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M. CRANSTON
Y. LE JAN

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On the noncoalescence of a two point Brownian motion reflecting on a circle (*)

by

M. CRANSTON
University of Rochester

and

Y. LE JAN
C.N.R.S., Paris

ABSTRACT. — We consider two particles driven by the same Brownian motion while in the interior of the unit disc in the complex plane and normally reflecting off the boundary. We show that with probability one the particles do not meet in finite time though the distance between them decreases to zero. They also do not meet almost surely when moving in the exterior of the disc with normal reflection off the boundary.

Key words : Skorohod equation, noncoalescence.

RÉSUMÉ. — On considère deux particules soumises au même mouvement brownien et réfléchies sur un cercle. Lorsque les particules sont à l'intérieur du cercle, leur distance mutuelle tend vers 0 mais les deux particules ne se rejoignent pas en temps fini.

It is well known (*see*, for example, McKeans'book [2]) that one may construct a strong solution to the Skorohod equation on

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$D=\{z:|z|\leqq 1\}\subseteq\mathbb{C}$

$$dX_t=dB_t-X_t\,d\varphi_t, \qquad X_0=x \tag{1}$$

Here B_t is Brownian motion on $\mathbb{R}^2$, φ_t an increasing process increasing only when X_t is on ∂D (the local time). Actually it has been shown by Tanaka [3] and Lions-Sznitman [1], one can construct a path-by-path solution to this equation.

Let x_1, x_2 be two different points in D, and X_t^1, X_t^2 the associated path-by-path solutions of (1) with $X_0^i=x_i$, $i=1,2$. Let φ^1 and φ^2 be the associated local times.

The processes X_t^1 and X_t^2 are driven by the same Brownian motion while they are both interior to D. In particular, the distance between X_t^1 and X_t^2 remains constant when neither of φ_t^1 and φ_t^2 are charging. If one considers such a two point process on a one dimensional interval or a square, one will see that the two points will meet and coalesce after a finite amount of time. In a *Ph. D. thesis* by Weerasinghe, it was shown that the distance $z_t=\|X_t^1-X_t^2\|/2$ goes to zero at least exponentially fast but whether the points meet in a finite time was left unanswered.

This question has been raised by Steven Orey. The answer is that $T=\inf\{t>0:z_t=0\}$ is a. s. infinite. We also consider the behavior of z_t when X_t^1 and X_t^2 are reflecting in the exterior of D.

1. The interior problem

The proof begins by reducing the number of dimensions as follows. Set $m_t=\frac{1}{2}(X_t^1+X_t^2)$ and for $t<T$, $i_t=\frac{X_t^2-X_t^1}{2z_t}$. Let $j_t=i_t^{\perp}$ where for $(x, y)\in\mathbb{R}^2$, $(x, y)^{\perp}=(y, -x)$. Notice then that $X_t^2=m_t+z_t i_t$, $X_t^1=m_t-z_t i_t$.

Finally define $x_t=\langle m_t, j_t\rangle$, $y_t=\langle m_t, i_t\rangle$, $\varphi_t=\varphi_t^1+\varphi_t^2$. Then the equations for dz_t, dx_t and dy_t are easily computed.

Since

$$z_t^2=\left\langle\frac{X_t^2-X_t^1}{2},\frac{X_t^2-X_t^1}{2}\right\rangle,$$

$$\begin{aligned}2z_t\,dz_t&=\left\langle\frac{X_t^2-X_t^1}{2}, dX_t^2-dX_t^1\right\rangle\\ &=z_t\langle i_t, -X_t^2\,d\varphi_t^2+X_t^1\,d\varphi_t^1\rangle\\ &=z_t\langle i_t, -(m_t+z_t i_t)\,d\varphi_t^2+(m_t-z_t i_t)\,d\varphi_t^1\rangle\\ &=z_t[-y_t\,d\varphi_t^2-z_t\,d\varphi_t^2+y_t\,d\varphi_t^1-z_t\,d\varphi_t^1]\\ &=-z_t[|y_t|+z_t]\,d\varphi_t.\end{aligned}$$

Indeed, $y_t \geqq 0$ if $X_t^2 \in \partial D$ and $y_t \leqq 0$ if $X_t^1 \in \partial D$ since $y_t = \langle m_t, i_t \rangle = \frac{1}{2 z_t} [\| X_t^2 \|^2 - \| X_t^1 \|^2]$. Thus, when $z_t > 0$,

$$dz_t = -\frac{1}{2} [| y_t | + z_t] \, d\varphi_t.$$

Since $\| i_t \| = 1$, di_t is parallel to j_t. Observe that

$$di_t = \frac{1}{2 z_t} (dX_t^2 - dX_t^1) - \frac{1}{z_t} i_t \, dz_t$$

and on taking the inner product with j_t

$$\begin{aligned}
\langle j_t, di_t \rangle &= \frac{1}{2 z_t} \langle dX_t^2 - dX_t^1, j_t \rangle \\
&= \frac{1}{2 z_t} \langle -X_t^2 \, d\varphi_t^2 + X_t^1 \, d\varphi_t^1, j_t \rangle \\
&= \frac{1}{2 z_t} \langle -(m_t + z_t i_t) \, d\varphi_t^2 + (m_t - z_t i_t) \, d\varphi_t^1, j_t \rangle \\
&= \frac{1}{2 z_t} [-x_t \, d\varphi_t^2 + x_t \, d\varphi_t^1] \\
&= -\frac{x_t}{2 z_t} (d\varphi_t^2 - d\varphi_t^1).
\end{aligned}$$

Thus $di_t = -j_t \frac{x_t}{2 z_t} (d\varphi_t^2 - d\varphi_t^1)$ and consequently $dj_t = i_t \frac{x_t}{2 z_t} (d\varphi_t^2 - d\varphi_t^1)$. Next

$$\begin{aligned}
dx_t &= \langle j_t, dm_t \rangle + \langle dj_t, m_t \rangle \\
&= \langle j_t, dB_t \rangle + \frac{1}{2} \langle j_t, -X_t^2 \, d\varphi_t^2 - X_t^1 \, d\varphi_t^1 \rangle + \langle i_t, m_t \rangle \frac{x_t}{2 z_t} (d\varphi_t^2 - d\varphi_t^1) \\
&= dW_t^1 + \frac{1}{2} (j_t, -(m_t + z_t i_t) \, d\varphi_t^2 - (m_t - z_t i_t) \, d\varphi_t^1 + \frac{y_t x_t}{2 z_t} (d\varphi_t^2 - d\varphi_t^1) \\
&= dW_t^1 + \frac{1}{2} [-\langle j_t, m_t \rangle \, d\varphi_t^2 - \langle j_t, m_t \rangle \, d\varphi_t^1] + \frac{| y_t | x_t}{2 z_t} d\varphi_t \\
&= dW_t^1 + \left(\frac{x_t | y_t |}{2 z_t} - \frac{x_t}{2} \right) d\varphi_t
\end{aligned}$$

Lastly,

$$
\begin{aligned}
dy_t &= d\langle i_t, m_t\rangle \\
&= \langle dm_t, i_t\rangle + \langle m_t, di_t\rangle \\
&= \langle dB_t, i_t\rangle + \langle -\frac{X_t^1}{2} d\varphi_t^1 - \frac{X_t^2}{2} d\varphi_t^2, i_t\rangle + \langle m_t, -j_t\rangle \frac{x_t}{2z_t}(d\varphi_t^2 - d\varphi_t^1) \\
&= dW_t^2 - \frac{1}{2}\langle (m_t - z_t i_t)\, d\varphi_t^1 + (m_t + z_t i_t)\, d\varphi_t^2, i_t\rangle - \frac{x_t^2}{2z_t}(d\varphi_t^2 - d\varphi_t^1) \\
&= dW_t^2 - \frac{1}{2}\langle m_t, i_t\rangle\, d\varphi_t - \frac{z_t}{2}(d\varphi_t^2 - d\varphi_t^1) - \frac{x_t^2}{2z_t}(d\varphi_t^2 - d\varphi_t^1) \\
&= dW_t^2 - \frac{y_t}{2} d\varphi_t - \left(\frac{x_t^2}{2z_t} + \frac{z_t}{2}\right)(d\varphi_t^2 - d\varphi_t^1).
\end{aligned}
$$

In the expressions for dx_t and dy_t, dW_t^1 and dW_t^2 are the differentials of independent Brownian motions.

The asymptotic growth of the $\varphi_t^i, i=1, 2$ is well-known. In general it is determined by solving the Neumann problem $\Delta u = 1$ in D, $\frac{\partial u}{\partial n} = c$ on ∂D, and noticing that Green's formula $\int_D \Delta u \, dm = -\int_{\partial D} \frac{\partial u}{\partial n} d\sigma$ forces the choice of $c = -\frac{m(D)}{\sigma(\partial D)}$. Applying Itô's formula for $i=1$ or 2,

$$
\begin{aligned}
u(X_t^i) &= u(x_i) + \int_0^t \nabla u(X_s^i)\, dB_s + \frac{1}{2}\int_0^t \Delta u(X_s^i)\, ds + \int_0^t \frac{\partial u}{\partial n}(X_s^i)\, d\varphi_s^i \\
&= u(x_i) + \int_0^t \nabla u(X_s^i)\, dB_s + \frac{t}{2} - \frac{m(D)}{\sigma(\partial D)} \varphi_t^i,
\end{aligned}
$$

on dividing by t, sending t to infinity and summing on i, $\lim_{t \to \infty} \frac{\varphi_t}{t} = \frac{\sigma(\partial D)}{m(D)}$. In the present case $u(r, \theta) = \frac{r^2}{4}$.

We wish to observe that the formula for dz_t implies $z_t = e^{-(1/2)\varphi_t}\left(z_0 - \frac{1}{2}\int_0^t |y_s| e^{(1/2)\varphi_s} d\varphi_s\right)$ for $t < T$. Thus $z_t \leqq z_0 e^{-(1/2)\varphi_t}$ and together with the above mentioned asymptotics for φ_t, z_t tends to zero at least exponentially fast, which was realized by Weerasinghe.

The process $\xi_t = (x_t, y_t, z_t)$ is a reflecting diffusion in $\mathscr{C} = \{(x, y, z) : z \geqq 0, x^2 + (|y| + z)^2 \leqq 1\}$. In the interior of $\mathscr{C}$, ξ_t is a two-dimensional Brownian motion in the first two components while the third z_t component is frozen until ξ_t hits $\partial\mathscr{C}$ when z_t again decreases.

Introduce the notations $L_t = x_t^2 + (y_t - 1)^2$, $N_t = x_t^2 + (y_t + 1)^2$, and $u(x_t, y_t) = u_t = \log(L_t N_t)$ for $t < T$.

LEMMA 1. — *On* $\{T < \infty\}$, $\inf_{t<T} u_t$ *is almost surely finite.*

Proof. — First observe $u(x, y)$ is harmonic on the horizontal $(z = c)$ sections of $\mathscr{C}$ so the second order terms in the Itô expansion for du_t vanish. For $t < T < \infty$, we have

$$du_t = 1_{\partial \mathscr{C}}(\xi_t)\left[\frac{2x_t\,dx_t + 2y_t\,dy_t - 2\,dy_t}{L_t} + \frac{2x_t\,dx_t + 2y_t\,dy_t + 2\,dy_t}{N_t}\right] + dM_t,$$

where M_t is a locally square integrable continuous local martingale.

Hence,

$$du_t = \sum_{i=1,2}\left[\frac{x_t^2 - y_t^2 - z_t|y_t| + y_t + (-1)^i(z_t + x_t^2/z_t)}{L_t}\right.$$
$$\left. + \frac{-x_t^2 - y_t^2 - z_t|y_t| - y_t - (-1)^i(z_t + (x_t^2/z_t)}{N_t}\right] d\varphi_t^i + dM_t$$

Note that $\dfrac{1}{L_t} - \dfrac{1}{N_t} = \dfrac{4y_t}{L_t N_t}$. Therefore

$$du_t = dM_t + \left[\frac{4(y_t^2 + z_t|y_t| + x_t^2|y_t|/z_t)}{L_t N_t}\right.$$
$$\left. - (x_t^2 + y_t^2 + z_t|y_t|)\left(\frac{1}{L_t} + \frac{1}{N_t}\right)\right] d\varphi_t$$

or $du_t = dM_t + (y_t^2 + z_t|y_t|)(L_t N_t)^{-1}(4 - (L_t + N_t))\,d\varphi_t$

$$+ x_t^2 (L_t N_t)^{-1}\left(4\frac{|y_t|}{z_t} - (L_t + N_t)\right) d\varphi_t.$$

The coefficient of the first $d\varphi_t$ term is positive. While the coefficient of the second $d\varphi_t$ term is not necessarily positive, it does hold that

$$x_t^2 (L_t N_t)^{-1}\left(4\frac{|y_t|}{z_t} - (L_t + N_t)\right)$$
$$\geqq -x_t^2 (L_t N_t)^{-1}(L_t + N_t) \geqq -\frac{x_t^2}{L_t} - \frac{x_t^2}{N_t} \geqq -2$$

since $\dfrac{x_t^2}{N_t}$ and $\dfrac{x_t^2}{L_t}$ are bounded by one. Since $\varphi_T \simeq \dfrac{\sigma(\partial D)}{m(D)} T$, $\inf_{t<T} u_t = -\infty$ can only occur in the case $T < \infty$ when $\inf_{t<T} M_t = -\infty$.

The last can only occur when $\sup_{t<T} M_t = \infty$. However, $u_t = \log L_t N_t \leqq \ln 4$ and the bounded variation term in the Itô expansion is bounded below

by $-2\varphi_T$. This makes $\sup_{t<T} M_t = \infty$ impossible and so $\inf_{t<T} M_t > -\infty$. This proves the Lemma. ■

THEOREM 1. — *The time* $T=\infty$ *a. s.*

Proof. — On $T<\infty$, $2\frac{dz_t}{z_t} = -\left(\frac{|y_t|}{z_t}+1\right)d\varphi_t$ and computing the stochastic differential for $\log(z_t^2 L_t N_t)$ one has

$$\begin{aligned}\frac{2\,dz_t}{z_t}+du_t = dM_t &\\ &+\left[-(x_t^2+y_t^2+z_t|y_t|)\left(\frac{1}{L_t}+\frac{1}{N_t}\right)+\frac{4(y_t^2+z_t|y_t|)}{L_t N_t}-1\right]d\varphi_t\\ &+\frac{|y_t|}{z_t}\,\frac{(4x_t^2-L_t N_t)}{L_t N_t}d\varphi_t \equiv C_t\,d\varphi_t + dM_t\end{aligned}$$

with the same dM_t as in the previous Lemma.

However,

$$\begin{aligned}4x_t^2 - L_t N_t &= 4x_t^2-(x_t^2+y_t^2+1)^2+4y_t^2\\ &= -(x_t^2+y_t^2-1)^2\\ &= -(2|y_t|+z_t)^2 z_t^2\end{aligned}$$

where we have used the fact that $x_t^2+(|y_t|+z_t)^2=1$ on the set of times charged by $d\varphi_t$. Thus,

$$\begin{aligned}C_t = -(x_t^2+y_t^2+z_t|y_t|)\left(\frac{1}{L_t}+\frac{1}{N_t}\right)+\frac{4(y_t^2+z_t|y_t|)}{L_t N_t}-1&\\ -(L_t N_t)^{-1}|y_t|\,z_t(2|y_t|+z_t)^2.&\end{aligned}$$

By Lemma 1, when $T(\omega)<\infty$ there is a $c_0(\omega)$ such that $(L_t N_t)^{-1}\leqq c_0(\omega)$ for all $t<T(\omega)$. This implies the existence of another constant $c_1(\omega)$ such that $C_t(\omega)\geqq -c_1(\omega)$ for $t<T(\omega)$ again when $T(\omega)<\infty$. Then for $t<T<\infty$,

$$2\log\frac{z_t}{z_0} = M_t + \int_0^t C_s\,d\varphi_s - \log\frac{L_t N_t}{L_0 N_0} \geqq M_t - c_1\,\varphi_1 - \log\frac{L_t N_t}{L_0 N_0}.$$

Now $T<\infty$ holding on a set of positive probability would by Lemma 1 imply $M_{T-}=-\infty$ which is impossible. ■

2. The exterior problem

Now the two particles satisfy the Skorohod equation $dX_t^i = dB_t + X_t^i d\varphi_t^i$, $i=1, 2$ and $X_0^1 = x_1 \neq x_2 = X_0^2$. As before $z_t = \left\| \frac{X_t^2 - X_t^1}{2} \right\|$ and $T = \inf\{t > 0 : z_t = 0\}$. Set $m_t = \frac{X_t^1 + X_t^2}{2}$ and for $t < T$, $i_t = \frac{X_t^1 - X_t^1}{2 z_t}$, $j_t = i_t^\perp$, $x_t = \langle m_t, j_t \rangle$, $y_t = \langle m_t, i_t \rangle$ and $\varphi_t = \varphi_t^1 + \varphi_t^2$. Then $\xi_t = (x_t, y_t, z_t)$ is a diffusion on $\mathscr{D} = \{(x, y, z) : z \geqq 0, x^2 + (|y| - z)^2 \geqq 1\}$. The stochastic equations of motion are

$$dx_t = dW_t^1 + \left(\frac{x_t |y_t|}{2 z_t} + \frac{x_t}{2} \right) d\varphi_t$$

$$dy_t = dW_t^2 + \left(-\frac{x_t^2}{2 z_t} + \frac{1}{2}(|y_t| - z_t) \right)(d\varphi_t^1 - d\varphi_t^2)$$

$$dz_t = \frac{1}{2}(z_t - |y_t|)\, d\varphi_t$$

with dW_t^1, dW_t^2 the increments of orthogonal standard Brownian motions. Also, $z_t = e^{(1/2)\varphi_t}\left(z_0 - \frac{1}{2} \int_0^t |y_s| e^{-(1/2)\varphi_s} d\varphi_s \right)$ holds for $t < T$.

Now set $L_t = x_t^2 + (y_t - 1)^2$, $N_t = x_t^2 + (y_t + 1)^2$, $u_t = \log L_t N_t = u(x_t, y_t)$. For $z_t > 2$ neither L_t nor N_t will be zero since this must happen outside the support of $d\varphi_t$ and the motion (x_t, y_t) is a Brownian motion there which will not hit the points (0, 1) or (0, −1). To avoid a possible problem at $(x, y, z) = (0, 1, 2)$ or $(0, -1, 2)$ introduce for $r < \frac{1}{4}$ fixed the stopping time

$$T_{r/2} = \inf\left\{ t > 0 : x_t^2 + (|y_t| - 1)^2 + (z_t - 2)^2 \leqq \frac{r^2}{4} \right\}.$$

Then for $t < T_{r/2}$, (x_t, y_t) will remain in the region where $u(x, y)$ is harmonic. Thus the second order terms in the expansion for du_t when $t < T_{r/2}$ vanish.

LEMMA 1. — *On* $\{T_{r/2} \wedge T < \infty\}$, $\inf\limits_{t < T_{r/2} \wedge T} u_t$ *is almost surely finite.*

Proof. — For $t < T_{r/2} \wedge T$ one has

$$du_t = L_t^{-1}(2x_t dx_t + 2y_t dy_t - 2dy_t) + N_t^{-1}(2x_t dx_t + 2y_t dy_t + 2dy_t)$$

$$= dM_t + (N_t^{-1} + L_t^{-1})\left(x_t^2 + \frac{x_t^2|y_t|}{z_t} - \frac{x_t^2|y_t|}{z_t} + |y_t|(|y_t| - z_t)\right)d\varphi_t$$

$$-(L_t^{-1} - N_t^{-1})\left(\frac{x_t^2}{z_t} + z_t\right)(d\varphi_t^2 - d\varphi_t^1) - (L_t^{-1} - N_t^{-1})y_t d\varphi_t$$

$$= dM_t + (L_t N_t)^{-1}\left[x_t^2\left(L_t + N_t + 4\frac{|y_t|}{z_t}\right)\right.$$

$$\left. + |y_t|(L_t + N_t - 4)(|y_t| - z_t)\right] d\varphi_t.$$

Convergence to minus infinity can only arise from the bounded variation term. Given $r>0$ as above, select ε with $\frac{r}{8} > \varepsilon > 0$ so that also $2 - \frac{r}{8} < 1 - \varepsilon + \sqrt{1-\varepsilon^2}$. We argue that u_t can not converge to minus infinity in finite time before $T_{r/2}$. In the expansion for du_t the term $(L_t N_t)^{-1} x_t^2 \left(L_t + N_t + 4\frac{|y_t|}{z_t}\right)$ is positive and therefore can not cause such behavior. Suppose $L_t N_t \geqq \varepsilon^4$. If $|y_t| > z_t$ the term $(L_t N_t)^{-1}|y_t|(L_t + N_t - 4)(|y_t| - z_t)$ is positive and no problem arises. If $|y_t| \leqq z_t$ then on the support of $d\varphi_t$, $||y_t| - z_t| < 1$, $|y_t| < z_t < z_0 e^{(1/2)\varphi_t}$, $(L_t N_t)^{-1} < \varepsilon^{-4}$ and $(L_t + N_t - 4) = 2z_t(2|y_t| - z_t) < 4e^{\varphi_t}$ so no explosion to minus infinity may occur in finite time for $t < T_{r/2}$. When $L_t N_t < \varepsilon^4$ either $L_t < \varepsilon^2$ or $N_t < \varepsilon^2$ and we suppose the former holds.

Thus $x_t^2 + (y_t - 1)^2 < \varepsilon^2$ so $|x_t| < \varepsilon$ and $1 - \varepsilon < y_t < 1 + \varepsilon$. In addition, $x_t^2 + |(|y_t| - z_t)^2 = 1$ holds on the support of $d\varphi_t$ so either $0 < z_t < 1 + \varepsilon - \sqrt{1-\varepsilon^2}$, i.e. z_t is small or $1 - \varepsilon + \sqrt{1-\varepsilon^2} < z_t < 2 + \varepsilon$. In the latter case (x_t, y_t, z_t) is within $\frac{r}{2}$ of (0, 1, 2) since $|x_t| < \varepsilon < \frac{r}{8}$, $|y_t - 1| < \varepsilon < \frac{r}{8}$ and $|z_t - 2| < \frac{r}{8}$ by the choice of ε. This contradicts $t < T_{r/2}$ so $z_t < 1 + \varepsilon - \sqrt{1-\varepsilon^2}$ holds. Consequently, $|y_t| > z_t$ and both the $d\varphi_t$ coefficients are positive implying no convergence to minus infinity before $T_{r/2}$. ■

Next we prove the two point process in the exterior of the disc does not coalesce.

THEOREM 2. — *For the two point motion reflecting in the complement of the disc,* $T = \infty$ *a. s.*

Proof. — Notice that

$$4x_t^2 - L_t N_t = 4x_t^2 - (x_t^2 + y_t^2 + 1)^2 + 4y_t^2 = -(x_t^2 + y_t^2 - 1)^2$$

and $x_t^2 + (|y_t| - z_t)^2 = 1$ on the support of $d\varphi_t$ so $x_t^2 + y_t^2 - 1 = 2|y_t| z_t - z_t^2$. Consequently, $4x_t^2 - L_t N_t = -z_t^2 (2|y_t| - z_t)^2$. This leads to, for $t < T_{r/2} \wedge T$,

$$du_t + 2\frac{dz_t}{z_t} = dM_t + (x_t^2 + y_t^2 - |y_t| z_t)(L_t^{-1} + N_t^{-1})\, d\varphi_t \\ - 4(L_t N_t)^{-1}(y_t^2 - |y_t| z_t)\, d\varphi_t - d\varphi_t \\ - (L_t N_t)^{-1} |y_t| z_t (2|y_t| - z_t)^2\, d\varphi_t.$$

By the same argument as in the previous case we can conclude that $T > T_{r/2}$ a. s. On $\{T_{r/2} < \infty\}$ define $T_1 = T_r \circ \theta_{T_{r/2}}$ where θ is the shift operator and

$$T_r = \inf\{t > 0 : x_t^2 + (|y_t| - 1)^2 + (z_t - 2)^2) = r^2\}.$$

Starting at ξ_{T_1}, apply the strong Markov property and the same argument to show $T > T_2 = T_{r/2} \circ \theta_{T_1}$.

Obviously z_t cannot hit zero on the interval $[T_{r/2}, T_1]$. Continuing in this manner on the various pieces of the path, $T = \infty$ a. s. follows. ■

Most natural questions remain open in this area: Find a noncoalescence condition for general piecewise smooth domains. Determine what kind of instability is created by concavity. Our work should be viewed as an invitation to go further.

Remark. — There cannot exist a flow of homeomorphism Φ_t such that $\Phi_t(x)$ would be equal to x_t a. s. Indeed, an homeomorphism of D or of $\bar{D}^c$ preserves the boundary ∂D.

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