Annales de l'I. H. P., section B

J. Vuolle-Apiala
S. E. Graversen

Duality theory for self-similar processes

Annales de l'I. H. P., section B, tome 22, nº 3 (1986), p. 323-332
<http://www.numdam.org/item?id=AIHPB_1986__22_3_323_0>

Ann. Inst. Henri Poincaré,
Vol. 22, n° 3, 1986, p. 323-332 *Probabilités et Statistiques*

Duality theory for self-similar processes

by

J. VUOLLE-APIALA and S. E. GRAVERSEN

Department of Mathematics, University of Aarhus, Ny Munkegade,
DK-8000 Aarhus C, Denmark

ABSTRACT. — Let (X(t)) be an α-self similar, rotation invariant Markov process on $R^n \setminus \{ 0 \}$. We show that there exists another α-self similar process of the same type, which is in a weak duality with X(t)) with respect to the measure $| x |^{1/\alpha - n} dx$. Two characterisations of the dual process are also given.

RÉSUMÉ. — Soit (X(t) un processus α-self similaire invariant par rotation et de Markov sur $R^n \setminus \{ 0 \}$. Nous montrons qu'il existe un second processus du même type qui est en dualité faible avec (X(t)) par rapport à la mesure $| x |^{1/\alpha - n}$. Deux caractérisations de ces processus duales sont également données.

INTRODUCTION

All processes considered in this note have time index set $R_+ = [0, \infty)$ and we will therefore suppress this in the notation.

α-self-similar Markov processes (α-s. s. M. P.) on R_+ were introduced by J. Lamperti in 1972 [*8*], where for each $\alpha > 0$ a process $(X(t), (P^x, x \in [0, \infty)))$ with state space R_+ is called an α-s. s. M. P. if there exists a Borel semi-group $(P_t(.,.))_{t \geqq 0}$ on $R_+ \times \mathscr{B}(R_+)$ satisfying

a) $$P_0(.,.) = I$$

b) $P_t(x, A) = P_{at}(a^\alpha x, a^\alpha A)$ for $t \geqq 0$, $x \in R_+$, $A \in \mathscr{B}(R_+)$ and $a > 0$

Classification AMS : 60-XX.

Annales de l'Institut Henri Poincaré - Probabilités et Statistiques - 0246-0203
Vol. 22/86/03/323/10/$ 3,00/© Gauthier-Villars

such that $(X(t), (P^x, x \in [0, \infty)))$ is a time homogeneous strong Markov process with transition function $(P_t(.,.))_{t \geqq 0}$ and with sample paths which are P^x-almost surely right continuous with left limits for all x in $[0, \infty)$.

α-s. s. M. P. with state space $(0, \infty)$, R^n, $R^n \backslash (0)$ or more generally cones in R^n are defined similarly. Lamperti used the word semistable instead of self-similar. In [*6*] [*7*] the authors proved that if an α-s. s. M. P. on $R^n \backslash (0)$ is rotation invariant, i. e. $(P_t(.,.))_{t \geqq 0}$ also satisfies

$$c) \quad P_t(x, A) = P_t(T(x), T(A)) \quad \text{for} \quad t \geqq 0, \ x \in R^n \backslash (0), \ A \in \mathscr{B}(R^n \backslash (0))$$

and $T \in \mathscr{O}(R^n)$ (the group of orthogonal transformations on R^n), it can be represented as the following skew product,

$$(X(t)) \sim P^x = (|X(t)| . \Theta(A_t)) \sim P^x x Q^{x/|x|} \quad \text{for} \quad x \in R^n \backslash (0)$$

($(Z(t)) \sim P^x$ denotes the distribution of the process $(Z(t))$ under the measure P^x), where $(A_t) = \left(\int_0^t |X(s)|^{-1/\alpha} ds \right)$ and $(\Theta(t), (Q^x, x \in S^{n-1}))$ is a time homogeneous Markov process on the unit sphere S^{n-1} in R^n having the following properties

1) $Q^x(\Theta(0) = x) = Q^x(\Theta(t) \in S^{n-1}) = 1$ for $t \geqq 0$ and $x \in S^{n-1}$,
2) $t \to \Theta(t)$ is right continuous with left limits Q^x-a. s. for $x \in S^{n-1}$,
3) $(\Theta(t)) \sim \tilde{Q}^x = (T^{-1}(\Theta(t))) \sim Q^{T(x)}$ for $x \in S^{n-1}$ and $T \in \mathscr{O}(R^n)$.

Furthermore, if $(X(t))$ is a diffusion, there exist parameters $\delta > 0$, $\mu \in R$, $\lambda \geqq 0$ and $\rho > 0$ such that $(\Theta(\rho t), (Q^x, x \in S^{n-1}))$ are Brownian Motions on S^{n-1} and the characteristic operator of $(X(t))$ restricted to $C_c^2(R^n \backslash (0), R)$ is equal to the differential operator

$$|x|^{-1/\alpha}\left(\frac{1}{2} \sum_{i=1}^{n} \left(\delta^2 x_i^2 + \sum_{\substack{j=1 \\ i \neq j}} \rho x_j^2\right) \partial^2/\partial x_i^2 + \frac{1}{2}(\delta^2 - \rho) \sum_{\substack{i,j=1 \\ i \neq j}}^{n} x_i x_j \frac{\partial^2}{\partial x_i \partial x_j} \right.$$
$$\left. + \left(\mu - \frac{n-1}{2}\rho\right) \sum_{i=1}^{n} x_i \partial/\partial x_i - \lambda \right).$$

The main object of this note is to prove that every rotation invariant α-s. s. M. P. on $R^n \backslash (0)$ has a strong Markov dual et least in the weak sense (see 5 for the definition of weak duality) which is also an α-s. s. M. P. This result is proved in Section 3). Section 4) contains representations of the dual process.

1. NOTATION

In [8] and [6] only $\alpha > 0$ was considered. In this paper, however, α will be allowed to vary in $R\backslash(0)$. But as we shall see, this does not bring about much new. Δ denotes a point used as graveyard for the process under consideration, and we will always assume that Δ is joined to the particular state space as a topological isolated point. We shall use the notation E_n for $R^n\backslash(0)$ for $n \geqq 2$ and E_1 for $(0, \infty)$. For $n \geqq 1$ Ω_n denotes the space of all functions ω from $R_+ \to E_n \cup \Delta$ if $n \geqq 2$ and from $R_+ \to E_1 \cup \Delta$ if $n = 1$, which satisfy

(1.1) $\quad \omega(t) = \Delta \quad \text{for} \quad t \geqq \zeta(\omega) = \inf\{ t \geqq 0 \mid \omega(t) = \Delta \}$

(1.2) ω is right continuous and ω or $\dfrac{\omega}{|\omega|^2}$ has left limits in R^n or $[0, \infty)$ at every t in $(0, \zeta(\omega)]$.

DEFINITION. — Let $\alpha \in R\backslash(0)$ and $n \geqq 2$ be given. A stochastic process $(X(t), (P^x, x \in E_n))$ with state space $E_n \cup \Delta$ is called a rotation invariant α-s. s. M. P. on $R^n\backslash(0)$ if what follows is satisfied:

There exists a Borel semigroup $(P_t(.,.))_{t \geqq 0}$ on $E_n \times \mathscr{B}(E_n)$ with the properties

(1.3) $\quad P_0(.,.) = I$

(1.4) $\quad P_t(x, A) = P_{at}(a^\alpha x, a^\alpha A) \quad \text{for} \quad t \geqq 0,\ x \in E_n,\ A \in \mathscr{B}(E_n) \text{ and } a > 0$

(1.5) $\quad P_t(x, A) = P_t(T(x), T(A)) \quad \text{for} \quad t \geqq 0,\ x \in E_n,\ A \in \mathscr{B}(E_n) \text{ and } T \in \mathscr{O}(R^n)$

such that $(X(t), (P^x, x \in E_n))$ is a time homogeneous Markov process with transition function $(P_t(.,.))_{t \geqq 0}$ and such that $t \to X(t) \in \Omega_n$ P^x-a. s. for $x \in E_n$.

α-s. s. M. P. on $(0, \infty)$ are defined similarly writing E_1 instead of E_n and omitting (1.5).

Notice that we do not require the strong Markov property. Because, as proved in [6], theorem 2.1, every rotation invariant α-s. s. M. P. on $R^n\backslash(0)$ and every α-s. s. M. P. on $(0, \infty)$ is automatically a strong Markov proces w. r. t. a right-continuous filter of σ-fields. In [6] this fact was proved only for positive α, but Lemma 1 below shows that it also holds in the case $\alpha < 0$.

Finally we shall use the notation $\mathscr{S}\mathscr{S}M(\alpha, E_n)$ and $\mathscr{S}\mathscr{S}M(\alpha, E_1)$ to denote all rotation invariant α-s. s. M. P. on $R^n\backslash(0)$ and all α-s. s. M. P. on $(0, \infty)$ respectively.

2. GENERALITIES

All results in this section are easily proved using the observation that a time homogeneous Markov process $(X(t), (P^x))$ with a Borel transition function and sample paths of the correct type is an element of $\mathcal{S}\mathcal{S}M(\alpha, E_n)$ if

(2.1) $(X(t)) \sim P^x = (a^{-\alpha}X(at)) \sim P^{a^\alpha x}$ for $x \in E_n$ and $a > 0$

(2.2) $(X(t)) \sim P^x = (T^{-1}(X(t))) \sim P^{T(x)}$ for $x \in E_n$ and $T \in \mathcal{O}(R^n)$,

and an element of $\mathcal{S}\mathcal{S}M(\alpha, E_1)$ if

(2.3) $(X(t)) \sim P^x = (a^{-\alpha}X(at)) \sim P^{a^\alpha x}$ for $x \in E_1$ *and* $a > 0$.

For p in R, denote by ϕ_p the mapping $x \rightarrow x\,|\,x\,|^{-p}$ for x in E_n.

LEMMA 1. — Let $\alpha \in R\backslash(0)$ be given.

$$(X(t), (P^x, x \in E_1)) \in \mathcal{S}\mathcal{S}M(\alpha, E_1) \Leftrightarrow (Y(t), (Q^x, x \in E_1)) \in \mathcal{S}\mathcal{S}M(-\alpha, E_1)$$

where $Y(t) = 1/X(t)$ for $t \geqq 0$ and $Q^x = P^{1/x}$ for $x \in E_1$.

$$(X(t), (P^x, x \in E_n)) \in \mathcal{S}\mathcal{S}M(\alpha, E_n) \Leftrightarrow (Y(t), (Q^x, x \in E_n)) \in \mathcal{S}\mathcal{S}M(-\alpha, E_n)$$

where $Y(t) = \phi_2(X(t))$ for $t \geqq 0$ and $Q^x = P^{\phi_2(x)}$ for $x \in E_n$.

COROLLARY. — If $(X(t), (P^x, x \in E_1)) \in \mathcal{S}\mathcal{S}M(\alpha, E_1)$ and $(X(t))$ is also a diffusion, then the characteristic operator of $(X(t))$ restricted to $\mathscr{C}_c^2((0, \infty), R)$ equals the differential operator

$$\frac{1}{2}\delta^2 x^{2-1/\alpha}\frac{d^2}{dx^2} + \mu x^{1-1/\alpha}\frac{d}{dx} - \lambda x^{-1/\alpha} \quad \text{for some} \quad \delta > 0, \quad \mu \in R \text{ and } \lambda \geqq 0.$$

In [6] several stability properties of $\mathcal{S}\mathcal{S}M(\alpha, E_1)$ and $\mathcal{S}\mathcal{S}M(\alpha, E_n)$ for $\alpha > 0$ were mentioned. It is easy to see by Lemma 1 that these extend to the case $\alpha < 0$. In this note we shall furthermore use the following result.

LEMMA 2. — Let $\alpha \in R\backslash(0)$ and $(X(t), (P^x, x \in E_n)) \in \mathcal{S}\mathcal{S}M(\alpha, E_n)$ for some $n \geqq 2$ be given. Then $(Y(t), (Q^x, x \in E_n)) \in \mathcal{S}\mathcal{S}M(\alpha(1-p), E_n)$ if $Y(t) = \phi_p(X(t))$ for $t \geqq 0$ and $Q^x = P^{\phi_q(x)}$ for $x \in E_n$, where p is a real number different from 1 and $q = p/p - 1$.

Likewise $(Y(t), (Q^x, x \in E_n)) \in \mathcal{S}\mathcal{S}M(\alpha(1+\alpha\beta)^{-1}, E_n)$ if $Y(t) = X(A_t^-)$, where (A_t^-) is the right continuous inverse of $(A_t) = \left(\int_0^t |X_s|^\beta ds\right)$, and $Q^x = P^x$ for $x \in E_n$, where β is a real number different from $-1/\alpha$.

Remark. — A similar result is true for $\mathcal{S}\mathcal{S}M(\alpha, E_1)$.

3. DUALITY

Let $\alpha > 0$ be fixed and let $(X(t), (P^x, x \in E_1))$ in $\mathscr{S}\mathscr{S}M(\alpha, E_1)$ be given. According to proposition 2.2 [6] there exists a Levy process (see [3] for definition) $(r(t), (Q^x, x \in R))$ with state space R and a $\lambda \geqq 0$ such that

$$(X(A_t^-)) \sim P^x = (\exp(r(t))) \sim Q^{\log x, \lambda} \qquad \text{for} \quad x \in E_1,$$

where (A_t^-) is the right continuous inverse of $(A_t) = \left(\int_0^t |X_s|^{-1/\alpha} ds \right)$ and $Q^{\log x, \lambda}$ denotes the measure corresponding to $(r(t), Q^{\log x})$ killed with an independent exponential distributed clock with mean $1/\lambda$.

Well known theory about Levy processes ensures the existence of another Levy process $(\hat{r}(t), (\hat{Q}^x, x \in R))$ with state space R with the property: $(r(t), (Q^{x,\lambda}, x \in R))$ and $(r(t), (\hat{Q}^{x,\lambda}, x \in R))$ are in weak duality w. r. t. Lebesgue measure dx on R, i. e.

$$\int_R e^{-\lambda t} E^x(f(r(t)))g(x)dx = \int_R f(x) e^{-\lambda t} \hat{E}^x(g(\hat{r}(t)))dx$$

for all f and g bounded real-valued Borel functions defined on R.

A simple substitution now gives that

$$(\exp(r(t)), (Q_1^x, x \in E_1)) \quad \text{and} \quad (\exp(\hat{r}(t)), (\hat{Q}_1^x, x \in E_1))$$

are in weak duality w. r. t. the measure $x^{-1}dx$ on E_1, where $Q_1^x = Q^{\log x, \lambda}$ and $\hat{Q}_1^x = \hat{Q}^{\log x, \lambda}$ for $x \in E_1$. Using a theorem of J. B. Walsh [9], we can conclude that

$$(\exp(r(T_t^-)), (Q_1^x, x \in E_1)) \quad \text{and} \quad (\exp(\hat{r}(\hat{T}_t^-)), (\hat{Q}_1^x, x \in E_1))$$

are in weak duality w. r. t. the measure $x^{-1+1/\alpha}dx$ on E_1, where (T_t^-), respectively $(\hat{T}_t^-)$ is the right continuous inverse of $(T_t) = \left(\int_0^t \exp\left(\frac{1}{\alpha} r(s)\right) ds \right)$ and $(\hat{T}_t) = \left(\int_0^t \exp\left(\frac{1}{\alpha} \hat{r}(s)\right) ds \right)$.

But theorems 2.3 and 2.4 in [6] imply that

$$(X(t)) \sim P^x = (\exp(r(T_t^-))) \sim Q_1^x \quad \text{for} \quad x \in E_1$$

and $(\exp(\hat{r}(\hat{T}_t^-)), (\hat{Q}_1^x, x \in E_1)) \in \mathscr{S}\mathscr{S}M(\alpha, E_1)$.

We have thus proved the following result:

THEOREM 1. — For $(X(t), (P^x, x \in E_1)) \in \mathcal{S}\mathcal{S}M(\alpha, E_1), \alpha > 0$, there exists $(Y(t), (Q^x, x \in E_1)) \in \mathcal{S}\mathcal{S}M(\alpha, E_1)$ such that $(X(t), (P^x, x \in E_1))$ and $(Y(t), (Q^x, x \in E_1))$ are in weak duality w. r. t. the measure $x^{-1+1/\alpha}dx$ on E_1.

COROLLARY 1. — Theorem 1 is also true for $\alpha < 0$.

Proof. — Immediate from Lemma 1.

COROLLARY 2. — Let $\alpha \in R\backslash(0)$ and $(X(t), (P^x, x \in E_1)) \in \mathcal{S}\mathcal{S}M(\alpha, E_1)$ be given. If furthermore $(X(t))$ is a diffusion with characteristic operator

$$\frac{1}{2}\delta^2 x^{2-1/\alpha} d^2/dx^2 + \mu x^{1-1/\alpha} d/dx - \lambda x^{-1/\alpha},$$

then $(Y(t), (Q^x, x \in E_1))$ can be chosen also to be a diffusion with characteristic operator

$$\frac{1}{2}\delta^2 x^{2-1/\alpha} d^2/dx^2 + (\delta^2 - \mu) x^{1-1/\alpha} d/dx - \lambda x^{-1/\alpha}.$$

Proof. — Straightforward calculations using the fact that the Levy process corresponding to $(X(t))$ is Brownian Motion with constant drift up to a time change of the form $t \to rt, r > 0$.

This result will now be generalized to rotation invariant α-s. s. M. P. on E_n. Like above we need only consider the case $\alpha > 0$. Therefore, let $\alpha > 0$ and $(X(t), (P^x, x \in E_n)) \in \mathcal{S}\mathcal{S}M(\alpha, E_n)$ for some $n \geqq 2$ be given. According to theorem 2.2 [*6*] there exists a time homogeneous Markov process $(\Theta(t), (Q^x \in S^{n-1}))$ with state space S^{n-1} fulfilling 1), 2) and 3) as stated in the introduction such that

$$(X(t)) \sim P^x = (|X(t)| . \Theta(A_t)) \sim P^x x Q^{x/|x|} \quad \text{for} \quad x \in E_n,$$

where
$$(A_t) = \left(\int_0^t |X(s)|^{-1/\alpha} ds \right).$$

By time change we therefore have

$$(X(A_t^-)) \sim P^x = (|X(A_t^-)| . \Theta_t) \sim P^x x Q^{x/|x|} \quad \text{for} \quad x \in E_n.$$

Since $(|X(t)|, (\tilde{P}^x, x \in E_1)) \in \mathcal{S}\mathcal{S}M(\alpha, E_1)$ if $\tilde{P}^x = P^{\tilde{x}}$ for some $\tilde{x}$ in E_n with $|\tilde{x}| = x$ for $x \in E_1$, we known from above how to handle the radial process. Concerning the angular process the following result is important.

LEMMA 3. — $(\Theta(t), (Q^x, x \in S^{n-1}))$ defined as above is in weak duality with itself w. r. t. the uniform measure $m_{n-1}(dx)$ on S^{n-1}.

Proof. — Let $t > 0$ be given. We shall show that for all f and g in $\mathscr{C}(S^{n-1}, R)$, the set of real-valued continuous functions defined on S^{n-1}, we have

$$(3.1) \quad \int_{S^{n-1}} H_t f(\Theta) g(\Theta) m_{n-1}(d\Theta) = \int_{S^{n-1}} f(\Theta) H_t q(\Theta) m_{n-1}(d\Theta),$$

where $H_t h(x) = E^x(h(\Theta_t))$ for $x \in S^{n-1}$ and $h \in \mathscr{C}(S^{n-1}, R)$. If $n = 2$, $(\Theta(t), (Q^x, x \in S^1))$ is a symmetric Levy process on the circle group in R^2 in which case (3.1) is clear. Therefore we may assume $n \geqq 3$. As proved in proposition 2.3 [*6*] there exists a probability measure $F_t(ds)$ on $[-1, 1]$ such that for all Θ in S^{n-1} and $h \in \mathscr{C}(S^{n-1}, R)$ we have

$$(3.2) \qquad H_t h(\Theta) = \int_{S^{n-2}(\Theta)} \int_{-1}^{1} h(s\Theta + \sqrt{1-s^2}\tilde{\Theta}) F_t(ds) m_{n-2}(d\tilde{\Theta}),$$

where $S^{n-2}(\Theta) = \{ \tilde{\Theta} \in S^{n-1} \mid \Theta . \tilde{\Theta} = 0 \}$. $S^{n-2}(\Theta)$ can be identified with S^{n-2} in a natural way and thus equipped with the measure m_{n-2}.

Since m_{n-1} is invariant under $\mathcal{O}(R^n)$, (3.1) is satisfied if $F_t(ds)$ is concentrated on the set $\{ 1, -1 \}$. A continuity and linearity argument therefore implies that it suffices to consider the case where $F_t(ds)$ is absolutely continuous w. r. t. the Lebesgue measure on $(-1, 1)$.

Similarly to (3.2) we have

$$(3.3) \quad \int_{S^{n-1}} h(x) m_{n-1}(dx) = \int_{S^{n-2}(\Theta)} \int_{-1}^{1} h(s . \Theta + \sqrt{1-s^2}\tilde{\Theta}) G(ds) m_{n-2}(d\tilde{\Theta})$$

for all $\Theta \in S^{n-1}$ and all $h \in \mathscr{C}(S^{n-1}, R)$, where $G(ds) = \mathscr{C}(1-s^2)^{n-3/2} ds$ where $1/\mathscr{C} = 2\pi^{n/2}/\Gamma(n/2)$ (see [*1*]).

(3.2) and (3.3) imply that for each $\Theta \in S^{n-1}$ and $h \in \mathscr{C}(S^{n-1}, R)$

$$(3.4) \qquad H_t h(\Theta) = \int_{S^{n-1}} \tilde{g}(\Theta, \eta) h(\eta) m_{n-1}(d\eta),$$

where $\tilde{g}(\Theta, \eta) = g(\Theta . \eta)$ for $\eta \in S^{n-1}$ and $s \to g(s)$ is the Radon-Nikodym derivative of $F_t(ds)$ w. r. t. $G(ds)$. (3.1) now follows from (3.4) and Fubini's theorem.

From above we know that there exists a $\lambda \geqq 0$ and a Levy process $(\hat{r}(t), (\hat{Q}^x, x \in R))$ with state space R such that $(| X(A_t^-) |, (\tilde{P}^x, x \in E_1))$ and $(\exp(\hat{r}(t)), (\hat{Q}_1^x, x \in E_1))$ are in weak duality w. r. t. the measure $x^{-1}dx$ on E_1, where for $x \in E_1$ $\tilde{P}^x = P^{\tilde{x}}$ for some $\tilde{x} \in E_n$ with $| \tilde{x} | = x$ and

$\hat{Q}_1^x = \hat{Q}^{\log x, \lambda}$. The independence of the radial and angular processes permits us to conclude by Lemma 3 that

$$(|X(A_t^-)|.\Theta(t), (P^x x Q^{x/|x|}, x \in E_n))$$

and

$$(\exp(\hat{r}(t)).\Theta(t), (\hat{Q}_1^{|x|} x Q^{x/|x|}, x \in E_n))$$

are in weak duality w. r. t. the measure $|x|^{-n} dx$ on E_n. By the afore-mentioned theorem of J. B. Walsh [9] we conclude that $(X(t), (P^x, x \in E_n))$ and $(\exp(\hat{r}(\hat{T}_t^-)).\Theta(\hat{T}_t^-), \hat{Q}_1^{|x|} x Q^{x/|x|}, x \in E_n))$, where $(\hat{T}_t^-)$ is the right continuous inverse of $\left(\int_0^t \exp(1/\alpha\, \hat{r}(s) ds\right)$, are in weak duality w. r. t. the measure $|x|^{-n+1/\alpha} dx$ on E_n. Referring to theorems 2.3 and 2.4 in [6] we have therefore proved the following result:

THEOREM 2. — For $(X(t), (P^x, x \in E_n)) \in \mathcal{S}\mathcal{S}M(\alpha, E_n)$, $\alpha > 0$, there exists $(Y(t), (Q^x, x \in E_n)) \in \mathcal{S}\mathcal{S}M(\alpha, E_n))$ such that $(X(t), (P^x, x \in E_n))$ and $(Y(t), (Q^x, x \in E_n))$ are in weak duality w. r. t. the measure $|x|^{-n+1/\alpha} dx$ on E_n.

COROLLARY 3. — Theorem 2 is also valid for $\alpha < 0$.

Proof. — Immediate from Lemma 1.

COROLLARY 4. — Let $\alpha \in R \backslash (0)$ and $(X(t), (P^x, x \in E_n)) \in \mathcal{S}\mathcal{S}M(\alpha, E_n)$ be given. If $(X(t))$ is a diffusion with characteristic operator on $\mathcal{C}_c^2(E_n, R)$ equal to

$$|x|^{-1/\alpha}\left(\frac{1}{2}\sum_{i=1}^{n}\left(\delta^2 x_i^2 + \sum_{\substack{j=1\\ i\neq j}}^{n} \rho x_j^2\right)\partial^2/\partial x_i^2 + \frac{1}{2}(\delta^2 - \rho)\sum_{\substack{i,j=1\\ i\neq j}}^{n} x_i x_j \partial^2/\partial x_i \partial x_j + \left(\mu - \frac{n-1}{2}\rho\right)\sum_{i=1}^{n} x_i \partial/\partial x_i - \lambda\right),$$

then $(Y(t), (Q^x, x \in E_n))$ can be chosen also to be a diffusion with characteristic operator

$$|x|^{-1/\alpha}\left(\frac{1}{2}\sum_{i=1}^{n}\left(\delta^2 x_i^2 + \sum_{\substack{j=1\\ i\neq j}}^{n} \rho x_j^2\right)\partial^2/\partial x_i^2 + \frac{1}{2}(\delta^2 - \rho)\right)\sum_{\substack{i,j=1\\ i\neq j}}^{n} x_i x_j \partial^2/\partial x_i \partial x_j + \left(\delta^2 - \mu - \frac{n-1}{2}\rho\right)\sum_{i=1}^{n} x_i \partial/\partial x_i - \lambda\,.$$

Proof. — Follows from Corollary 2 and Lemma 3 and the fact that the spherical process corresponding to a diffusion is the spherical Brownian motion up to a time change of the form $t \to rt$, $r > 0$.

4. CHARACTERISATIONS OF THE DUAL PROCESS

In this section we shall give two characterisations of the dual process. Let $\alpha \in \mathrm{R}\backslash(0)$ and $(\mathrm{X}(t), (\mathrm{P}^x, x \in \mathrm{E}_n)) \in \mathscr{S}\mathscr{S}\mathrm{M}(\alpha, \mathrm{E}_n)$ be given. Assume for convenience that $(\mathrm{X}(t))$ is a diffusion.

The first characterisation uses h-transform theory [4]. Let h be an excessive function of $(\mathrm{X}(t), (\mathrm{P}^x, x \in \mathrm{E}_n))$ and let $(\mathrm{X}^h(t), (\mathrm{P}^x, x \in \mathrm{E}_n))$ denote the corresponding h-process. General theory implies that if h is continuous, then this process is a time homogeneous Markov process with continuous sample paths and governed by the transition function

$$\mathrm{P}_t^h(x, \mathrm{A}) = h(x)^{-1} \int_{\mathrm{A}} \mathrm{P}_t(x, dy)h(y)\,. \tag{4.1}$$

From this formula it is seen that the self-similarity property will be preserved if h is of the form $x \to |x|^k$ for an appropriate k in R. In this connection we have the following result which we state without proof.

THEOREM 3. — Let α and $(\mathrm{X}(t), (\mathrm{P}^x, x \in \mathrm{E}_n))$ be as above. Then h: $x \to |x|^{1-2\mu/\delta^2}$ is excessive and $(\mathrm{X}^h(t), (\mathrm{P}^x, x \in \mathrm{E}_n))$ is an element of $\mathscr{S}\mathscr{S}\mathrm{M}(\alpha, \mathrm{E}_n)$ and is the weak dual of $(\mathrm{X}(t), (\mathrm{P}^x, x \in \mathrm{E}_n))$ w. r. t. the measure $|x|^{-n+1/\alpha}dx$ on E_n.

δ^2 and μ are coefficients in the characteristic operator of $(\mathrm{X}(t))$ (see the introduction).

The second characterisation of the dual process is contained in the following construction which in the case of Brownian Motion was used by M. Yor [10]. By Lemma 2 we have that for each $p \in \mathrm{R}\backslash(1)$ and $\beta \in \mathrm{R}\backslash(1/\alpha(p-1))$ $(\mathrm{Y}^{p,\beta}(t), (\mathrm{Q}^x, x \in \mathrm{E}_n)) \in \mathscr{S}\mathscr{S}\mathrm{M}(\alpha(1-p)(1+\alpha(1-p)\beta)^{-1}, \mathrm{E}_n)$, where $\mathrm{Y}^{p,\beta}(t) = \phi_p(\mathrm{X}(\mathrm{A}_t^-))$ for $t \geq 0$ and (A_t^-) is the right continuous inverse of $(\mathrm{A}_t) = \left(\int_0^t |\phi_p(\mathrm{X}(s))|^\beta ds\right)$, and $\mathrm{Q}^x = \mathrm{P}^{\phi_q(x)}$ for $x \in \mathrm{E}_n$ with $q = p/p-1$.

Straightforward calculations now show

THEOREM 4. — Let α and $(\mathrm{X}(t), (\mathrm{P}^x, x \in \mathrm{E}_n))$ be as above. If $p = 2$ and

$\beta = 2/\alpha$, then $(X(t), (P^x, x \in E_n)$ and $(Y^{p,\beta}(t), (Q^x, x \in E_n))$ are in weak duality with respect to the measure $| x |^{-n+1/\alpha} dx$ on E_n.

Remark. — We have above concentrated on weak duality, but in many cases, e. g. in the diffusion case, we will indeed have strong duality.

REFERENCES

[1] N. H. BINGHAM, Random walk on spheres, *Z. Warsch. Verw. Gebiete*, t. **22**, 1972, p. 169-192.

[2] R. M. BLUMENTHAL, R. K. GETOOR, *Markov processes and potential theory*, New York, Academic Press, 1968.

[3] K. L. CHUNG, *Lectures from Markov processes to Brownian motion*, New York, Heidelberg, Berlin, Springer, 1982.

[4] J. L. DOOB, Conditional Brownian motion and the boundary limits of harmonic functions, *Bull. Soc. Math. France*, t. **85**, 1957, p. 431-458.

[5] R. K. GETOOR, M. J. SHARPE, Naturality, standardness and weak duality for Markov processes, *Z. Warsch. Verw. Gebiete*, t. **67**, 1984, p. 1-62.

[6] S. E. GRAVERSEN, J. VUOLLE-APIALA, α-self-similar Markov processes. To appear in *Z. Warsch. Verw. Gebiete.*

[7] S. E. GRAVERSEN, J. VUOLLE-APIALA, *α-semi stable Markov processes*, Matematisk Institut, Aarhus Universitet, Preprint Ser., 1982-1983, no. 24.

[8] J. W. LAMPERTI, *Semi-stable Markov processes I*, *Z. Warsch. Verw. Gebiete*, t. **22**, 1972, p. 205-225.

[9] J. B. WALSH, Markov processes and their functionals in duality, *Z. Warsch. Verw. Gebiete*, t. **24**, 1972, p. 222-246.

[10] M. YOR, A propos de l'inverse du mouvement Brownien dans R^n ($n \geqq 3$), *Ann. Inst. Henri Poincaré, Probabilités et Statistiques*, t. **21** (1), 1985, p. 27-38.

(Manuscrit reçu le 17 septembre 1985)