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Asymptotic expansions of the invariant density of a Markov process with a small parameter

by

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ABSTRACT. — We consider the asymptotic behavior of the invariant density of a Markov process on $\mathscr{R}^d$ which is a perturbation of a dynamical system.

Key words : Asymptotic expansion, invariant density, S. Watanabe's theory.

RÉSUMÉ. — Nous considérons le comportement asymptotique de la densité invariante d'un processus de Markov sur $\mathscr{R}^d$ qui est une perturbation d'un système dynamique.

INTRODUCTION

Let $(X^\varepsilon(t), P_x)$, $0 \leqq t$, $x \in \mathscr{R}^d$ be Markov processes on $\mathscr{R}^d$ solving the following stochastic differential equation:

$$dX^\varepsilon(t) = b(X^\varepsilon(t))\,dt + \varepsilon\,dW(t)$$
$$X^\varepsilon(0) = x$$

where $W(t)$ is an d-dimensional Wiener process, $b(x)$ is a function of $\mathscr{R}^d$ to $\mathscr{R}^d$ and $\varepsilon > 0$ is a small parameter. Then the following result is known; if $b(.) : \mathscr{R}^d \to \mathscr{R}^d$ is Lipschitz continuous,

$$P_x(\lim_{\varepsilon \to 0} \sup_{0<t<T} |X^\varepsilon(t) - \mathfrak{x}(t)| = 0) = 1 \qquad (x \in \mathscr{R}^d)$$

where we denote by $\mathfrak{x}(t)$ the dynamical system solving the next differential equation:

$$\frac{d}{dt}\mathfrak{x}(t)=b(\mathfrak{x}(t))$$
$$\mathfrak{x}(0)=x.$$

In this paper we study, by way of S. Watanabe's theory [6], the asymptotic behavior of the invariant density $\mathfrak{p}^\varepsilon(x)$ of $X^\varepsilon(t)$ of the following type:

$$\varepsilon^d \exp(V(x)/\varepsilon^2)\,\mathfrak{p}^\varepsilon(x)=\mathfrak{p}^0(x)+\varepsilon^2\mathfrak{p}^1(x)+\varepsilon^4\mathfrak{p}^2(x)+\ldots \quad (\text{as } \varepsilon\to 0)$$

where $V(x)$ is the Wentzell-Freidlin quasi-potential. We also note that the invariant density $\mathfrak{p}^\varepsilon(x)$ is the solution of the following equation:

$$\frac{\varepsilon^2}{2}\sum_{i=1}^{d}\frac{\partial^2}{\partial x_i^2}\mathfrak{p}^\varepsilon(x)\text{-div}(b(x)\,\mathfrak{p}^\varepsilon(x))=0 \qquad (x\in\mathscr{R}^d)$$

$$\int_{\mathscr{R}^d}\mathfrak{p}^\varepsilon(x)\,dx=1.$$

The study is important when we study the exit distribution of $X^\varepsilon(t)$ from the bounded domain in $\mathscr{R}^d$ (*cf.* M. V. Day [1], [2]).

With respect to this problem M. V. Day [3] got results only in the case $n=0$ but under weaker conditions than we give in this paper. As the special case, if $b(x)=\nabla U(x)$ for an infinitely differentiable function U of $\mathscr{R}^d$ to $\mathscr{R}$ such that $\int_{\mathscr{R}^d}\exp(2U(x)/\varepsilon^2)\,dx<+\infty$, then

$$\mathfrak{p}^\varepsilon(x)=C(\varepsilon)\exp(2U(x)/\varepsilon^2)$$

where we put $C(\varepsilon)=\left(\int_{\mathscr{R}^d}\exp(2U(x)/\varepsilon^2)\,dx\right)^{-1}$.

In section 1 we state our results. In section 2 we give lemmas necessary for the proofs of our results. In section 3 we prove our results.

At last we give some notations which we use in this paper. For (d, d)-matrix $A=(a_{ij})_{i,j=1}^d$, we put $|A|=$ determinant of A, $\|A\|=\left(\sum_{i,j=1}^{d}(a_{ij})^2\right)^{1/2}$ and $A^*=(a_{ji})_{i,j=1}^d$.

For $u=(u_i)_{i=1}^d$ of $\mathscr{R}^d$, we put $|u|=\left(\sum_{i=1}^{d}(u_i)^2\right)^{1/2}$.

1. MAIN RESULTS

In this section we state our results.

We introduce the following conditions.

(A.0) $b(x)$ is an d-dimensional C^∞-vector field on $\mathcal{R}^d$ with bounded derivatives of all orders.

(A.0)′ $b(x)$ is a function of $\mathcal{R}^d$ to $\mathcal{R}^d$ with Hölder continuous first derivatives for some exponent $\gamma (0<\gamma\leqq 1)$ and is C^∞-class in a small neighborhood of $0\in\mathcal{R}^d$.

$$b(0)=0. \tag{A.1}$$

$$\sup\left(\sup\left(\sum_{i,j=1}^{d}\frac{\partial b^i}{\partial x_j}(x)\,e^i e^j;\ \sum_{i=1}^{d}(e^i)^2=1\right);\ x\in\mathcal{R}^d\right)<0. \tag{A.2}$$

$$\left\{\begin{aligned}&\sup\left(\sum_{i,j=1}^{d}\frac{\partial b^i}{\partial x_j}(0)\,e^i e^j;\ \sum_{i=1}^{d}(e^i)^2=1\right)<0,\\&\lim_{t\to\infty}\mathfrak{x}(t)=0\qquad(\mathfrak{x}(0)\in\mathcal{R}^d).\end{aligned}\right. \tag{A.2}'$$

(A.3) There exist positive constants ε_0, R and a nonnegative C^2-function $w(x)$ which tends to ∞ as $|x|\to\infty$ so that

$$\frac{\varepsilon^2}{2}\sum_{i=1}^{d}\frac{\partial^2}{\partial x_i^2}w(x)+\sum_{i=1}^{d}b^i(x)\frac{\partial w}{\partial x_i}(x)\leqq -1\qquad(|x|\geqq R,\ 0<\varepsilon\leqq\varepsilon_0).$$

Remark. — The assumptions (A.0), (A.1) and (A.2) imply the assumptions (A.0)′, (A.1), (A.2)′ and (A.3). Moreover the assumption (A.0)′ imply the regularity assumptions of M. V. Day [3] and (A.1), (A.2)′ and (A.3) imply the stability assumptions of M. V. Day [3].

The following proposition shows how our assumptions are strong.

PROPOSITION 1.1. — *Under the conditions* (A.0), (A.1) *and* (A.2), *Markov processes* $(X^\varepsilon(t), P_x)$, $0\leqq t$, $x\in\mathcal{R}^d$ *are positively recurrent and*

$$|\mathfrak{x}(t)|\leqq\exp(-\lambda t)\,|\mathfrak{x}(0)|\qquad(t>0) \tag{1.1}$$

where we put the quantity in (A.2) $-\lambda$.

Before we state other results we give some notations. Let $V_T(y)$ denote the minimum of $\frac{1}{2}\int_0^T|\dot\varphi(t)-b(\varphi(t))|^2\,dt$ over all absolutely continuous functions φ of $[0,T]$ to $\mathcal{R}^d$ such that $\varphi(0)=0$ and $\varphi(T)=y$. Let us put $V(y)=\text{infimum}\,(V_T(y);\ T>0)$. For a minimal path φ of $V_T(y)$, let $Y^\varepsilon(t)$, $0\leqq t\leqq T$, be the solution of the following S.D.E.

$$\left\{\begin{aligned}&dY^\varepsilon(t)=(b(Y^\varepsilon(t))-b(\varphi(t))+\dot\varphi(t))\,dt+\varepsilon\,dW(t)\\&Y^\varepsilon(0)=0\end{aligned}\right. \tag{1.2}$$

and let $Y^i(t)$, $0\leqq t\leqq T$, $(i=0,1,2,\dots)$ be determined by the following formal expansion

$$Y^\varepsilon(t)=\sum_{k=0}^{\infty}\varepsilon^k Y^k(t)\qquad(\text{as }\varepsilon\to 0). \tag{1.3}$$

To avoid confusion, we sometimes write $Y^{\varepsilon}(t)=Y^{\varepsilon, T}(t)$ and $Y^{i}(t)=Y^{i, T}(t)$.
The following theorems show how the meaning of the expansion (1.3) is strong.

THEOREM 1.2. — *Under the conditions* (A.0), (A.1) *and* (A.2), *for any natural number m and n*

$$(1.4)\quad \sup_{\substack{0<T\\ \varepsilon<1}}\left(\sup_{\substack{0<t<T\\ \varphi(T)=y\in\mathscr{R}^d}}\left(\varepsilon^{-(n+1)}\left\|Y^{\varepsilon, T}(t)-\sum_{i=0}^{n}\varepsilon^{i}Y^{i, T}(t)\right\|_{m}\right)\right)<+\infty$$

where we denote by $\|.\|_m$ L^m*-norm.*

THEOREM 1.3. — *Under the conditions* (A.0), (A.1) *and* (A.2), *for any natural number m, n and sufficiently large q*

$$(1.5)\quad \sup_{\substack{0<T\\ \varepsilon<1}}\left(\sup_{\substack{0<t<T\\ \varphi(T)=y\in\mathscr{R}^d}}\left(\varepsilon^{-(n+1)}\left\|\left|D^{q}\left(Y^{\varepsilon, T}(t)-\sum_{i=0}^{n}\varepsilon^{i}Y^{i, T}(t)\right)\right|_{HS}\right\|_{m}\right)\right)<+\infty$$

where we denote by $|.|_{HS}$ *Hilbert Schmidt norm.*
The following proposition shows the uniform integrability of exponential moments which usually appear in such arguments.

PROPOSITION 1.4. — *Under the conditions* (A.0), (A.1) *and* (A.2), *there exists a constant* $p>1$ *such that*

$$(1.6)\quad \overline{\lim_{T\to\infty}}\sup\left(E\left[\exp\left(p\left|\left\langle\dot{\varphi}(T)-b(\varphi(T)), \frac{Y^{\varepsilon}(T)-Y^{0}(T)-\varepsilon Y^{1}(T)}{\varepsilon^{2}}\right\rangle\right|\right)\right]\right)<+\infty$$

where the supremum is over all $\varepsilon\,(0<\varepsilon<1)$ *and all* $\varphi(T)=y\in\mathscr{R}^d$ *for which* $V(y)<4\lambda^3(d^6\|\partial^2 b\|_{\infty}^2)^{-1}$. *Here we denote by* $\langle .,. \rangle$ *the inner product in* $\mathscr{R}^d$ *and we put*

$$\|\partial^2 b\|_{\infty}=\sup\left(\left|\frac{\partial^2 b^i(x)}{\partial x_j\,\partial x_k}\right|;\, x\in\mathscr{R}^d, i, j, k=1, \ldots, d\right).$$

The following results are what we want.

THEOREM 1.5. — *Suppose the conditions* (A.0), (A.1) *and* (A.2). *Then there exist functions* $p^k(x)$ $(k\geqq 0)$ *such that for all* $n\geqq 0$

$$(1.7)\quad \overline{\lim_{\varepsilon\to 0}}\left|\varepsilon^{d}\exp(V(x)/\varepsilon^2)\,p^{\varepsilon}(x)-\sum_{i=0}^{n}\varepsilon^{2i}p^{i}(x)\right|\varepsilon^{-2(n+1)}<+\infty$$

uniformly for all x for which $|x|$ is sufficiently small.

From theorem 3 in M. V. Day [3] and theorem 1.5, we have the following corollary.

COROLLARY 1.6. — *Under the conditions* (A.0)′, (A.1), (A.2)′ *and* (A.3), *there exist functions* $\mathfrak{p}^k(x)$ $(k \geqq 0)$ *such that for all* $n \geqq 0$

$$\overline{\lim_{\varepsilon \to 0}} \left| \varepsilon^d \exp(V(x)/\varepsilon^2)\, \mathfrak{p}^\varepsilon(x) - \sum_{i=0}^{n} \varepsilon^{2i} \mathfrak{p}^i(x) \right| \varepsilon^{-2(n+1)} < +\infty \tag{1.8}$$

uniformly for all x for which $|x|$ is sufficiently small.

2. LEMMAS

In this section we state the lemmas necessary for the proofs of our results. The next lemma is technically essential.

LEMMA 1. — *Let $f(t)$ be a continuous function of* $[0, \infty)$ *to* $\mathscr{R}$. *If there exist a positive constant* α *and a measurable function g of* $[0, \infty)$ *to* $\mathscr{R}$ *such that for any s, t* $(0 \leqq s < t)$

$$f(t) - f(s) \leqq -\alpha \int_s^t f(u)\, du + \int_s^t g(u)\, du, \tag{2.1}$$

then we have

$$f(t) \leqq \exp(-\alpha t)\left(\int_0^t \exp(\alpha s)\, g(s)\, ds + f(0)\right). \tag{2.2}$$

Proof. — If $t = 0$, then (2.2) holds. Suppose that there exists a positive constant t_0 such that (2.2) does not hold. Put

$$s_0 = \max\left\{ u < t_0;\ f(u) \leqq \exp(-\alpha u)\left(\int_0^u \exp(\alpha v)\, g(v)\, dv + f(0)\right)\right\}. \tag{2.3}$$

Then for $u\,(s_0<u<t_0)$, (2.2) does not hold. Therefore we have

$$\exp(-\alpha t_0)\left(\int_0^{t_0}\exp(\alpha u)\,g(u)\,du+f(0)\right)$$

$$-\exp(-\alpha s_0)\left(\int_0^{s_0}\exp(\alpha u)\,g(u)\,du+f(0)\right)$$

$$<f(t_0)-f(s_0)$$

$$<-\alpha\int_{s_0}^{t_0}\exp(-\alpha u)\left(\int_0^{u}\exp(\alpha v)\,g(v)\,dv+f(0)\right)du+\int_{s_0}^{t_0}g(u)\,du$$

$$=\exp(-\alpha t_0)\left(\int_0^{t_0}\exp(\alpha u)\,g(u)\,du+f(0)\right)$$

$$-\exp(-\alpha s_0)\left(\int_0^{s_0}\exp(\alpha u)\,g(u)\,du+f(0)\right)$$

which is a contradiction.

Q.E.D.

The following lemma can be proved by lemma 1 and is used to prove theorem 1.5.

LEMMA 2. — *Let $f^n(t)$ $(n=1, 2, \ldots)$ be continuous functions of* $[0, \infty)$ *to* $\mathcal{R}$ *such that $f^n(0)=0$ $(n=1, 2, \ldots)$. Suppose that there exist positive constants c and $c_n\,(n=1, 2, \ldots)$ such that for any s, t $(0\leqq s<t)$ and natural number n*

$$f^n(t)-f^n(s)\leqq -cn\int_s^t f^n(u)\,du+c_n\int_s^t f^{n-1}(u)\,du \tag{2.4}$$

where we put $f^0(t)\equiv f^0=\text{constant}$. Then we have

$$f^n(t)\leqq f^0\,(c^n\,n!)^{-1}\,(1-\exp(-ct))^n\prod_{k=1}^{n}c_k \tag{2.5}$$

for all $t\geqq 0$ and natural numbers n.

Proof. — We prove by induction.

When $n=1$, by lemma 1 we have

$$f^1(t)\leqq\exp(-ct)\int_0^t\exp(cs)\,(c_1 f^0)\,ds=f^0\,c^{-1}\,(1-\exp(-ct))\,c_1.$$

Assume that (2.5) holds for $n=k$. Then we have, by lemma 1,

$$
\begin{aligned}
f^{k+1}(t) &\leqq \exp(-c(k+1)t)\int_0^t \exp(c(k+1)s)\,c_{k+1}f^k(s)\,ds \\
&\leqq \exp(-c(k+1)t)\int_0^t \exp(c(k+1)s)\,c_{k+1}f^0 \\
&\quad \times (c^k k!)^{-1}(1-\exp(-cs))^k \prod_{j=1}^{k} c_j\,ds \\
&= f^0 (c^{k+1}(k+1)!)^{-1}(1-\exp(-ct))^{k+1}\prod_{j=1}^{k+1} c_j.
\end{aligned}
$$

Q.E.D.

We use the following lemma to prove theorem 1.3.

LEMMA 3. — *Let* $\psi(u)$ *be a measurable function of* $[0, \infty)$ *to* $\mathscr{R}$ *and let* $Y(t)$ $(0 \leqq t < +\infty)$ *be the solution of the following differential equation:*

$$
(2.6) \qquad \begin{aligned}
&\frac{d}{dt}Y(t)=\partial b(\psi(t))\,Y(t)\\
&Y(0)=I
\end{aligned}
$$

where I *denote an* (d, d)*-identity matrix. Then under the conditions* (A.0) *and* (A.2), *for any positive constants* s, t $(s<t)$, *we have*

$$
(2.7) \qquad \|Y(t)\,Y(s)^{-1}\| \leqq d\{\exp(-\lambda(t-s))\}
$$

Proof. — We put $Y(t)\,Y(s)^{-1}=Z^{s,t}=(Z_{ij}^{s,t})_{i,j=1}^d$.

Then for any t_0, $(t>t_0>s)$, we have

$$
\|Z^{s,t}\|^2-\|Z^{s,t_0}\|^2 \leqq -2\lambda\int_{t_0}^t \|Z^{s,u}\|^2\,du
$$

since

$$
\sum_{i,j=1}^d (Z_{ij}^{s,t})^2-\sum_{i,j=1}^d (Z_{ij}^{s,t_0})^2=\int_{t_0}^t 2\sum_{i,j=1}^d Z_{ij}^{s,u}\frac{d}{du}Z_{ij}^{s,u}\,du,
$$

$$
\begin{aligned}
\sum_{i,j=1}^d Z_{ij}^{s,u}\frac{d}{du}Z_{ij}^{s,u} &= \sum_{i,j=1}^d Z_{ij}^{s,u}\left(\sum_{k=1}^d \frac{\partial b^i}{\partial x_k}(\psi(u))\,Z_{kj}^{s,u}\right)\\
&= \sum_{j=1}^d \langle Z_j^{s,u}, \partial b(\psi(u))\,Z_j^{s,u}\rangle\\
&\leqq -\lambda\sum_{j=1}^d |Z_j^{s,u}|^2=-\lambda\|Z^{s,u}\|^2 \quad [\text{from (A.2)}]
\end{aligned}
$$

where we put $Z_j^{s,u}=(Z_{ij}^{s,u})_{i=1}^d \in \mathscr{R}^d$ $(j=1, \ldots, d)$.

Therefore by lemma 1, we have

$$\|Z^{s,t}\|^2 \leqq \exp(-2\lambda(t-s))\|Z^{s,s}\|^2 = \exp(-2\lambda(t-s))\,d^2.$$

Q.E.D.

Remark. — It is easy to see that $Y(t)^{-1}$ exists.

We use the next lemma when we prove theorem 1.2.

LEMMA 4. — *Under the conditions* (A.0) *and* (A.2), *for any natural number m and n,*

(2.8) $$\sup(\sup(\|Y^{n,T}(t)\|_m;\ 0\leqq t\leqq T,\ \varphi(T)=y\in\mathscr{R}^d);\ 0\leqq T)<+\infty$$

Proof. — We prove by induction. Put $Y^{n,T}(t)=Y^n(t)$.

(When $n=1$.)

$$E[|Y^1(t)|^{2n}]\leqq((2\lambda)^n n!)^{-1}\prod_{k=1}^{n}(kd+2k(k-1)),$$

since

$$dY^1(t)=\partial b(Y^0(t))\,Y^1(t)\,dt+dW(t),$$

and by Ito's formula,

$$\begin{aligned}
&E[|Y^1(t)|^{2n}]-E[|Y^1(s)|^{2n}]\\
&\quad=2n\int_s^t E[|Y^1(u)|^{2(n-1)}\langle Y^1(u),\partial b(Y^0(u))\,Y^1(u)\rangle]\,du\\
&\qquad+(nd+2n(n-1))\int_s^t E[|Y^1(u)|^{2(n-1)}]\,du\\
&\quad\leqq -2\lambda n\int_s^t E[|Y^1(u)|^{2n}]\,du\\
&\qquad+(nd+2n(n-1))\int_s^t E[|Y^1(u)|^{2(n-1)}]\,du
\end{aligned}$$

[from (A.2)], therefore from lemma 2, we have

$$E[|Y^1(t)|^{2n}]\leqq((2\lambda)^n n!)^{-1}(1-\exp(-2\lambda t))^n\prod_{k=1}^{n}(kd+2k(k-1)).$$

(When $n\geqq 2$.) Since

$$Y^n(t)=\int_0^t[\partial b(Y^0(s))\,Y^n(s)+R_n(s)]\,ds,$$

where we put

$$R_n(s)=\frac{1}{2!}\sum_{\substack{i+j=n\\ i,j\geqq 1}}\partial^2 b(Y^0(s))\,Y^i(s)\otimes Y^j(s)+\ldots+\frac{1}{n!}\partial^n b(Y^0(s))\overset{n}{\otimes}Y^1(s)$$

and for $a_i=(a_i^j)_{j=1}^d \in \mathcal{R}^d\,(i=1,\ldots,n)$, we put

$$\partial^n b(x)\, a_1 \otimes \ldots \otimes a_n = \left(\sum_{i_1,\ldots,i_n=1}^{d} \frac{\partial^n b^i(x)}{\partial x_{i_1}\ldots \partial x_{i_n}} a_1^{i_1}\ldots a_n^{i_n} \right)_{i=1}^{d} \in \mathcal{R}^d,$$

we have, for any s, t $(s<t)$ and $\alpha>0$,

$$\begin{aligned}(|\mathrm{Y}^n(t)|^2+\alpha)^{1/2}-(|\mathrm{Y}^n(s)|^2+\alpha)^{1/2} &= \int_s^t (|\mathrm{Y}^n(u)|^2+\alpha)^{-1/2}\langle \mathrm{Y}^n(u), \partial b(\mathrm{Y}^0(u))\mathrm{Y}^n(u)+\mathrm{R}_n(u)\rangle\, du \\ &\leqq -\lambda \int_s^t (|\mathrm{Y}^n(u)|^2+\alpha)^{-1/2}|\mathrm{Y}^n(u)|^2\, du + \int_s^t |\mathrm{R}_n(u)|\, du.\end{aligned}$$

Let α tend to 0 then we have

$$|\mathrm{Y}^n(t)|-|\mathrm{Y}^n(s)| \leqq -\lambda \int_s^t |\mathrm{Y}^n(u)|\, du + \int_s^t |\mathrm{R}_n(u)|\, du.$$

Therefore from lemma 1,

$$|\mathrm{Y}^n(t)| \leqq \int_0^t \exp(-\lambda(t-s))\, |\mathrm{R}_n(s)|\, ds.$$

Hence, by Hölder's inequality,

$$|\mathrm{Y}^n(t)|^m \leqq \left(\int_0^t \exp\left(-\frac{m\lambda(t-s)}{2(m-1)}\right) ds \right)^{m-1} \times \left(\int_0^t \exp\left(-\frac{m\lambda(t-s)}{2}\right) |\mathrm{R}_n(s)|^m\, ds \right),$$

where we consider $\lambda(t-s)=\dfrac{\lambda(t-s)}{2}+\dfrac{\lambda(t-s)}{2}$ and $m^{-1}+(m/m-1)^{-1}=1$, which completes the proof.

At last we give the next lemma.

LEMMA 5. — *Under the assumptions* (A.0) *and* (A.2), *we have, for the Malliavin's covariance* $\left\langle \dfrac{\mathrm{D}(\mathrm{Y}^\varepsilon(\mathrm{T})-\mathrm{Y}^0(\mathrm{T}))}{\varepsilon}, \dfrac{\mathrm{D}(\mathrm{Y}^\varepsilon(\mathrm{T})-\mathrm{Y}^0(\mathrm{T}))}{\varepsilon} \right\rangle_{\mathrm{HS}}$,

$$\sup\left(\left\| \left\langle \frac{\mathrm{D}(\mathrm{Y}^\varepsilon(\mathrm{T})-\mathrm{Y}^0(\mathrm{T}))}{\varepsilon}, \frac{\mathrm{D}(\mathrm{Y}^\varepsilon(\mathrm{T})-\mathrm{Y}^0(\mathrm{T}))}{\varepsilon} \right\rangle_{\mathrm{HS}} \right\|;\ \mathrm{T}>0, \varepsilon>0, \varphi(\mathrm{T})=y\in\mathcal{R}^d \right) < +\infty \tag{2.9}$$

$$\lim_{\overline{\mathrm{T}\to\infty}} \inf(\lambda_y^{\varepsilon,\mathrm{T}};\ \varepsilon>0, y\in\mathcal{R}^d)>0 \tag{2.10}$$

where we denote by $\lambda_y^{\varepsilon, T}$ *the minimal eigenvalue of*

$$\left\langle \frac{D(Y^\varepsilon(T)-Y^0(T))}{\varepsilon}, \frac{D(Y^\varepsilon(T)-Y^0(T))}{\varepsilon} \right\rangle_{HS} \quad \textit{for} \quad y=\varphi(T)\in\mathscr{R}^d.$$

Proof [proof of (2.9)]. – Let $Y^\varepsilon(t)$ be the solution of (2.6) for $\psi(u)=Y^\varepsilon(u)$ and put $Z^{s,t,\varepsilon}=(Z_{ij}^{s,t,\varepsilon})_{i,j=1}^d=Y^\varepsilon(t)Y^\varepsilon(s)^{-1}$ $(s\leqq t\leqq T)$. Since

$$\left\langle \frac{D(Y^\varepsilon(T)-Y^0(T))}{\varepsilon}, \frac{D(Y^\varepsilon(T)-Y^0(T))}{\varepsilon} \right\rangle_{HS} = \int_0^T Z^{s,T,\varepsilon}(Z^{s,T,\varepsilon})^* ds,$$

$$\left|\sum_{k=1}^d \int_0^T Z_{ik}^{s,T,\varepsilon} Z_{jk}^{s,T,\varepsilon} ds\right| \leqq \int_0^T \| Z^{s,T,\varepsilon}\|^2 ds$$

$$\leqq \int_0^T d^2 \exp(-2\lambda(T-s))\, ds \quad \text{(from lemma 3)}.$$

Hence we conclude

$$\sup\left(\left\| \left\langle \frac{D(Y^\varepsilon(T)-Y^0(T))}{\varepsilon}, \frac{D(Y^\varepsilon(T)-Y^0(T))}{\varepsilon} \right\rangle_{HS}\right\|;\; T>0, \varepsilon>0, \varphi(T)=y\in\mathscr{R}^d\right) \leqq \frac{d^3}{2\lambda}.$$

[proof of (2.10)].

We put $B_\varepsilon^{s,T}=Z^{s,T,\varepsilon}(Z^{s,T,\varepsilon})^*$ and denote by $\lambda_{1,\varepsilon}^{s,T}$ the minimal eigenvalue of $B_\varepsilon^{s,T}$ and $\|\partial b\|_\infty=\sup\left(\left|\frac{\partial b^i}{\partial x_j}(x)\right|; i,j=1,\ldots,d, x\in\mathscr{R}^d\right)$.

Then $\lambda_{1,\varepsilon}^{s,T}\geqq \exp(-2d\|\partial b\|_\infty (T-s))$, since

$$\lambda_{1,\varepsilon}^{s,T}=\inf\{|Z^{s,T,\varepsilon}e|; e=(e_i)_{i=1}^d\in\mathscr{R}^d \text{ and } |e|=1\}$$

and for t_1, t_2 $(s<t_1,\, t_2<T)$,

$$|Z^{s,t_2,\varepsilon}e|^2-|Z^{s,t_1,\varepsilon}e|^2=\int_{t_1}^{t_2}\sum_{i=1}^d 2\left(\sum_{k=1}^d Z_{ik}^{s,u,\varepsilon}e_k\right)\left(\sum_{j=1}^d \frac{d}{du}Z_{ij}^{s,u,\varepsilon}e_j\right)du$$

$$=2\int_{t_1}^{t_2}\langle Z^{s,u,\varepsilon}e, \partial b(Y^\varepsilon(u))Z^{s,u,\varepsilon}e\rangle du$$

therefore $\frac{d}{dt}|Z^{s,t,\varepsilon}e|^2\geqq -2d\|\partial b\|_\infty |Z^{s,t,\varepsilon}e|^2$ and

$$|Z^{s,t,\varepsilon}e|^2\geqq \exp(-2d\|\partial b\|_\infty (t-s))|e|^2.$$

Hence

$$\varliminf_{T\to\infty} \inf(\lambda_y^{\varepsilon, T}; \varepsilon>0, y\in\mathscr{R}^d)$$

$$\geqq \varliminf_{T\to\infty} \int_0^T \exp(-2d\|\partial b\|_\infty(T-s))\,ds$$

$$=(2d\|\partial b\|_\infty)^{-1}.$$

Q.E.D.

3. PROOFS OF MAIN RESULTS

In this section we give the proofs of our results.

Proof of proposition 1.1. [proof of the positively recurrent property of $X^\varepsilon(t)$]. – Since

$$\langle b(x), x\rangle=\langle b(x), x\rangle-\langle b(0), x\rangle$$

$$=\int_0^1 \langle \partial b(ux)x, x\rangle\,du \leqq -\lambda|x|^2 \quad \text{[from (A.2)]},$$

$X^\varepsilon(t)$ is positively recurrent (*cf.* Has'minskii [5]), [proof of $|\mathfrak{x}(t)|\leqq\exp(-\lambda t)|\mathfrak{x}(0)|$]. Since

$$|\mathfrak{x}(t)|^2-|\mathfrak{x}(s)|^2=2\int_s^t \langle \mathfrak{x}(u), b(\mathfrak{x}(u))\rangle du \leqq -2\lambda\int_s^t |\mathfrak{x}(u)|^2\,du,$$

$$|\mathfrak{x}(t)|^2\leqq\exp(-2\lambda t)|\mathfrak{x}(0)|^2 \quad \text{(from lemma 1)}.$$

Q.E.D.

Next we prove theorem 1.2.

Proof of theorem 1.2. – We put $R_n^\varepsilon(t)=Y^\varepsilon(t)-\sum_{i=0}^n \varepsilon^i Y^i(t)$.

For any $t\,(0\leqq t\leqq T)$ and $\alpha>0$, since

$$R_n^\varepsilon(t)=\int_0^t\left[b(Y^\varepsilon(s))-b\left(\sum_{i=0}^n \varepsilon^i Y^i(s)\right)\right]ds$$

$$+\int_0^t\left[b\left(\sum_{i=0}^n \varepsilon^i Y^i(s)\right)-\sum_{k=0}^n \frac{1}{k!}\frac{d^k}{d\varepsilon^k}b\left(\sum_{i=0}^n \varepsilon^i Y^i(s)\right)\bigg|_{\varepsilon=0}\varepsilon^k\right]ds,$$

we have for any s, $t\,(0 \leqq s < t \leqq \mathrm{T})$,

$$
\begin{aligned}
&(|\mathrm{R}_n^\varepsilon(t)|^2+\alpha)^{1/2}-(|\mathrm{R}_n^\varepsilon(s)|^2+\alpha)^{1/2}\\
&= \int_s^t \left\langle \mathrm{R}_n^\varepsilon(u), \frac{d}{du}\mathrm{R}_n^\varepsilon(u) \right\rangle (|\mathrm{R}_n^\varepsilon(u)|^2+\alpha)^{-1/2}\,du\\
&= \int_s^t (|\mathrm{R}_n^\varepsilon(u)|^2+\alpha)^{-1/2} \left\langle \mathrm{R}_n^\varepsilon(u), b(\mathrm{Y}^\varepsilon(u)) - b\left(\sum_{i=0}^n \varepsilon^i \mathrm{Y}^i(u)\right) \right\rangle du + \int_s^t (|\mathrm{R}_n^\varepsilon(u)|^2+\alpha)^{-1/2}\\
&\quad \times \left\langle \mathrm{R}_n^\varepsilon(u), b\left(\sum_{i=0}^n \varepsilon^i \mathrm{Y}^i(u)\right) - \sum_{i=0}^n \frac{1}{k!}\frac{d^k}{d\varepsilon^k} b\left(\sum_{i=0}^n \varepsilon^i \mathrm{Y}^i(u)\right)\Bigg|_{\varepsilon=0} \varepsilon^k \right\rangle du\\
&= \int_s^t (|\mathrm{R}_n^\varepsilon(u)|^2+\alpha)^{-1/2} \left\langle \mathrm{R}_n^\varepsilon(u), \partial b\left(\sum_{i=0}^n \varepsilon^i \mathrm{Y}^i(u) + \theta(u)\,\mathrm{R}_n^\varepsilon(u)\right) \mathrm{R}_n^\varepsilon(u) \right\rangle du\\
&\quad + \int_s^t (|\mathrm{R}_n^\varepsilon(u)|^2+\alpha)^{-1/2} \left\langle \mathrm{R}_n^\varepsilon(u), \frac{1}{(n+1)!}\frac{d^{n+1}}{d\gamma^{n+1}} b\left(\sum_{i=0}^n \gamma^i \mathrm{Y}^i(u)\right)\Bigg|_{\gamma=\varepsilon\tilde{\theta}(u)} \varepsilon^{n+1} \right\rangle du\\
&\leqq -\lambda \int_s^t (|\mathrm{R}_n^\varepsilon(u)|^2+\alpha)^{-1/2} |\mathrm{R}_n^\varepsilon(u)|^2\,du\\
&\quad + \int_s^t (|\mathrm{R}_n^\varepsilon(u)|^2+\alpha)^{-1/2} |\mathrm{R}_n^\varepsilon(u)| \left| \frac{1}{(n+1)!}\frac{d^{n+1}}{d\gamma^{n+1}} b\left(\sum_{i=0}^n \gamma^i \mathrm{Y}^i(u)\right)\right|_{\gamma=\varepsilon\tilde{\theta}(u)} \varepsilon^{n+1}\,|\,du
\end{aligned}
$$

for some $\theta(u)$, $\tilde{\theta}(u)$ such that $0 \leqq \theta(u), \tilde{\theta}(u) \leqq 1$ $(0 \leqq u \leqq \mathrm{T})$.

Let $\alpha \to 0$, then we have, for any s, $t\,(0 \leqq s < t \leqq \mathrm{T})$

$$
|\mathrm{R}_n^\varepsilon(t)| - |\mathrm{R}_n^\varepsilon(s)| \leqq -\lambda \int_s^t |\mathrm{R}_n^\varepsilon(u)|\,du + \int_s^t \left| \frac{1}{(n+1)!}\frac{d^{n+1}}{d\gamma^{n+1}} b\left(\sum_{i=0}^n \gamma^i \mathrm{Y}^i(u)\right)\right|_{\gamma=\varepsilon\tilde{\theta}(u)} |\,\varepsilon^{n+1}\,du.
$$

Hence from lemma 1,

$$|R_n^\varepsilon(t)| \leqq \exp(-\lambda t) \int_0^t \exp(\lambda s) \times \left| \frac{1}{(n+1)!} \frac{d^{n+1}}{d\gamma^{n+1}} b\left(\sum_{i=0}^n \gamma^i Y^i(s)\right)\bigg|_{\gamma=\varepsilon\tilde{\theta}(s)} \right| \varepsilon^{n+1} ds.$$

which completes the proof by lemma 4.

We prove theorem 1.3 by induction.

Proof of theorem 1.3. – It is easy to see that for any natural number n there exists a natural number $q(n)$ such that if $q \geqq q(n)$, then

$$D^q Y^i(h_1, \ldots, h_q)(t) = 0$$
$$(0 \leqq i \leqq n, 0 \leqq t \leqq T, h_1, \ldots, h_q \in H)$$

where H denote Cameron Martin space. Hence we only have to prove

$$(3.1) \quad \sup_{0 \leqq T, \varepsilon < 1} (\sup(\varepsilon^{-(n+1)} \| |D^q Y^{\varepsilon, T}(t)|_{HS} \|_m; \; 0 \leqq t \leqq T, \varphi(T) = y \in \mathscr{R}^d)) < +\infty$$

for sufficiently large q. From the following proposition, (3.1) holds.

Q.E.D.

PROPOSITION 3.1. – *For each natural number n, $t(0 \leqq t \leqq T)$ and $h_1, \ldots, h_n \in H$,*

$$\varepsilon^{-n} D^n Y^{\varepsilon, T}(h_1, \ldots, h_n)(t) = \int_0^t ds_1 \ldots \int_0^t ds_n g^{n, T}(s_1, \ldots, s_n; t) \dot{h}_1(s_1) \otimes \ldots \otimes \dot{h}_n(s_n)$$

for some

$$g^{n, T}(s_1, \ldots, s_n; t) = (g^{n, i, T}_{i_1, \ldots, i_n}(s_1, \ldots, s_n; t))^d_{i, i_1, \ldots, i_n = 1}$$

where we put

$$g^{n, T}(s_1, \ldots, s_n; t) \dot{h}_1(s_1) \otimes \ldots \otimes \dot{h}_n(s_n) = \left(\sum_{i_1, \ldots, i_n = 1}^d g^{n, i, T}_{i_1, \ldots, i_n}(s_1, \ldots, s_n; t) \times \dot{h}_1^{i_1}(s_1) \ldots \dot{h}_n^{i_n}(s_n)\right)_{i=1}^d \in \mathscr{R}^d.$$

Moreover there exist nonrandom constants C_n $(n = 1, 2, \ldots)$ such that

$$\sup_{\substack{0 \leqq T \\ \varepsilon < 1}} \left(\sup_{\substack{0 \leqq t \leqq T \\ \varphi(T) = y \in \mathscr{R}^d}} \left(\int_0^t ds_1 \ldots \int_0^t ds_n \times \sum_{i, i_1, \ldots, i_n = 1}^d |g^{n, i, T}_{i_1, \ldots, i_n}(s_1, \ldots, s_n; t)|^2 \right) \right) < C_n.$$

Proof of proposition 3.1. — It is easy to see that

$$(3.2)\qquad DY^{\varepsilon}(h)(t)=\varepsilon h(t)+\int_0^t \partial b(Y^{\varepsilon}(s))\,DY^{\varepsilon}(h)(s)\,ds \qquad (h\in H)$$

$$(3.3)\qquad D^2Y^{\varepsilon}(h_1,h_2)(t)=\int_0^t [\partial b(Y^{\varepsilon}(s))\,D^2Y^{\varepsilon}(h_1,h_2)(s) + \partial^2 b(Y^{\varepsilon}(s))\,DY^{\varepsilon}(h_1)(s)\otimes DY^{\varepsilon}(h_2)(s)]\,ds\,(h_1,h_2\in H).$$

Inductively, in the same way, we can show that for all $n\geqq 2$

$$(3.4)\qquad \varepsilon^{-n}D^nY^{\varepsilon}(\mathbf{h})(t)=\int_0^t [\partial b(Y^{\varepsilon}(s))\,\varepsilon^{-n}D^nY^{\varepsilon}(\mathbf{h})(s) + f^n(\varepsilon^{-1}DY^{\varepsilon}(s),\ldots,\varepsilon^{-(n-1)}D^{n-1}Y^{\varepsilon}(s))(\mathbf{h})]\,ds$$

where we put $\mathbf{h}=(h_1,\ldots,h_n)$ and f^n is a polynomial of

$$\varepsilon^{-1}DY^{\varepsilon}(s),\ldots,\varepsilon^{-(n-1)}D^{n-1}Y^{\varepsilon}(s)$$

with coefficient $\partial b(Y^{\varepsilon}(s)),\ldots,\partial^n b(Y^{\varepsilon}(s))$ and

$$\partial^k b(x)=\left(\frac{\partial^k b^i(x)}{\partial x_{i_1}\ldots\partial x_{i_k}}\right)^d_{i,i_1,\ldots,i_k=1} \qquad (1\leqq k\leqq n).$$

(When $n=1$.) We can put $g^{1,T}(s,t)=Z^{s,t,\varepsilon}$. In fact, it is easy to see that $\varepsilon^{-1}DY^{\varepsilon}(h)(t)=\int_0^t g^{1,T}(s,t)\,\dot h(s)\,ds$ from (3.2) and we have

$$\sup_{\substack{0\leqq T\\ \varepsilon<1}}\left(\sup_{\substack{0\leqq t\leqq T\\ \varphi(T)=y\in\mathscr{R}^d}}\int_0^t\sum_{i,j=1}^d |g_j^{1,i,T}(s,t)|^2\,ds\right)\leqq d^2(2\lambda)^{-1},$$

since

$$\sum_{i,j=1}^d |g_j^{1,i,T}(s,t)|^2=\|Z^{s,t,\varepsilon}\|^2\leqq d^2\exp(-2\lambda(t-s))$$

from lemma 3.

(When $n=2$.) We put, for i, i_1, i_2 $(=1,\ldots,d)$,

$$g^{2,i,T}_{i_1,i_2}(u_1,u_2;t)=\int_{u_1 v u_2}^t \langle (g^{1,T}(s,t))^i,\partial^2 b(Y^{\varepsilon}(s)) \times(g^{1,T}(u_1,s))_{i_1}\otimes(g^{1,T}(u_2,s))_{i_2}\rangle\,ds$$

where we put, for i, j $(=1,\ldots,d)$,

$$(g^{1,T}(u,s))^i=(g_k^{1,i,T}(u,s))^d_{k=1}$$

and

$$(g^{1,T}(u,s))_j=(g_j^{1,k,T}(u,s))^d_{k=1}.$$

Then from (3.3), it is easy to see that

$$\varepsilon^{-2} D^2 Y^\varepsilon(h_1, h_2)(t) = \int_0^t du_1 \int_0^t du_2\, g^{2,T}(u_1, u_2; t)\, \dot{h}_1(u_1) \otimes \dot{h}_2(u_2),$$

$$\sup_{\substack{0 \le T \\ \varepsilon < 1}} \left(\sup_{\substack{0 \le t \le T \\ \varphi(T) = y \in \mathscr{R}^d}} \left(\int_0^t du_1 \int_0^t du_2 \sum_{i, i_1, i_2 = 1}^d |g^{2, i, T}_{i_1, i_2}(u_1, u_2; t)|^2 \right) \right) < C_2$$

for some nonrandom constant C_2. In fact

$$\sum_{i, i_1, i_2 = 1}^d |g^{2, i, T}_{i_1, i_2}(u_1, u_2; t)|^2$$

$$= \sum \left| \int_{u_1 \vee u_2}^t \langle (g^{1,T}(s, t))^i, \partial^2 b(Y^\varepsilon(s)) \times (g^{1,T}(u_1, s))_{i_1} \otimes (g^{1,T}(u_2, s))_{i_2} \rangle\, ds \right|^2$$

$$= \sum \int_{u_1 \vee u_2}^t \langle (g^{1,T}(s_1, t))^i, \partial^2 b(Y^\varepsilon(s_1)) \times (g^{1,T}(u_1, s_1))_{i_1} \otimes (g^{1,T}(u_2, s_1))_{i_2} \rangle\, ds_1$$

$$\times \int_{u_1 \vee u_2}^t \langle (g^{1,T}(s_2, t))^i, \partial^2 b(Y^\varepsilon(s_2)) \times (g^{1,T}(u_1, s_2))_{i_1} \otimes (g^{1,T}(u_2, s_2))_{i_2} \rangle\, ds_2$$

$$\leqq \left(\int_{u_1 \vee u_2}^t \left(\sum \langle (g^{1,T}(s, t))^i, \partial^2 b(Y^\varepsilon(s)) \times (g^{1,T}(u_1, s))_{i_1} \otimes (g^{1,T}(u_2, s))_{i_2} \rangle^2 \right)^{1/2} ds \right)^2$$

where Σ is over all $i, i_1, i_2 (= 1, \ldots, d)$ and

$$|\langle (g^{1,T}(s, t))^i, \partial^2 b(Y^\varepsilon(s)) \times (g^{1,T}(u_1, s))_{i_1} \otimes (g^{1,T}(u_2, s))_{i_2} \rangle|$$

$$\leqq |(g^{1,T}(s, t))^i| \cdot |\partial^2 b(Y^\varepsilon(s)) \times (g^{1,T}(u_1, s))_{i_1} \otimes (g^{1,T}(u_2, s))_{i_2}|$$

$$\leqq |(g^{1,T}(s, t))^i| \cdot \| \partial^2 b(Y^\varepsilon(s)) \| \cdot | \times (g^{1,T}(u_1, s))_{i_1} | \cdot | (g^{1,T}(u_2, s))_{i_2} |$$

where we put

$$\| \partial^2 b(y) \| = \left(\sum_{i, j, k = 1}^d \left| \frac{\partial^2 b^i(y)}{\partial x_j \partial x_k} \right|^2 \right)$$

and hence

$$\int_0^t du_1 \int_0^t du_2 \sum_{i, i_1, i_2 = 1}^{d} | g^{2, i, \mathrm{T}}_{i_1, i_2}(u_1, u_2; t) |^2$$

$$\leqq \int_0^t du_1 \int_0^t du_2 \| \partial^2 b \|_\infty d^3 \left(\int_{u_1 \vee u_2}^t \| g^{1, \mathrm{T}}(s, t) \| \right.$$

$$\left. \times \| g^{1, \mathrm{T}}(u_1, s) \| . \| g^{1, \mathrm{T}}(u_2, s) \| ds \right)^2$$

$$\leqq \| \partial^2 b \|_\infty d^3 \sup_{\substack{0 \leqq \mathrm{T} \\ \varepsilon < 1}} \left(\left[\sup_{\substack{0 \leqq t \leqq \mathrm{T} \\ \varphi(\mathrm{T}) \in \mathscr{R}^d}} \left(\int_0^t \| g^{1, \mathrm{T}}(s, t) \| ds \right)^2 \right] \right.$$

$$\left. \times \left[\sup_{\substack{0 \leqq t \leqq \mathrm{T} \\ \varphi(\mathrm{T}) \in \mathscr{R}^d}} \left(\int_0^t \| g^{1, \mathrm{T}}(s, t) \|^2 ds \right)^2 \right] \right)$$

$$\leqq \| \partial^2 b \|_\infty d^9 (4 \lambda^4)^{-1}$$

since $\| g^{1, \mathrm{T}}(s, t) \| \leqq d \exp(-\lambda (t - s))$.

(Assume that proposition 3.1 holds when $n = k$.) From (3.4)

$$\varepsilon^{-(k+1)} \mathrm{D}^{k+1} \mathrm{Y}^\varepsilon (h_1, \ldots, h_{k+1})(t)$$

$$= \int_0^t \mathrm{Z}^{s, t, \varepsilon} f^{k+1} (\varepsilon^{-1} \mathrm{DY}^\varepsilon(s), \ldots,$$

$$\varepsilon^{-k} \mathrm{D}^k \mathrm{Y}^\varepsilon(s)) (h_1, \ldots, h_{k+1}) ds$$

and f^{k+1} can be written as the following;

$$\int_0^s ds_1 \ldots \int_0^s ds_{k+1} \tilde{g}^{k, \mathrm{T}}(s_1, \ldots, s_{k+1}; s) \dot{h}_1(s_1) \otimes \ldots \otimes \dot{h}_{k+1}(s_{k+1})$$

for some

$$\tilde{g}^{k, \mathrm{T}}(s_1, \ldots, s_{k+1}; s) = (\tilde{g}^{k, i, \mathrm{T}}_{i_1, \ldots, i_{k+1}}(s_1, \ldots, s_{k+1}; s))^d_{i, i_1, \ldots, i_{k+1} = 1}$$

such that

$$\sup_{\substack{0 \leqq \mathrm{T} \\ \varepsilon < 1}} \left(\sup_{\substack{0 \leqq t \leqq \mathrm{T} \\ \varphi(\mathrm{T}) = y \in \mathscr{R}^d}} \left(\int_0^t ds_1 \ldots \int_0^t ds_{k+1} \right. \right.$$

$$\left. \left. \times \Sigma | \tilde{g}^{k, i, \mathrm{T}}_{i_1, \ldots, i_{k+1}}(s_1, \ldots, s_{k+1}; t) |^2 \right) \right) < \tilde{\mathrm{C}}_{k+1}$$

for some nonrandom constant $\tilde{C}_{k+1}$, where $\sum$ is over all i, $i_1, \ldots, i_{k+1} (=1, \ldots, d)$. Hence we have to prove that the following quantity is bounded by some nonrandom constant C_n

$$\sup_{\substack{0 \leqq T \\ \varepsilon < 1}} \left(\sup_{\substack{0 \leqq t \leqq T \\ \varphi(T) = y \in \mathscr{R}^d}} \left(\int_1 \sum \left| \int_2 \sum_{j=1}^{d} g_j^{1, i, T}(u, t) \tilde{g}_{i_1, \ldots, i_{k+1}}^{k, j, T}(s_1, \ldots, s_{k+1}; u) \right|^2 \right) \right)$$

where $\int_1$ is over all $s_1, \ldots, s_{k+1}$ $(0 \leqq s_1, \ldots, s_{k+1} \leqq t)$ and $\int_2$ is over all $u (s_1 \vee \ldots \vee s_{k+1} \leqq u \leqq t)$ and $\sum$ is over all $i, i_1, \ldots, i_{k+1}$ $(=1, \ldots, d)$. In fact

$$\int_1 \sum \left| \int_2 \sum_{j=1}^{d} g_j^{1, i, T}(u, t) \tilde{g}_{i_1, \ldots, i_{k+1}}^{k, j, T}(s_1, \ldots, s_{k+1}; u) \right|^2$$

$$\leqq d^{k+2} \int_1 \left(\int_2 \| g^{1, T}(u, t) \| . \| \tilde{g}^{k, T}(s_1, \ldots, s_{k+1}; u) \| \right)^2$$

$$\leqq d^{k+2} \sup \left(\left[\sup \left\{ \left(\int_0^t \| g^{1, T}(s, t) \| \, ds \right)^2 ; 0 \leqq t \leqq T, \varphi(T) \in \mathscr{R}^d \right\} \right] \right.$$

$$\times \left[\sup \left\{ \int_0^t ds_1, \ldots \int_0^t ds_{k+1} \| \tilde{g}^{k, T}(s_1, \ldots, s_{k+1}; t) \|^2 ; \right. \right.$$

$$\left. \left. \left. 0 \leqq t \leqq T, \varphi(T) \in \mathscr{R}^d \right\} \right] ; 0 \leqq T, \varepsilon < 1 \right) \leqq d^{k+4} \lambda^{-2} \tilde{C}_{k+1}.$$

Q.E.D.

Next we prove proposition 1.4.

Proof of proposition 1.4. — Since

$$\langle \dot{\varphi}(T) - b(\varphi(T)), Y^\varepsilon(T) - Y^0(T) - \varepsilon Y^1(T) \rangle$$

$$= \int_0^T \langle \dot{\varphi}(s) - b(\varphi(s)), b(Y^\varepsilon(s)) - b(Y^0(s))$$

$$- \partial b(Y^0(s))(Y^\varepsilon(s) - Y^0(s)) \rangle \, ds,$$

$$|\langle \dot{\varphi}(T) - b(\varphi(T)), Y^\varepsilon(T) - Y^0(T) - \varepsilon Y^1(T) \rangle| \varepsilon^{-2}$$

$$\leqq \int_0^T |\dot{\varphi}(s) - b(\varphi(s))| \, d^2 \| \partial^2 b \|_\infty 2^{-1} (|Y^\varepsilon(s) - Y^0(s)| \varepsilon^{-1})^2 \, ds.$$

From this we have

$$\mathrm{E}[\exp(p|\langle \dot{\varphi}(\mathrm{T})-b(\varphi(\mathrm{T})), \mathrm{Y}^{\varepsilon}(\mathrm{T})-\mathrm{Y}^0(\mathrm{T})-\varepsilon\mathrm{Y}^1(\mathrm{T})\rangle|\varepsilon^{-2})]$$

$$\leqq \sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{1}{2}pd^2\|\partial^2 b\|_{\infty}\right)^n \mathrm{E}\left[\left(\int_0^{\mathrm{T}}|\dot{\varphi}(s)-b(\varphi(s))|\times|(\mathrm{Y}^{\varepsilon}(s)-\mathrm{Y}^0(s))\varepsilon^{-1}|^2 ds\right)^n\right]$$

$$\leqq \sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{1}{2}pd^2\|\partial^2 b\|_{\infty}\right)^n \left(\int_0^{\mathrm{T}}|\dot{\varphi}(s)-b(\varphi(s))|\times\|(\mathrm{Y}^{\varepsilon}(s)-\mathrm{Y}^0(s))\varepsilon^{-1}\|_{2n}^2 ds\right)^n$$

$$\leqq \sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{1}{2}pd^2\|\partial^2 b\|_{\infty}\right)^n \left(\prod_{k=1}^{n}\frac{2k+d-2}{2\lambda}\right)\left(\frac{4}{\lambda}\mathrm{V}_{\mathrm{T}}(y)\right)^{n/2}$$

$$\leqq \sum_{n=0}^{\infty}\left(\frac{1}{2}pd^2\|\partial^2 b\|_{\infty}\frac{d}{2\lambda}\left(\frac{4}{\lambda}\mathrm{V}_{\mathrm{T}}(y)\right)^{1/2}\right)^n<+\infty$$

if $\mathrm{V}_{\mathrm{T}}(y)<4\lambda^3(p^2 d^6\|\partial^2 b\|_{\infty}^2)^{-1}$, since

$$\mathrm{E}[|(\mathrm{Y}^{\varepsilon}(t)-\mathrm{Y}^0(t))\varepsilon^{-1}|^{2n}]\leqq(2\lambda)^{-n}\prod_{k=1}^{n}(2k+d-2) \tag{3.5}$$

$$\int_0^{\mathrm{T}}|\dot{\varphi}(s)-b(\varphi(s))|\,ds\leqq 2(\lambda^{-1}\mathrm{V}_{\mathrm{T}}(y))^{1/2}. \tag{3.6}$$

Under the conditions (A.0), (A.1) and (A.2), the limit of $\mathrm{V}_{\mathrm{T}}(y)$ as $\mathrm{T}\to\infty$ exists and equals to $\mathrm{V}(y)$ for each y (*cf.* M. I. Freidlin and A. D. Wentzell [4]). Therefore we only have to prove (3.5) and (3.6).

[Proof of (3.5).] We put $\mathrm{Z}^{\varepsilon}(t)=\mathrm{Y}^{\varepsilon}(t)-\mathrm{Y}^0(t)$. Since

$$\mathrm{Y}^{\varepsilon}(t)-\mathrm{Y}^0(t)=\varepsilon\mathrm{W}(t)+\int_0^{\mathrm{T}}[b(y^{\varepsilon}(s))-b(\mathrm{Y}^0(s))]\,ds,$$

by Ito's formula, we have

$$|Z^\varepsilon(t)|^{2n}-|Z^\varepsilon(s)|^{2n}$$

$$=2n\int_s^t |z^\varepsilon(u)|^{2(n-1)}\langle \bar{Z}^\varepsilon(u), \varepsilon\, dW(u)+\{b(Y^\varepsilon(u))-b(Y^0(u))\}\rangle du$$

$$+\varepsilon^2(nd+2n(n-1))\int_s^t |Z^\varepsilon(u)|^{2(n-1)}du$$

$$\leqq 2n\int_s^t |z^\varepsilon(u)|^{2(n-1)}\langle Z^\varepsilon(u), \varepsilon dW(u)\rangle - 2n\lambda\int_s^t |Z^\varepsilon(u)|^{2n}du$$

$$+\varepsilon^2(nd+2n(n-1))\int_s^t |Z^\varepsilon(u)|^{2(n-1)}du \quad [\text{from (A.2)}].$$

Hence

$$E[|(Y^\varepsilon(t)-Y^0(t))\varepsilon^{-1}|^{2n}]\leqq(2\lambda)^{-n}\prod_{k=1}^{n}(2k+d-2),$$

since for any s, t $(0\leqq s<t\leqq T)$,

$$E[|Z^\varepsilon(t)|^{2n}]-E[|Z^\varepsilon(s)|^{2n}]$$

$$\leqq -2n\lambda\int_s^t E[|Z^\varepsilon(u)|^{2n}]du+\varepsilon^2(nd+2n(n-1))$$

$$\times\int_s^t E[|Z^\varepsilon(u)|^{2(n-1)}]du$$

and from lemma 2 we have

$$E[|(Y^\varepsilon(t)-Y^0(t))\varepsilon^{-1}|^{2n}]\leqq(2\lambda)^{-n}(n!)^{-1}$$

$$\times(1-\exp(-2\lambda t))^n\prod_{k=1}^{n}(2k(k-1)+kd)$$

$$\leqq(2\lambda)^{-n}\prod_{k=1}^{n}(2k+d-2).$$

[Proof of (3.6).] Since we have

$$|\dot\varphi(t)-b(\varphi(t))|^2-|\dot\varphi(0)|^2$$

$$=\int_0^t 2\left\langle \dot\varphi(s)-b(\varphi(s)), \frac{d}{ds}(\dot\varphi(s)-b(\varphi(s)))\right\rangle ds$$

$$=-2\int_0^t \langle \dot\varphi(s)-b(\varphi(s)), \partial b(\varphi(s))^*(\dot\varphi(s)-b(\varphi(s))\rangle ds$$

$$\geqq 2\lambda\int_0^t |\dot\varphi(s)-b(\varphi(s))|^2 ds \quad [\text{from (A.2)}]$$

where we use Euler's equation for $\varphi(t)$ and from this for $\rho>0$,

$$\int_0^T |\dot\varphi(s)-b(\varphi(s))|^2(|\dot\varphi(s)-b(\varphi(s))|^2+\rho)^{-1/2}\,ds$$

$$\leqq \int_0^T |\dot\varphi(s)-b(\varphi(s))|^2\left(2\lambda\int_0^s |\dot\varphi(u)-b(\varphi(u))|^2\,du+\rho\right)^{-1/2} ds$$

$$=\left[\lambda^{-1}\left(2\lambda\int_0^s |\dot\varphi(u)-b(\varphi(u))|^2\,du+\rho\right)^{1/2}\right]_{s=0}^T$$

$$=\lambda^{-1}[(4\lambda V_T(y)+\rho)^{1/2}-\rho^{1/2}]\to\left(\frac{4}{\lambda}V_T(y)\right)^{1/2} \qquad (\rho\to 0),$$

$$\int_0^T |\dot\varphi(s)-b(\varphi(s))|\,ds\leqq \varliminf_{\rho\to 0}\int_0^T |\dot\varphi(s)-b(\varphi(s))|^2$$

$$\times(|\dot\varphi(s)-b(\varphi(s))|^2+\rho)^{-1/2}\,ds\leqq 2(\lambda^{-1}V_T(y))^{1/2}.$$

Q.E.D.

Now let us prove theorem 1.5.

Proof of theorem 1.5. – From lemma 5, theorems 1.2, 1.3 and proposition 1.4, by S. Watanabe's theory, for the transition probability density $p^\varepsilon(t,x,y)$ of $X^\varepsilon(t)$, there exist functions $p^i(T,0,y)$ $(i\geqq 0)$ and constants $C^i_{T,y}$ $(i\geqq 0)$ such that for all $n(=0,1,\ldots)$

$$(3.7)\quad \varepsilon^{-2(n+1)}\left|\varepsilon^d\exp(V_T(y)/\varepsilon^2)\,p^\varepsilon(T,0,y)-\sum_{i=0}^n \varepsilon^{2i}p^i(T,0,y)\right|<C^i_{T,y}$$

if $V_T(y)<4\lambda^3(d^6\|\partial^2 b\|_\infty^2)^{-1}$. Moreover for any α $(0<\alpha<1)$,

$$(3.8)\quad \left\{\varlimsup_{T\to\infty}\sup(C^i_{T,y};\,V_T(y)<\alpha[4\lambda^3(d^6\|\partial^2 b\|_\infty^2)^{-1}])<+\infty \quad (i=0,1,\ldots).\right.$$

Since $X^\varepsilon(t)$ is positively recurrent from proposition 1.1, the limits of $p^\varepsilon(T,0,y)$ as $T\to\infty$ exist for each $\varepsilon>0$ and $y\in\mathscr{R}^d$ (*cf.* Has'minskii [5]) and from this the limits of $p^i(T,0,y)$ as $T\to\infty$ exist for all $i(=0,1,\ldots)$ and y for which $V(y)<4\lambda^3(d^6\|\partial^2 b\|_\infty^2)^{-1}$. In fact, for $n=0$, since

$$\varepsilon^{-2}\left|\varepsilon^d\exp(V_T(y)/\varepsilon^2)\,p^\varepsilon(T,0,y)-p^0(T,0,y)\right|<C^0_{T,y},$$

$$-\varepsilon^2\varlimsup_{T\to\infty}C^0_{T,y}+\varepsilon^d\exp(V(y)/\varepsilon^2)\,p^\varepsilon(y)\leqq\varliminf_{T\to\infty}p^0(T,0,y)$$

$$\leqq\varlimsup_{T\to\infty}p^0(T,0,y)\leqq\varepsilon^2\varlimsup_{T\to\infty}C^0_{T,y}+\varepsilon^d\exp(V(y)/\varepsilon^2)\,p^\varepsilon(y).$$

Therefore

$$(3.9)\quad \lim_{\varepsilon \to 0} \varepsilon^d \exp(V(y)/\varepsilon^2)\, p^\varepsilon(y) = \underline{\lim}_{T \to \infty} p^0(T, 0, y) = \overline{\lim}_{T \to \infty} p^0(T, 0, y).$$

In the same way we can prove inductively that the limits of $p^i(T, 0, y)$ as $T \to \infty$ exist for all $i (=0, 1, \ldots)$.

Q.E.D.

At last we prove corollary 1.6.

Proof of corollary 1.6. — Let $X_1^\varepsilon(t)$, $0 \leqq t$ be the solution of the following stochastic differential equation:

$$(3.10)\quad \begin{cases} dX_1^\varepsilon(t) = b_1(X_1^\varepsilon(t))\, dt + \varepsilon\, dW(t) \\ X_1^\varepsilon(0) = x_1 \qquad (x_1 \in \mathscr{R}^d) \end{cases}$$

where $b_1(x)$ satisfies the conditions (A.0), (A.1) and (A.2) and $b_1(x) = b(x)$ if $|x| < r$ for some $r > 0$. Then from theorem 3 in M. V. Day [3], we have the following:

$$(3.11)\quad \overline{\lim}_{\varepsilon \to 0} \sup_{y \in \mathscr{K}} (\varepsilon \log |p^\varepsilon(x) - p_1^\varepsilon(x)|) \leqq -\min(V(x); |x| = r)$$

for any compact subset $\mathscr{K}$ of the set $(x; |x| < r)$ where we denote by $p_1^\varepsilon(x)$ the invariant density of $X_1^\varepsilon(t)$ and from theorem 1.5, if r is sufficiently small, (1.7) holds for $p^\varepsilon(x) = p_1^\varepsilon(x)$ uniformly for $|x| < r$. Therefore for any $\alpha < \min(V(x); |x| = r)$, (1.7) also holds for $p^\varepsilon(x)$ uniformly for x of the set $\{y; V(y) \leqq \alpha\}$, since

$$\begin{aligned} \varepsilon^{-2(n+1)} \left| \varepsilon^d \exp(V(y)/\varepsilon^2)\, p^\varepsilon(y) - \sum_{i=0}^{n} \varepsilon^{2i} p^i(y) \right| & \\ \leqq \varepsilon^{-2(n+1)} \left| \varepsilon^d \exp(V(y)/\varepsilon^2) (p^\varepsilon(y) - p_1^\varepsilon(y)) \right| & \\ + \varepsilon^{-2(n+1)} \left| \varepsilon^d \exp(V(y)/\varepsilon^2)\, p_1^\varepsilon(y) - \sum_{i=0}^{n} \varepsilon^{2i} p^i(y) \right| & \end{aligned}$$

and the first term is bounded by

$$\varepsilon^{d-2(n+1)} \exp\left(\frac{\alpha - \min(V(x); |x| = r)}{2\varepsilon^2} \right)$$

for sufficiently small ε, uniformly for $x \in \{y; V(y) \leqq \alpha\}$.

Q.E.D.

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