ANNALES DE L'I. H. P., SECTION B

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Annales de l'I. H. P., section B, tome 22, nº 3 (1986), p. 333-345

<http://www.numdam.org/item?id=AIHPB_1986__22_3_333_0>

Ann. Inst. Henri Poincaré,
Vol. 22, n° 3, 1986, p. 333-345 *Probabilités et Statistiques*

Weights for ergodic square functions

by

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ABSTRACT. — In this paper we study the relation between good weights for the maximal ergodic function and good weights for ergodic square functions associated to an invertible measure preserving transformation. More precisely, if w is a positive function and μ is an invariant measure we prove that ergodic square functions are of weak type (1, 1) with respect to $wd\mu$ if and only if the maximal ergodic function is of weak type (1,1) with respect to the same measure, i. e. if and only if w satisfies A'_1. We also prove that if the maximal function is bounded in $L^p(wd\mu)$ $(1 < p < \infty)$, i. e. if w satisfies A'_p, then square ergodic functions are bounded in $L^p(wd\mu)$; however the converse is not true.

Key-words and phrases: Ergodic square functions, Ergodic maximal function, Weighted inequalities, Weights.

RÉSUMÉ. — Dans cet article, nous étudions la relation entre les poids qui sont bons pour la fonction maximale ergodique et ceux qui sont bons pour les fonctions carrées ergodiques associées à une transformation inversible qui laisse la mesure invariante. Plus précisément, pour une fonction positive w et une mesure invariante μ, nous prouvons que les fonctions carrées ergodiques sont de type faible (1, 1) par rapport à $wd\mu$ si et seulement si la fonction maximale ergodique est de type faible (1, 1), toujours par rapport à la même mesure $wd\mu$, c'est-à-dire si et seulement si w satisfait A'_1.

Classification AMS : 60-XX.
AMS subject classification (1980) : Primary 28D05, Secondary 47A35.

Nous prouvons aussi que si la fonction maximale est bornée dans $L^p(wd\mu)(1 < p < \infty)$, c'est-à-dire si w satisfait A'_p, alors les fonctions carrées ergodiques sont bornées dans $L^p(wd\mu)$; néanmoins la réciproque n'est pas vraie.

1. INTRODUCTION

Let $(X, \mathscr{F}, \mu)$ be a σ-finite measure space and T an invertible measure preserving transformation from X onto itself. For each pair of non negative integers, n and k, we define the operators $T_{n,k}$, acting on measurable functions by

$$T_{n,k}f(x) = (n + k + 1)^{-1} \sum_{i=-n}^{k} f(T^i x)$$

and the ergodic square function

$$\bar{S}f = \left[\sum_{k=0}^{\infty} |T_{k+1,0}f - T_{k,0}f|^2 + |T_{0,k+1}f - T_{0,k}f|^2 \right]^{1/2}.$$

It is easy to see that there exists a constant $C > 0$ independent of f such that

$$\bar{S}f \leqslant C(Sf + Mf) \tag{1.1}$$

where

$$Sf(x) = \left[\sum_{k=-\infty}^{\infty} (|k| + 1)^{-2} |f(T^k x)|^2 \right]^{1/2}$$

and M is the ergodic maximal operator, i. e.,

$$Mf = \sup_{n,k>0} T_{n,k} |f| .$$

It is well known that the maximal operator M is of weak type (1, 1) and strong type (p, p) for any $p > 1$ with respect to the measure μ. On the other hand, if $(X, \mathscr{F}, \mu)$ is a non atomic probability space and T is ergodic then the results in [*4*] insure us that S has the same properties. Thus, by (1.1), if T is ergodic and $(X, \mathscr{F}, \mu)$ a non atomic probability space then $\bar{S}$ is of weak type (1, 1) and strong type (p, p), $1 < p$, with respect to the measure μ.

Our aim is to study the boundedness of the operators $\bar{S}$ and S with respect to the measure $w d\mu$ where w is a positive measurable function, T a transformation not necessarily ergodic and $(X, \mathscr{F}, \mu)$ any σ-finite measure space. In fact we shall work with the operators

$$\bar{S}_r f = \left[\sum_{k=0}^{\infty} | T_{k+1,0} f - T_{k,0} f |^r + | T_{0,k+1} f - T_{0,k} f |^r \right]^{1/r}$$

and

$$S_r f(x) = \left[\sum_{k=-\infty}^{\infty} (| k | + 1)^{-r} | f(T^k x) |^r \right]^{1/r},$$

2. BOUNDEDNESS OF THE OPERATORS $\bar{S}_r$ AND S_r

Throughout this section, $(X, \mathscr{F}, \mu)$ will be a σ-finite measure space, T an invertible measure preserving transformation from X onto itself, w a positive measurable function and C a constant not necessarily the same one at each occurrence.

We shall consider two sets as equal if they agree up to a set of measure zero.

The main results of this paper give us a necessary and sufficient condition for the operators $\bar{S}_r$ and S_r to be of weak type (1, 1) with respect to the measure $w d\mu$ and a sufficient condition to be of strong type (p, p) $(1 < p < \infty)$. We shall establish that the good weights w for the maximal operator M are good weights for $\bar{S}_r$ and S_r. The study of the maximal operator can be found in [*5*] (theorems (2.7) and (4.1)) where the following was proved:

(2.1) Let $1 < p < \infty$ and q the conjugate exponent of p (letter q will have always the same meaning in this paper). The maximal operator M is of strong type (p, p) with respect to the measure $w d\mu$ if and only if w satisfies the following condition:

A'_p : « There exists a constant $C > 0$ such that for every non negative integer k

$$T_{0,k} w(x) [T_{0,k} w^{-q/p}(x)]^{p/q} \leqslant C \qquad \text{a. e. »}$$

(2.2) The maximal operator M is of weak type (1, 1) with respect to $w d\mu$ if and only if w satisfies the following condition:

A'_1 : « There exists a constant $C > 0$ such that

$$Mw(x) \leqslant Cw(x) \quad \text{a. e. ».}$$

The result (2.1) can be found also in [*6*].

Once conditions A'_p have been established, we state and prove our results.

(2.3) THEOREM. — Let $1 < r < \infty$. The following conditions are equivalent :

a) There exists a constant $C > 0$ such that for every $\lambda > 0$ and for all measurable functions f

$$\int_{\{x \in X : S_r f(x) > \lambda\}} w d\mu \leqslant C\lambda^{-1} \int_X |f| w d\mu$$

b) There exists a constant $C > 0$ such that for every $\lambda > 0$ and for all measurable functions f

$$\int_{\{x \in X : \bar{S}_r f(x) > \lambda\}} w d\mu \leqslant C\lambda^{-1} \int_X |f| w d\mu$$

c) w satisfies A'_1.

(2.4) THEOREM. — Let $1 < r < \infty$ and $1 < p < \infty$. If w satisfies A'_p then $\bar{S}_r$ and S_r are of strong type (p, p) with respect to the measure $w d\mu$, i. e., there exists a constant $C > 0$ such that

$$\int_X |\bar{S}_r f|^p w d\mu \leqslant C \int_X |f|^p w d\mu$$

$$\int_X |S_r f|^p w d\mu \leqslant C \int_X |f|^p w d\mu$$

for every measurable function f.

In the proofs we shall use the concept of (ergodic) rectangle and the following lemma which can be found in [5].

(2.5) DEFINITION. — Let k be a non negative integer. The measurable set $B \subset X$ is the base of a rectangle of length $k+1$ if $T^i B \cap T^j B = \phi$, $i \neq j$, $0 \leqslant i, j \leqslant k$. In such a case the set $R = \bigcup_{i=0}^{k} T^i B$ will be called a rectangle with base B and length $k+1$.

(2.6) LEMMA. — Let Y be a measurable subset of X and let k be a non negative integer. Then there exists a countable family $\{B_i\}_{i=0}^{\infty}$ of sets of finite measure such that

i) $Y = \bigcup_{i=0}^{\infty} B_i$

ii) $B_i \cap B_j = \phi$ if $i \neq j$.

iii) For every i, B_i is the base of a rectangle of length $1 + s(i)$ with $0 \leqslant s(i) \leqslant k$ and such that if $s(i) < k$ then $T^{1+s(i)}A = A$ for every measurable set $A \subset B_i$. Consequently, for every measurable set $A \subset B_i$, if χ_A denotes the characteristic function of A,

$$(2.7) \qquad \sum_{j=0}^{k} \chi_{T^jA} \leqslant C(i) \sum_{j=0}^{s(i)} \chi_{T^jA} = C(i)\chi_{\bigcup_{j=0}^{s(i)} T^jA} \leqslant 2 \sum_{j=0}^{k} \chi_{T^jA}$$

where $C(i)$ is the least integer not smaller than $(k+1)(1+s(i))^{-1}$.

Proof of theorem (2.3).

(*a*) $\Rightarrow$ (*c*). Let's fix a non negative integer k and let $\{B_i\}_{i=0}^{\infty}$ be the sequence given by lemma (2.6) for the set X and the number k. Let's fix B_i. For any measurable set $A \subset B_i$ let $R = \bigcup_{j=0}^{s(i)} T^jA$ $\left(\text{observe that } R = \bigcup_{j=0}^{k} T^jA\right)$. If $x \in A$, $0 \leqslant h \leqslant s(i)$, we have

$$S_r\chi_A(T^hx) \geqslant (h+1)^{-1}\chi_A(T^{-h}T^hx) \geqslant (s(i)+1)^{-1}.$$

Therefore, $R \subset \{ x \in X : S_r\chi_A(x) \geqslant (s(i)+1)^{-1} \}$. Then by (*a*) we get

$$\int_A \sum_{j=0}^{s(i)} w(T^jx)d\mu = \int_R wd\mu \leqslant C(s(i)+1) \int_A wd\mu.$$

Now by (2.7)

$$\int_A \sum_{j=0}^{k} w(T^jx)d\mu \leqslant C(i) \int_A \sum_{j=0}^{s(i)} w(T^jx)d\mu \leqslant C(i)C(s(i)+1) \int_A wd\mu$$
$$\leqslant C((k+1)(1+s(i))^{-1}+1)(s(i)+1) \int_A wd\mu \leqslant 2C(k+1) \int_A wd\mu.$$

Since this inequality holds for any measurable subset A if B_i and $X = \bigcup_{i=0}^{\infty} B_i$ we deduce that $T_{0,k}w(x) \leqslant 2Cw(x)$ *a. e.* In the same way we get $T_{k,0}w(x) \leqslant 2Cw(x)$ *a. e.* Now, since 2C is independent of k we have that w satisfies A'_1.

(*c*) $\Rightarrow$ (*a*). Let λ be a positive number. Let f be a measurable function and let's define for every integer k the function f_k,

$$f_k(x) = f(T^kx)$$

and let $g_k(x) = | f_k(x) |$ if $| f_k(x) | < \lambda(| k | + 1)$ and $g_k(x) = 0$ if $| f_k(x) | \geqslant \lambda(| k | + 1)$. Finally let $b_k = | f_k | - g_k$. Then

$$S_r f(x) \leqslant \left[\sum_{k=-\infty}^{\infty} (| k | + 1)^{-r} | g_k(x) |^r \right]^{1/r} + \left[\sum_{k=-\infty}^{\infty} (| k | + 1)^{-r} | b_k(x) |^r \right]^{1/r}.$$

Therefore, in order to prove (*a*) it will suffice to establish the following inequalities:

$$\int_{G_\lambda} w d\mu \leqslant C\lambda^{-1} \int_X | f | w d\mu . \tag{2.8}$$

$$\int_{B_\lambda} w d\mu \leqslant C\lambda^{-1} \int_X | f | w d\mu \tag{2.9}$$

where

$$G_\lambda = \left\{ x \in X : 2^r \sum_{k=-\infty}^{\infty} (| k | + 1)^{-r} | g_k(x) |^r > \lambda^r \right\}$$

and

$$B_\lambda = \left\{ x \in X : 2^r \sum_{k=-\infty}^{\infty} (| k | + 1)^{-r} | b_k(x) |^r > \lambda^r \right\}$$

Proof of inequality (2.8). — By the definition of G_λ we have

$$\begin{aligned} \int_{G_\lambda} w d\mu &\leqslant 2^r \lambda^{-r} \sum_{k=-\infty}^{\infty} (| k | + 1)^{-r} \int_X | g_k |^r w d\mu \\ &= 2^r \lambda^{-r} \sum_{k=-\infty}^{\infty} (| k | + 1)^{-r} \int_0^\infty r t^{r-1} \int_{G_{k,t}} w d\mu dt \end{aligned} \tag{2.10}$$

where $G_{k,t} = \{ x \in X : g_k(x) > t \}$. Now, by the definition of g_k and since T preserves the measure μ we get

$$\begin{aligned} &2^r \lambda^{-r} \sum_{k=-\infty}^{\infty} (| k | + 1)^{-r} \int_0^\infty r t^{r-1} \int_{G_{k,t}} w d\mu dt \\ &\qquad \leqslant 2^r \lambda^{-r} \int_0^\infty r t^{r-1} \int_{F_t} \sum_{|k| \geqslant [t/\lambda]} (| k | + 1)^{-r} w(T^k x) d\mu dt \end{aligned} \tag{2.11}$$

where $F_t = \{ x \in X : | f(x) | > t \}$ and $[t/\lambda]$ is the integer part of t/λ. To finish the proof of inequality (2.8) we need the following lemma:

(2.12) LEMMA. — If $1 < p < \infty$ and w satisfies A'_p then there exists $C > 0$ such that for every non negative integer i we have

$$\sum_{|k|>i} (2i + 1)^p \, |k|^{-p} w(T^k x) \leqslant C \sum_{|k|\leqslant i} w(T^k x) \quad a.e.$$

A proof of lemma (2.12) can be found in [1]. Although in [1] T is ergodic and $(X, \mathscr{F}, \mu)$ is a non atomic probability space, the proof is valid in our setting because these properties are not used.

Now, we shall continue with the proof of inequality (2.8). Since w satisfies A'_r (A'_1 implies A'_r) lemma (2.12) applied to w with $i = [t/\lambda]+1$ and $p=r$ gives us

$$\sum_{|k|>[t/\lambda]+1} (|k|+1)^{-r} w(T^k x) \leqslant C(2[t/\lambda]+3)^{-r} \sum_{|k|\leqslant[t/\lambda]+1} w(T^k x)$$

and therefore

$$(2.13) \quad \sum_{|k|\geqslant[t/\lambda]} (|k|+1)^{-r} w(T^k x) \leqslant C(2[t/\lambda]+3)^{-r} \sum_{|k|\leqslant[t/\lambda]+1} w(T^k x) \leqslant C(2[t/\lambda]+3)^{1-r} Mw(x).$$

If we put together (2.10), (2.11) and (2.13) we get

$$(2.14) \quad \int_{G_\lambda} w d\mu \leqslant C\lambda^{-r} \int_0^\infty t^{r-1}(2[t/\lambda]+3)^{1-r} \int_{F_t} Mw d\mu dt \leqslant C\lambda^{-1} \int_0^\infty \int_{F_t} Mw d\mu dt = C\lambda^{-1} \int_X |f| \, Mw d\mu.$$

Now remembering that w satisfies A_1 we get (2.8).

Proof of inequality (2.9). — Let

$$P = \left\{ x \in X : \sum_{k=-\infty}^{\infty} (|k| + 1)^{-r} |b_k(x)|^r > 0 \right\}$$

$P_k = \{ x \in X : b_k(x) > 0 \}$ and $E_k = \{ x \in X : |f(x)| \geqslant \lambda(|k| + 1) \}$.

It is clear that

$$(2.15) \quad \int_{B_\lambda} w d\mu \leqslant \int_P w d\mu \leqslant \sum_{k=-\infty}^{\infty} \int_{P_k} w d\mu.$$

Since $P_k = T^{-k}E_k$ and T preserves the measure μ

$$(2.16)\quad \sum_{k=-\infty}^{\infty} \int_{P_k} w d\mu = \int_X \sum_{k=-\infty}^{\infty} w(T^k x)\chi_{E_k}(x) d\mu$$

$$\leqslant \int_X \sum_{|k| \leqslant [|f(x)|/\lambda]} w(T^k x)\chi_{E_k}(x) d\mu \leqslant \int_X \sum_{|k| \leqslant [|f(x)|/\lambda]} w(T^k x)\chi_{E_0}(x) d\mu$$

$$\leqslant \int_X (2[|f(x)|/\lambda] + 1) Mw(x)\chi_{E_0}(x) d\mu .$$

It is clear that if $x \in E_0$ then $2[|f(x)|/\lambda] + 1 \leqslant 3|f(x)|/\lambda$. This observation, (2.15) and (2.16) give us

$$(2.17)\quad \int_{B_\lambda} w d\mu \leqslant 3\lambda^{-1} \int_X |f| Mw d\mu .$$

Then, inequality (2.9) follows from (2.17) since w satisfies A'_1.

(*b*) $\Rightarrow$ (*c*). Let's fix a non negative integer k and let $\{B_i\}_{i=0}^{\infty}$ be the family given by lemma (2.6) for the set X and the number k. For B_i fixed with $s(i) \neq 0$ and for any measurable subset $A \subset B_i$ let $R = \bigcup_{j=0}^{s(i)} T^j A$. If $x \in A$, $0 \leqslant h \leqslant s(i)$, $h \neq 0$, we have

$$\bar{S}_r \chi_A(T^h x) \geqslant |T_{h,0}\chi_A(T^h x) - T_{h-1}\chi_A(T^h x)| \geqslant (h+1)^{-1} \geqslant (s(i)+1)^{-1} .$$

Therefore $R - A \subset \{ x \in X : \bar{S}_r \chi_A(x) \geqslant (s(i)+1)^{-1} \}$. Then, by (*b*) we get

$$\int_A \sum_{j=1}^{s(i)} w(T^j x) d\mu \leqslant C(s(i) + 1) \int_A w d\mu .$$

Then

$$(2.18)\quad \int_A \sum_{j=0}^{s(i)} w(T^j x) d\mu \leqslant C(s(i) + 1) \int_A w d\mu .$$

Inequality (2.18) is clear if $s(i) = 0$. Thus, (2.18) holds for every i and for any measurable subset $A \subset B_i$. Then, by (2.7)

$$\int_A \sum_{j=0}^{k} w(T^j x) d\mu \leqslant C(i)C(s(i) + 1) \int_A w d\mu \leqslant 2C(k + 1) \int_A w d\mu .$$

Since this inequality is valid for any measurable subset A of B_i and $X = \bigcup_{i=0}^{\infty} B_i$ we have $T_{0,k}w(x) \leqslant 2Cw(x)$ a. e. In the same way we get $T_{k,0}w(x) \leqslant 2Cw(x)$ a. e. and therefore w satisfies A'_1.

(*c*) $\Rightarrow$ (*b*). It is clear that there exists a constant $C > 0$ such that

$$\bar{S}_r f \leqslant C(Mf + S_r f). \tag{2.19}$$

M and S_r are of weak type (1, 1) by (2, 2) and by what we have already shown. Then (*b*) follows from (2.19).

Proof of theorem (2.4). — Let's assume that w satisfies A'_p. Then, M is of strong type (p, p) by (2.1). Thus, keeping in mind inequality (2.19), to prove (*c*) it will suffice to establish the statement for the operator S_r.

In the proof we follow the idea of an extrapolation theorem (see [*3*] for example) and we shall need the following lemmas:

(2.20) LEMMA. — Let $1 < p < \infty$. If w satisfies A'_p then there exists $\varepsilon > 0$, $0 < \varepsilon < p - 1$, such that w satisfies $A'_{p-\varepsilon}$.

(2.21) LEMMA. — Let $1 < p < \infty$ and let g be a non negative function in $L^q(wd\mu)$. If w satisfies A'_p then there exists $G \geqslant g$ with $\| G \|_{q,w} \leqslant C \| g \|_{q,w}$ and such that Gw satisfies A'_1 (where C is independent of g).

A proof of lemma (2.20) can be found in [*2*] where T is ergodic and $(X, \mathscr{F}, \mu)$ is a non atomic probability space; the proof is valid in our setting because these properties are not used.

Proof of lemma (2.21). — It is clear that w satisfies A'_p if and only if w^{1-q} satisfies A'_q. Therefore, (2.1) guarantees the existence of a constant $C > 0$ such that $\| Mf \|_{q,w^{1-q}} \leqslant C \| f \|_{q,w^{1-q}}$ for every f. Then $Pf = w^{-1}M(fw)$ is bounded in $L^q(wd\mu)$ with the same constant. As a consequence the function

$$G = \sum_{j=0}^{\infty} (2C)^{-j} P^j g$$

satisfies the conditions required in the lemma.

Now we shall prove theorem (2.4) for S_r. First, we shall see that if w satisfies A'_p then S_r is of weak type (p, p) with respect to $wd\mu$, i. e., there exists $C > 0$ such that for every measurable function f and for each $\lambda > 0$

$$\int_{O_\lambda} wd\mu \leqslant C\lambda^{-p} \int_X | f |^p wd\mu \tag{2.22}$$

where $O_\lambda = \{ x \in X : S_r f(x) > \lambda \}$. Let's fix $\lambda > 0$ and f. Then

$$\int_{O_\lambda} w d\mu = \| \chi_{O_\lambda} \|_{p,w}^p = \left[\int_X g \chi_{O_\lambda} w d\mu \right]^p \tag{2.23}$$

where g is a function such that $\| g \|_{q,w} = 1$. Let G be a function given by (2.21) for g. Then

$$\left[\int_X g \chi_{O_\lambda} w d\mu \right]^p \leqslant \left[\int_{O_\lambda} G w d\mu \right]^p . \tag{2.24}$$

Since Gw satisfies A'_1, inequality (2.24), (2.23) and theorem (2.3) ensure us that

$$\int_{O_\lambda} w d\mu \leqslant C \lambda^{-p} \left[\int_X G \, | f | \, w d\mu \right]^p . \tag{2.25}$$

Then (2.22) follows from (2.25) since $\| G \|_{q,w} \leqslant C \| g \|_{q,w} = C$.

Once (2.22) has been proved, we get theorem (2.4) for S_r by a standard argument. If w satisfies A'_p then w satisfies $A'_{p-\varepsilon}$ (lemma (2.20)). Therefore, by what we have already shown, S_r is of weak type $(p - \varepsilon, p - \varepsilon)$ with respect to $w d\mu$. Now, since S_r is of type (∞, ∞), Marcinkiewicz's interpolation theorem gives us that S_r is of strong type (p, p) with respect to the measure $w d\mu$.

(2.26) EXAMPLE. — The converse of theorem (2.4) is not true. In order to see this we consider $X = \mathbb{Z}$, the set of the integers, $\mathscr{F}$ the power set of $\mathbb{Z}$, μ the counting measure, $Tx = x + 1$ and $w(x) = (1 + |x|)^{-r}$ $(r > 1)$. We shall prove that S_r and $\bar{S}_r$ are bounded in $L^r(w d\mu)$. However w does not satisfy A'_r.

w does not satisfy A'_r. If w satisfies A'_r then there exists $C > 0$ such that

$$\sum_{i=0}^{k} (1 + |x+i|)^{-r} \left[\sum_{i=0}^{k} (1 + |x+i|)^{r/(r-1)} \right]^{r-1} \leqslant C(k+1)^r \tag{2.27}$$

for every integer x and for any non negative integer k. If we take $x = 0$ in (2.27) and observe that $1 \leqslant \sum_{i=0}^{k} (1 + |x+i|)^{-r}$ we get

$$\left[\sum_{i=0}^{k} (1 + |i|)^{r/(r-1)} \right]^{r-1} \leqslant C(k+1)^r .$$

On the other hand $\sum_{i=0}^{k}(1+|i|)^{r/(r-1)} \geqslant (r-1)(1+k)^{2r/(r-1)}(2r)^{-1}$ and hence

$$(r-1)(1+k)^{2r/(r-1)}(2r)^{-1} \leqslant C(k+1)^{r/(r-1)}$$

for every k. This is a contradiction. Therefore w does not satisfy A'_r.

S_r *is bounded in* $L^r(wd\mu)$. Let f be a function in $L^r(wd\mu)$. Then

$$(2.28) \qquad \sum_{x=-\infty}^{\infty} |S_r f(x)|^r w(x) = \sum_{y=-\infty}^{\infty} |f(y)|^r \sum_{k=-\infty}^{\infty} (|k|+1)^{-r}(1+|y-k|)^{-r}.$$

Let's fix y. If $2|k| \geqslant |y|$ then $(|k|+1)^{-r} \leqslant 2r(|y|+1)^{-r}$ and therefore

$$(2.29) \qquad \sum_{2|k|\geqslant|y|} (|k|+1)^{-r}(1+|y-k|)^{-r} \leqslant C(|y|+1)^{-r}.$$

If $2|k| < |y|$ we have $1+|y-k| > (|y|+1)/2$ and then

$$(2.30) \qquad \sum_{2|k|<|y|} (|k|+1)^{-r}(1+|y-k|)^{-r} \leqslant C(|y|+1)^{-r}$$

Then (2.29) and (2.30) give us

$$(2.31) \qquad \sum_{k=-\infty}^{\infty} (|k|+1)^{-r}(1+|y-k|)^{-r} \leqslant C(|y|+1)^{-r}.$$

Putting (2.31) in (2.28) we get

$$\sum_{x=-\infty}^{\infty} |S_r f(x)|^r w(x) \leqslant C \sum_{y=-\infty}^{\infty} |f(y)|^r (|y|+1)^{-r} = C \sum_{y=-\infty}^{\infty} |f(y)|^r w(y)$$

as we wished to prove.

$\bar{S}_r$ *is bounded in* $L^r(wd\mu)$. Let f be a function in $L^r(wd\mu)$. Then

$$\sum_{x=-\infty}^{\infty} |\bar{S}_r f(x)|^r w(x) \leqslant C \sum_{x=-\infty}^{\infty} |S_r f(x)|^r w(x)$$
$$+ C \sum_{x=-\infty}^{\infty} \sum_{k=0}^{\infty} (k+2)^{-r}(|T_{0,k} f(x)|^r + |T_{k,0} f(x)|^r) w(x).$$

Since S_r is a bounded operator in $L^r(w\,d\mu)$ we have

$$(2.32)\quad \sum_{x=-\infty}^{\infty} |\bar{S}_r f(x)|^r w(x) \leq C \sum_{x=-\infty}^{\infty} |f(x)|^r w(x)$$
$$+ C \sum_{x=-\infty}^{\infty} \sum_{k=0}^{\infty} (k+2)^{-r}(|T_{0,k}f(x)|^r + |T_{k,0}f(x)|^r)w(x).$$

Now we shall bound the second term on the right-hand side of inequality (2.32). It is clear that

$$|T_{0,k} f(x)|^r \leq T_{0,k}|f|^r(x), \qquad |T_{k,0} f(x)|^r \leq T_{k,0}|f|^r(x).$$

Then

$$(2.33)\quad C \sum_{x=-\infty}^{\infty} \sum_{k=0}^{\infty} (k+2)^{-r}(|T_{0,k}f(x)|^r + |T_{k,0}f(x)|^r)w(x)$$
$$\leq C \sum_{y=-\infty}^{\infty} |f(y)|^r \sum_{k=0}^{\infty} (k+1)^{-r-1} \sum_{i=-k}^{k} (1+|y-i|)^{-r}$$
$$\leq C \sum_{y=-\infty}^{\infty} |f(y)|^r \sum_{i=-\infty}^{\infty} (1+|i|)^{-r}(1+|y-i|)^{-r}$$
$$\leq C \sum_{y=-\infty}^{\infty} |f(y)|^r(1+|y|)^{-r}$$

where the last inequality follows from (2.31). If we put (2.33) together (2.32) we get

$$\sum_{x=-\infty}^{\infty} |\bar{S}_r f(x)|^r w(x) \leq C \sum_{x=-\infty}^{\infty} |f(x)|^r w(x)$$

as we wished to prove.

(2.34). Now we will give an example which shows that the converse of theorem (2.4) is not true even if $\mu(X) < \infty$. In order to see this, it suffices to consider $X = \bigcup_{n=0}^{\infty} \bigcup_{i=-n}^{n} \{(n, i)\}$, $\mathcal{F}$ the power set of X, μ the measure determined by $\mu(n, i) = 2^{-n-1}(2n+1)^{-1}$, T the transformation defined by

$$T(n, i) = (n, i+1) \quad \text{if} \quad -n \leq i < n$$
$$T(n, n) = (n, -n)$$

and $w(i) = (1 + |i|)^{-r}$. As in example (2.26) it is clear that w does not satisfy A'_r. On the other hand, the proof of the boundedness of S_r and $\bar{S}_r$ in $L^p(wd\mu)$ follows from the proof of example (2.26).

(2.35) FINAL REMARK. — The A'_p classes are not trivial. A characterization of weights in A'_p classes can be found in [*6*] (see [*7*] also). It was shown that a weight w belongs to the A'_p $(1 < p)$ class if and only if there exist w_1 and w_2 satisfying A'_1 and such that $w = w_1 w_2^{1-p}$. On the other hand, a weight w is in A'_1 if and only if $w = g.u$ where g is essentially constant $(0 < C_1 \leqslant g \leqslant C_2)$ and $u = \Sigma\, 2^{-j} u_j < \infty$ with $u_0 > 0$ and $Cu_j = Mu_{j-1}$ (observe that to generate functions u we can take, for example, any positive function u_0 in $L^p(d\mu)$ and choose C to be the constant in the theorem of A. Ionescu-Tulcea).

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(Manuscrit reçu le 17 septembre 1985)