i_n} |$. Then, we have $P(i_1 + j_1, \ldots, i_d + j_d) \leqq P(i_1, \ldots, i_d) P(j_1, \ldots, j_d)$, and thus Dunford-Schwartz technique shows that there exists a continuous sub-semigroup S such that

$$r^{-n} \sum_{i_1=0}^{r-1} \cdots \sum_{i_n=0}^{r-1} P(i_1, \ldots, i_n) | f | \leqq c_n r^{-n} \int_0^r \cdots \int_0^r S_{(x_1, \ldots, x_n)} | f | \, dx_1 \ldots dx_n$$
$$\leqq c'_n \bar{r} - 1 \int_0^{\bar{r}} U_t | f | \, dt,$$

(where (U_t) is the one-dimensional strongly continuous semigroup obtained in the proof of 4.3)

$$\leqq c''_n (1/_{r_n+1}) \sum_{j=0}^{r_n} U_1^j \left(\int_0^1 U_s | f | \, ds \right)$$

where r_n is the integer part of $\bar{r} + 1$.

It then suffices to put $U = U_1$ and $Af = \int_0^1 \overline{U}_s f \, ds$ to obtain the Corollary 4.4.

4.5 Proof of the local ergodic theorem 4.1.

Let $f \in L_1$, and for any $s > 0$ let $M_s f = s^{-n} \int_0^s \cdots \int_0^s T_{(t_1, \ldots, t_n)} f \, dt_1 \ldots dt_n$.

For any $s > 0$, $M_s f$ verifies the conclusion of the Theorem 4.1 (see U. Krengel [*15*] and T. R. Terrell [*21*]).

If we note that $M_r f = M_r(T_0 f) = T_0(M_r f)$, we then obtain

$$\begin{aligned}
0 \leqq f^* &= \limsup_{r \to 0^+} | M_r f - T_0 f | \\
&\leqq \limsup_{r \to 0^+} | M_r(T_0 f - M_s f) | \\
&\quad + \limsup_{r \to 0^+} | M_r(M_s f) - T_0(M_s f) | + | M_s f - T_0 f | \\
&\leqq \limsup_{r \to 0^+} | M_r(T_0 f - M_s f) | + | M_s f - T_0 f | \\
&\leqq c_n \limsup_{t \to 0^+} t^{-1} \int_0^t U_x(| T_0 f - M_s f |) dx + | M_s f - T_0 f | \quad \text{(by (4.3))} \\
&= c_n | T_0 | (| T_0 f - M_s f |) + | M_s f - T_0 f |
\end{aligned}$$

(by U. Krengel's theorem [*15*])

It then follows that

$$\| f^* \|_1 \leqq (c_n + 1) \| T_0 f - M_s f \|_1 \qquad \text{for any} \quad s > 0 .$$

Since T is continuous at zero, we have

$$\lim_{s \to 0^+} \| T_0 f - M_s f \|_1 = 0 \qquad \text{and thus} \quad f^* = 0 \text{ a. e.}$$

Thus $\lim_{r \to 0^+} M_r f = T_0 f$ a. e.

Note that the proof is given for complex L_1 spaces.

REFERENCES

[0] M. A. AKCOGLU and A. BRUNEL, Contractions on L_1-spaces. *Trans. Amer. Math. Soc.*, t. **155**, 1971, p. 315-325.

[1] M. A. AKCOGLU and R. V. CHACON, A local ratio theorem. *Canad. J. Math.*, t. **22**, 1970, p. 545-552.

[2] M. A. AKCOGLU and A. DEL-JUNCO, Differentiation of *n*-dimensional additive processes. *Canad. J. Math.*, t. **33**, n° 3, 1981, p. 749-768.

[3] M. A. AKCOGLU and U. KRENGEL, Differentiation of additive processes. *Math. Z.*, t. **163**, 1978, p. 199-210.

[4] M. A. AKCOGLU and L. SUCHESTON, A superadditive ratio theorem. *Z. Wahrs. verw. Gebiete*, t. **44**, 1978, p. 268-278.

[5] A. BRUNEL, Théorème ergodique ponctuel pour un semi-groupe commutatif finement engendré de contractions de L_1. *Ann. Inst. Henri Poincaré.* Vol. IX, n° 4, 1973, p. 327-343.

[6] R. V. CHACON and U. KRENGEL, Linear modulus of a linear operator. *Proc. Amer. Math. Soc.*, t. **15**, 1964, p. 553-559.

[7] N. DUNFORD and J. T. SCHWARTZ, *Linear Operators*, Vol. I. Interscience, 1958.

[8] R. ÉMILION, Convergence locale des processus sur-additifs dans L_p. *C. R. Acad. Sc. Paris*, t. **295**, 15 novembre 1982, Série I, p. 547-549.

[9] R. ÉMILION, Continuity at 0 of semigroups of L_1 and differentiation of additive processes (to appear). *Ann. Inst. Henri Poincaré.*

[10] R. ÉMILION and B. HACHEM, Un théorème ergodique local sur-additif. *C. R. Acad. Sc. Paris*, t. **294**, 8 mars 1982, série I, p. 337-340.

[11] D. FEYEL, Sur une classe remarquable de processus abéliens. *Math. Z.* (to appear).

[12] D. FEYEL, Convergence locale des processus sur-abéliens et sur-additives. *C. R. Acad. Sc. Paris*, t. **295**, 1982, Série I, p. 301-303.

[13] S. A. MCGRATH, Some ergodic theorems for commuting operators. *Stud. Math.*, t. **70**, 1980, p. 153-160.

[14] C. KIPNIS, Majoration de semi-groupes de contractions de L_1 et applications. *Ann. Inst. Henri Poincaré*, Vol. X, n° 4, 1974, p. 369-384.

[15] U. KRENGEL, A local ergodic theorem. *Invent. Math.*, t. **6**, 1969, p. 329-333.

[16] U. KRENGEL, Recent progress in ergodic theorem. *Soc. Math. de France. Astérique*, t. **50**, 1977, p. 151-192.

[17] Y. KUBOKAWA, Ergodic theorems for contraction semigroups. *J. Math. Soc. Japan*, t. **27**, 1975, p. 184-193.

[*18*] M. Lin, On local ergodic convergence of semigroups and additive processes. *Israel J. Math.*, t. **42**, 1982, p. 300-308.

[*19*] R. Sato, On a local ergodic theorem. *Stud. Math.*, t. **LVIII**, 1976, p. 1-5.

[*20*] R. Sato, On local ergodic theorems for positive semigroups. *Studs. Math.*, t. **LXIII**, 1978, p. 45-55.

[*21*] R. T. Terrell, Local ergodic theorems for *n*-parameters semigroups of operators. *Lecture Notes in Math.*, n° 160, 1970, p. 262-278, Springer-Verlag.

(Manuscrit reçu le 30 mai 1983)

(révisé le 24 juin 1985)