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T. KOMOROWSKI

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Asymptotic periodicity of some stochastically perturbed dynamical systems

by

T. KOMOROWSKI

Michigan State University East Lansing, MI 48824, U.S.A

ABSTRACT. – In the paper we investigate asymptotic behavior of a Markov chain given by a difference stochastic equation $X_{n+1} = S(X_n) + \xi_n$. We prove asymptotic periodicity of the Markov chain under the following assumptions: (H1) $S: V \to V$ is a Borel measurable transformation defined on a cone $V \subseteq \mathbb{R}^d$, bounded on bounded subsets of V, (H2) there is a norm $|.|$ defined in $\mathbb{R}^d$ such that $\liminf_{|x| \to +\infty} [|x| - |S(x)|] = m > 0$ and (H3) $\{\xi_n\}_{n \geq 0}$ is a sequence of i.i.d., random variables such that the distribuant of ξ_0 has a density $g: V \to \mathbb{R}_+$ having the first absolute moment $\int_V |x| g(x)\, dx < m$. Furthermore sufficient conditions for asymptotic stability are given. Obtained theorems are applied to prove asymptotic stability of a model of cell cycle given in [11] by J. Tyrcha.

Key words : Markov Chain, Markov Operator, Asymptotic Periodicity, Asymptotic Stability.

RÉSUMÉ. – Dans ce texte, j'étudie la conduite asymptotique d'une chaîne de Markov donnée par l'équation $X_{n+1} = S(X_n) + \xi_n$. Je prouve la périodicité asymptotique de la chaîne de Markov avec les hypothèses suivantes: (H1) $S: V \to V$ est une transformation mesurable de Borel définie sur un cône $V \subseteq \mathbb{R}^d$, bornée sur les sous-ensembles bornés de V, (H2) il existe une norme $|.|$ définie sur $\mathbb{R}^d$ telle que

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$\liminf_{|x| \to +\infty} [|x| - |S(x)|] = m > 0$ and (H3) $\{\xi_n\}_{n \geqq 0}$ est une suite de variables aléatoires indépendantes identiquement distribuées avec une loi de densité $g: V \to \mathbb{R}_+$ ayant le premier moment absolu $\int_V |x| g(x) \, dx < m$ des conditions de stabilité asymptotique sont données. Les théorèmes obtenus sont appliqués pour prouver la stabilité asymptotique d'un modèle de cycle d'une cellule, donné dans [11] par J. Tyrcha.

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1. INTRODUCTION

The subject of interest of this article is a difference equation with stochastic perturbation of the form

$$X_{n+1} = S(X_n) + \xi_n, \qquad n \geqq 0. \tag{1.1}$$

Here S is a transformation of a certain cone V in $\mathbb{R}^d$ into itself and $\{\xi_n\}_{n \geqq 0}$ is so-called "white noise" *i.e.* a sequence of independent identically distributed (i.i.d.) in the cone random variables. The system defined by equation (1.1) generates a Markov chain $\{X_n\}_{n \geqq 0}$ provided that X_0 and $\{\xi_n\}_{n \geqq 0}$ are independent. The problem of asymptotic behavior of Markov chains was extensively investigated by many authors *see* [1], [2], [7], [9] for reference. The main tool for finding some stability properties of (1.1) is the method of Lyapunov function. Generally saying this method consists in constructing a positive function $\mathscr{L}: V \to \mathbb{R}_+$, called a Lyapunov function, such that

(i) $\lim_{|x| \to +\infty} \mathscr{L}(x) = +\infty$

(ii) $\{\mathscr{L}(X_n)\}_{n \geqq 1}$ is a supermartingale *i.e.* in particular it decreases in average sense.

Whenever we can find such a function and verify that

$$\sup_{n \in \mathbb{N}} E \mathscr{L}(X_n) < +\infty \tag{1.2}$$

then $\{X_n\}_{n \geqq 1}$ must be recurrent to a sufficiently large ball. Under some additional hypotheses about the chain as for example that $\{X_n\}_{n \geqq 1}$ is Harris (*see* [7]) we can conclude that $\{X_n\}_{n \geqq 1}$ is asymptotically periodic in the sense of [7]. For more details about the method of Lyapunov function *see* [2] and [3]. A typical assumption one may admit to show the existence of Lyapunov function for the above system is some growth condition imposed on S. One of the simplest possible may be

$$|S(x)| \leqq \alpha |x| + \beta, \qquad \text{for} \quad x \in V$$

where $0<\alpha<1$ and $\beta \in \mathbb{R}$. Then the Lyapunov function $\mathscr{L}(x)=|x|$ will satisfy the condition of theorem 3, p. 86 in [2]. By adding some assumption about the white noise, for example one may suppose that the distribuant of ξ_0 has nonvanishing Radon derivative with respect to the Lebesgue measure, asymptotic periodicity in the sense of [7] can be established.

In our paper we investigate the case $\alpha=1$. We prove asymptotic periodicity of the Markov chain given by system (1.1) under the following assumptions:

(H1) $S: V \to V$ is a Borel measurable transformation defined on a cone $V \subseteq \mathbb{R}^d$, bounded on any bounded subset of V.

(H2) There is a norm $|.|$ defined in $\mathbb{R}^d$ such that

$$\liminf_{|x| \to +\infty} [|x| - |S(x)|] = m > 0$$

(H3) $\{\xi_n\}_{n \geqq 0}$ is a sequence of i.i.d. random variables such that the distribuant of ξ_0 has a density $g: V \to \mathbb{R}_+$ satisfying

$$\int_V |x| g(x)\, dx < m$$

In this case $\mathscr{L}(x)=|x|$, $x \in V$ does not have to satisfy (1.2) any more. However by studying the sequence of averages $\frac{1}{N}\sum_{n=0}^{N} E\mathscr{L}(X_n)$ we can prove that

$$\liminf_{N \to +\infty} \frac{1}{N} \sum_{n=0}^{N} \int_{[|X_n| \leqq A]} \mathscr{L}(X_n)\, dP > 0$$

if A is sufficiently large. This proves that $\{X_k\}_{k \geqq 0}$ is recurrent to a ball with a sufficiently large radius. Combining this result with the fact that transitions of probability for $\{X_n\}_{n \geqq 0}$ are absolutely continuous with respect to the Lebesgue measure we can show that $\{X_n\}_{n \geqq 0}$ is positive Harris recurrent and therefore periodic. This is the contents of our main result stated in theorem 4.3. In section 5 we apply this theorem in order to get some asymptotic stability results about system (1.1) under various conditions concerning the transformation S and density g. Some of these results are related to previous results obtained by other authors *see* [5], [6] and numerous papers due to H. J. Kushner. For instance theorem 5.2 corresponds to proposition 5.1 in [5] which concerns a more general system given by:

$$X_{n+1} = S(X_n, \xi_n), \qquad n \geqq 0. \tag{1.3}$$

It is proven there that (1.3) is asymptotically stable provided the following hold:

(i) S is C^1 and the deterministic system $X_{n+1}=S(X_n, 0)$ is freely evolving.

(ii) $\{\xi_n\}_{n\geqq 0}$ are i.i.d. random variables such that ξ_0 possesses a density g which is lower semicontinuous, $0\in \operatorname{supp} g$.

(iii) (1.3) is weakly stochastically controllable (*see* [5] for a definition).

In our case we obtain a similar result for (1.1), which, for simplicity, is stated on the real line. We do not have to assume that the system is weakly stochastically controllable. Moreover the hypotheses about lower semicontinuity of the density and differentiability of S can also be dropped.

The main difference between this paper and those mentioned above is that the method we used here does not require any topological type considerations. This yields the fact that we do not have to make assumptions about topological regularity of the transformation S and density g. Instead we use only some fairly general hypotheses about measurability of S and g to obtain asymptotic periodicity of the corresponding Markov chain.

2. NOTATIONS

Suppose that (X, Σ, m) is a σ-finite measure space. By $L^1(X)$ we denote the set of all real functions integrable with respect to m defined apart from a set of measure zero. All nonnegative elements $f\in L^1(X)$ satisfying

$$\int_X f\, dm = 1$$

are called densities. The set of all densities is denoted by $D(X)$. A linear operator $P: L^1(X)\to L^1(X)$ is called a Markov operator if $P(D(X))\subseteq D(X)$. A density f is said to be invariant for P if $Pf=f$.

Suppose that $V\subseteq \mathbb{R}^d$, V is called a cone if for any $x, y\in V$, λ, $\mu\geqq 0$ we have $\lambda x+\mu y\in V$. Since now all sets denoted by V will be cones, so we will not repeat this assumption in the sequel.

Assume that $S: V\to V$ is a transformation. Let $\{\xi_n\}_{n\geqq 0}$ be a sequence of independent random variables identically distributed in V with a density g having a finite first absolute moment $\int_V |x| g(x)\, dx < +\infty$. If X_0 is independent of all ξ_n, $n\geqq 0$ we can generate a sequence of random variables

$$X_{n+1}=S(X_n)+\xi_n, \qquad n\geqq 0$$

being a Markov chain. Its transition operator $P: L^1(V) \to L^1(V)$ is defined by the formula

$$Pf(x) = \int_V \bar{g}(x - S(y)) f(y)\, dy,$$

where $\bar{g}(x) = \begin{cases} 0, & x \in V \\ g(x), & x \in V \end{cases}$.

3. A FIXED POINT THEOREM

The theorem presented below resembles somewhat classical theorems concerning the existence of invariant densities for Markov operators given in [1], [10]. However, mainly for the sake of transparency, we formulate and prove it below.

THEOREM 3.1. – Suppose that (X, Σ, m) is a σ-finite measure space and $P: L^1(X) \to L^1(X)$ is a Markov operator. Assume that P satisfies the following two conditions

(C1) there is $f_0 \in D$, $A \subseteq X$ with $m[A] < +\infty$ such that

$$\limsup_{N \to +\infty} \frac{1}{N} \sum_{n=0}^{N-1} \int_A P^n f_0\, dm > 0. \tag{3.1}$$

(C2) for every $\varepsilon > 0$ there is $\delta > 0$ such that

$$\int_E P^n f_0\, dm < \varepsilon, \qquad n \in \mathbb{N},$$

if only $m[E] < \delta$.

Then there exists a density f_* such that

(i) $Pf_* = f_*$

(ii) $m[\operatorname{supp} f_* \cap A] > 0$.

Proof. – Consider a subspace $L \subseteq l^\infty$ spanned by two elements $\mathbf{u} = (1, 1, \ldots)$, $\mathbf{a} = (a_1, a_2, \ldots)$, where

$$a_k = \int_A P^k f_0\, dm$$

Define a linear functional $\Phi: L \to \mathbb{R}$ by

$$\Phi(s.\mathbf{u} + t.\mathbf{a}) = s + t.\limsup_{N \to +\infty} \frac{1}{N} . \sum_{n=0}^{N-1} a_n.$$

It is obvious that $\Phi(s\,\mathbf{u}+t\,\mathbf{a}) \leqq \limsup_{N \to +\infty} x_n$, provided $\{x_n\}_{n \geqq 0} = s\,\mathbf{u}+t\,\mathbf{a}$. According to the Hahn-Banach theorem Φ can be extended to the entire space l^∞ in such a way that the condition

$$\Phi(\{x_n\}_{n \geqq 0}) \leqq \limsup_{N \to +\infty} \frac{1}{N} \sum_{n=0}^{N-1} x_n$$

is preserved. From (3.1) we deduce

$$\Phi(a) > 0.$$

Setting $\varepsilon = \frac{1}{2}\Phi(\mathbf{a})$ and choosing δ according to condition (C2) we see that if $m[E] < \delta$ then

$$\Phi\left(\left\{\int_{E \cup (X \setminus A)} P^k f_0 \, dm\right\}_{k \geqq 0}\right) \leqq 1 - \frac{1}{2}\Phi(\mathbf{a}) < 1.$$

Using the arguments identical to those used in [10] we conclude our proof. □

4. ASYMPTOTIC PERIODICITY OF ITERATES

First we prove the following

THEOREM 4.1. – *Assume that* $g: V \to \mathbb{R}^+$ *is a density having the first absolute moment* $m = \int_V |x| g(x)\,dx$ *finite. Suppose that* S *satisfies the following conditions:*

(i) $S: V \to V$ *and it is bounded in any bounded subset of a cone* V.

(ii) $\liminf_{|x| \to +\infty} [|x| - |S(x)|] > m$.

Then a Markov operator $P: L'(V) \to L'(V)$ *given by*

$$P f(x) = \int_V \bar{g}(x - S(y)) f(y)\,dy \tag{4.1}$$

has an invariant density.

Remark. – A similar result was obtained by A. G. Pakes in [8]. However the author considered only the case of irreducible and aperiodic Markov chains with a discrete state space. □

Proof. – On the contrary suppose that P has no invariant density. Then choose $f_0 \in D(V)$ and consider a sequence of iterates $f_n = P^n f_0$.

Denote by:

$$M_N=\frac{1}{N+1}\sum_{n=0}^{N} f_n \tag{4.2}$$

$$V_n=\int_V |x|\, f_n(x)\,dx \tag{4.3}$$

$$W_N=\int_V |x|\, M_N(x)\,dx, \tag{4.4}$$

we have

$$\begin{aligned} V_{n+1} &= \int_V |x|\, f_{n+1}(x)\,dx \\ &= \int_V |x|\,dx\left\{\int_V \bar{g}(x-S(y))f_n(y)\,dy\right\} \\ &= \int_V f_n(y)\,dy\left\{\int_V \bar{g}(x-S(y))\,|x|\right\} \\ &\leqq \int_V f_n(y)\,dy\left\{\int_V |x-S(y)|\,\bar{g}(x-S(y))\,dx\right\} \\ &\quad + \int_V |S(y)|\, f_n(y)\,dy \leqq m-\int_V [|y|-|S(y)|]\, f_n(y)\,dy+V_n. \end{aligned} \tag{4.5}$$

From (4.5)

$$\begin{aligned} \sum_{n=0}^{N+1} V_n = \sum_{n=0}^{N} V_{n+1}+V_0 \leqq V_0+\sum_{n=0}^{N} V_n+(N+1)\,m \\ -\int_V [|y|-|S(y)|]\left(\sum_{n=0}^{N} f_n(y)\right)dy \end{aligned}$$

and

$$\begin{aligned} W_{N+1} \leqq \frac{N+2}{N+1}\, W_{N+1} \leqq W_N+\frac{V_0}{N+1} \\ +m-\int_V [|y|-|S(y)|]\, M_N(y)\,dy,\ N\geqq 1 \end{aligned} \tag{4.6}$$

From (ii) we can find such $A>0$ that for $x\in V$ and $|x|\geqq A$

$$|x|-|S(x)|\geqq\sigma>m.$$

For any $\varepsilon>0$ we can find $\delta>0$ such that

$$\int_E \bar{g}(x)\,dx<\varepsilon \text{ provided that } m[E]<\delta.$$

Then for any $f \in D(V)$ we have

$$\int_E Pf(x)\,dx = \int_E dx \int_V \bar{g}(x - S(y))\, f(y)\,dy = \int_V f(y)\,dy \int_E \bar{g}(x - S(y))\,dx < \varepsilon.$$

Hence P satisfies condition (C2). Using Theorem 3.1 we deduce that

$$\lim_{N \to +\infty} \int_{[|y| < A] \cap V} [|y| - S(y)|] M_N(y)\,dy = 0$$

and in consequence

$$\liminf_{N \to +\infty} \int_V [|y| - |S(y)|] M_N(y)\,dy$$

$$= \liminf_{N \to +\infty} \int_{[|y| \geqq A] \cap V} [|y| - |S(y)|] M_N(y)\,dy \geqq \sigma > m$$

Let $m < \rho < \sigma$. For a certain N_0 we have

$$\frac{V_0}{N+1} + m - \int_V [|y| - |S(y)|] M_N(y)\,dy \leqq m - \rho, \tag{4.7}$$

for $N \geqq N_0$. Compiling (4.6) and (4.7) we obtain

$$0 \leqq W_{N+1} \leqq W_N + m - \rho, \qquad N \geqq N_0.$$

The above condition cannot be met by any sequence $\{W_N\}_{N \geqq 1}$. This contradiction shows that P has to posses an invariant density. □

Suppose now that $\mathcal{M}$ is a set of all invariant densities of P. Let $f_0 \in D(V)$ be strictly positive in V. Set

$$a = \sup\left\{ \int_X f_0(x)\,dx : X = \operatorname{supp} f,\ f \in \mathcal{M} \right\}. \tag{4.8}$$

There exists such a $f_* \in \mathcal{M}$ that

$$\int_{X_*} f_*(x)\,dx = a, \qquad \text{where} \quad X_* = \operatorname{supp} f_*.$$

To see this one may apply a standard procedure which consists in taking a sequence of densities $\{f_n\}_{n \geqq 1} \subseteq \mathcal{M}$ whose supports S_n satisfy $\int_{S_n} f_0(x)\,dx \to a$. Writing $f_* = \sum_{n=1}^{+\infty} \frac{f_n}{2^n}$ we obtain a required density.

The following lemma will be essential for us in the sequel.

LEMMA 4.2. – For any density $f \in D(V)$ we have

$$\lim_{n \to +\infty} \int_{V \setminus X_*} P^n f\,dx = 0.$$

Proof. – First notice that for any f such that supp $f \subseteq X_*$ we have supp $Pf \subseteq X_*$ (*see* [12], p. 132, lemma 3.11). From the above we conclude that $U\,1_{V\setminus X_*} \leqq 1_{V\setminus X_*}$ where $U = P^*$. Hence we obtain

$$\int_{V\setminus X_*} P^{n+1} f\,dx \leqq \int_{V\setminus X_*} P^n f\,dx.$$

Suppose that

$$\lim_{n \to +\infty} \int_{V\setminus X_*} P^n f\,dx = c > 0.$$

For any $\varepsilon > 0$ we can find such a n_0 that

$$c \leqq \int_{V\setminus X_*} P^n f\,dx \leqq c + \varepsilon, \qquad \text{for} \quad n \geqq n_0.$$

Let

$$h = \frac{P^{n_0} f}{\displaystyle\int_{V\setminus X_*} P^{n_0} f\,dx}\,\mathbf{1}_{V\setminus X_*}.$$

We have

$$\begin{aligned}\int_{X_*} P^n h\,dx &= \frac{\displaystyle\int_{X_*} P^n (P^{n_0} f . \mathbf{1}_{V\setminus X_*})\,dx}{\displaystyle\int_{V\setminus X_*} P^{n_0} f\,dx} \\ &\leqq \frac{1}{c} \int_{X_*} P^n (P^{n_0} f . \mathbf{1}_{V\setminus X_*})\,dx \\ &= \frac{1}{c}\left[\int_{V\setminus X_*} P^{n_0} f\,dx - \int_{V\setminus X_*} P^n (P^{n_0} f . \mathbf{1}_{V\setminus X_*})\right] dx \\ &= \frac{1}{c}\left(\int_{V\setminus X_*} P^{n_0} f\,dx - \int_{V\setminus X_*} P^{n+n_0} f\,dx\right) \leqq \frac{\varepsilon}{c}.\end{aligned}$$

Notice that

$$\begin{aligned}\limsup_{N\to\infty}\frac{1}{N+1}\sum_{n=0}^{N}\int_{[|x|\leqq A,\,x\in V]}P^n h\,dx &\leqq \limsup_{N\to+\infty}\frac{1}{N+1}\sum_{n=0}^{N}\int_{[|x|\leqq A,\,x\in X_*]}P^n h\,dx\\ &+\limsup_{N\to+\infty}\frac{1}{N+1}\sum_{n=0}^{N}\int_{[|x|\leqq A,\,x\in V\setminus X_*]}P^n h\,dx\\ &\leqq \frac{\varepsilon}{c}+\limsup_{N\to+\infty}\frac{1}{N+1}\sum_{n=0}^{N}\int_{[|x|\leqq A,\,x\in V\setminus X_*]}P^n h\,dx \quad (4.9)\end{aligned}$$

here A is such that $|y|-|S(y)|\geqq\sigma>m$, if $|y|\geqq A$, $y\in V$. However the last term in (4.9) is equal to zero by virtue of Theorem 3.1 and the fact that X_* is maximal in the sense of (4.8). Thus

$$\limsup_{N\to+\infty}\frac{1}{N+1}\sum_{n=0}^{N}\int_{[|x|\leqq A,\,x\in V]}P^n h\,dx\leqq\frac{\varepsilon}{c}. \qquad (4.10)$$

Applying once again considerations from the proof of Theorem 4.1 we obtain

$$\bar{W}_{N+1}\leqq\bar{W}_N+\frac{\bar{V}_0}{N+1}+m-\int_V[|y|-|S(y)|]\bar{M}_N(y)\,dy,\ N\geqq 1$$

where $\bar{W}_N$, $\bar{M}_N$, $\bar{V}_N$ are defined identically as their correspondents W_N, M_N, V_N in (4.2)-(4.4). The only difference is that h is put instead of f_0 in those formulas. Hence

$$\begin{aligned}\bar{W}_{N+1}\leqq\bar{W}_N&+\frac{\bar{V}_0}{N+1}+m\\ &+\sup_{z\in V\cap[|y|\leqq A]}[|z|-|S(z)|]\int_V\bar{M}_N(y)\,dy-\sigma\int_{V\cap[|y|\geqq A]}\bar{M}_N(y)\,dy\end{aligned}$$

In view of (4.10) we see that there is N_0 such that for $N\geqq N_0$

$$\bar{W}_{N+1}\leqq\bar{W}_N+\frac{\bar{V}_0}{N+1}+m-\sigma\left(1-\frac{\varepsilon}{c}\right)+\frac{\varepsilon}{c}\sup_{z\in V\cap[|y|\leqq A]}[|z|-|S(z)|]$$

Choosing $\varepsilon>0$ so small that

$$\sigma_1=\sigma\left(1-\frac{\varepsilon}{c}\right)+\frac{\varepsilon}{c}\sup_{z\in V\cap[|y|\leqq A]}[|z|-|S(z)|]>m$$

we can find such a N_1 that for $N\geqq N_1$

$$0\leqq\bar{W}_{N+1}\leqq\bar{W}_N+m-\rho,\quad \text{where } m<\rho<\sigma_1.$$

This shows that such a sequence does not exist. The above contradiction concludes the proof. □

Now we confine our considerations to $V \cap X_*$. Define an operator

$$P_* : L^1(V \cap X_*) \to L^1(V \cap X_*)$$

where $P_* f = P f$. Since $P_* f_* = f_*$, the operator P_* is conservative. P_* is an operator given by a stochastic kernel. Summing up these facts we infer that P_* is a Harris operator. From the general theory of Harris operators (*see* for instance [1], p. 98, [7], pp. 109-119) there exist densities $g_1, g_2, \ldots, g_d$, functions $k_1, k_2, \ldots, k_d \in L^\infty_+(V \cap X_*)$ and a permutation $\alpha : \{1, 2, \ldots, d\} \to \{1, 2, \ldots, d\}$ such that P_* may be written in the form

$$P_* f = \sum_{i=1}^{d} \lambda_i(f) . g_i + Q f \tag{4.11}$$

where $\lambda_i(f) = \int_{V \cap X_*} f k_i \, dx$. The functions $g_1, g_2, \ldots, g_d$ and the operator Q have the following properties

(P1) $g_i g_j \equiv 0$, $i \neq j$ so g_i have disjoint supports.

(P2) $P_* g_i = g_{\alpha(i)}$, $i = 1, \ldots, d$.

(P3) $P^n_* Q f \to 0$, $n \to +\infty$.

Markov operators having the decomposition as in (4.11) are called asymptotically periodic (*see* [13], pp. 86-90). In case when $d = 1$ an operator is said to be asymptotically stable. From the foregoing and Lemma 4.2 we obtain.

THEOREM 4.3. – *Under the assumptions made in Theorem* 4.1 *the Markov operator* P *given by* (4.1) *is asymptotically periodic.*

Proof. – We have

$$\lim_{n \to +\infty} \operatorname{dist}(P^n f, \operatorname{Conv}\{g_1, \ldots, g_d\}) = 0.$$

However this condition is equivalent to the fact that P may be represented in the form given in (4.11). □

5. ASYMPTOTIC STABILITY

Below we give some conditions which allow us to admit $d = 1$ in (4.11). Throughout this paragraph we assume that S and g satisfy assumptions given in Theorem 4.1. We start with the following

THEOREM 5.1. – *Suppose that there is* M_0 *such that* $g(x) > 0$ *for* $|x| \geqq M_0$. *Then* P *given by* (4.1) *is asymptotically stable.*

Proof. – By virtue of Theorem 4.3 we have $g_1, \ldots, g_d$ as in (4.11). Let $L \geqq M_0$ be so large that

$$\int_{[|x| \leqq L] \cap V} g_i(x)\,dx > 0, \qquad i=1, \ldots, d.$$

For $|x| \geqq M_0 + \sup_{|y| \leqq L} |S(y)| + 1$ and $|z| \leqq L$ we have $|x - S(z)| \geqq M_0$. Hence

$$P g_i(x) = \int_V \bar{g}(x - S(y)) . g_i(y)\,dy$$

$$\geqq \int_{V \cap [|y| \leqq L]} \bar{g}(x - S(y)) . g_i(y)\,dy > 0, \qquad i=1, \ldots, d$$

The above shows that $d=1$. □

Another theorem concerns the case $V = \mathbb{R}^+$.

THEOREM 5.2. – *Suppose that* S *is continuous,* $S(0)=0$, $S(x)<x$ *for* $x>0$. *Assume additionally that there is* $\delta>0$ *such that* $g(x)>0$, $x \in [0, \delta]$. *Then* P *given by* (4.1) *is asymptotically stable.*

Proof. – Let $g_1, g_2, \ldots, g_d$ be as in (4.11). There is such a $k \in \mathbb{N}$ that

$$\left.\begin{aligned} g_1(x) = P^k g_1(x) = \int_{\mathbb{R}^+} \cdots \int_{\mathbb{R}^+} \bar{g}(x - S(y_1))\, \bar{g}(y_1 - S(y_2)) \ldots \\ \bar{g}(y_{k-1} - S(y_k)) . g_1(y_k)\,dy_1 \ldots dy_k \end{aligned}\right\} \quad (5.1)$$

where the equality is satisfied up to a certain set of the Lebesgue measure zero. Let $x_0 \in \mathbb{R}^+$ be such that for any $h>0$

$$\int_{x_0 - h}^{x_0} g_1(y_k)\,dy_k > 0 \quad \text{and} \quad \int_{x_0}^{x_0+h} g_1(y_k)\,dy_k > 0. \qquad (5.2)$$

Consider an open subset G of $(\mathbb{R}^+)^{d+1}$ consisting of $(x, y_1, \ldots, y_k)$ satisfying

$$\begin{gathered} x_0 - \varepsilon < y_k < x_0 + \varepsilon \\ S(y_k) < y_{k-1} < S(y_k) + \delta \\ \vdots \\ S(y_2) < y_1 < S(y_2) + \delta \\ S(y_1) < x < S(y_1) + \delta. \end{gathered}$$

For any $(x, y_1, \ldots, y_k) \in G$

$$\bar{g}(x - S(y_1))\, \bar{g}(y_1 - S(y_2)), \ldots, \bar{g}(y_{k-1} - S(y_k)) > 0.$$

Additionally if for x_1 we can choose $y_1^0, \ldots, y_k^0$ such that $(x_1, y_1^0, \ldots, y_k^0) \in G$ then

$$\int_{\mathbb{R}^+} \cdots \int_{\mathbb{R}^+} \bar{g}(x_1 - S(y_1))\, \bar{g}(y_1 - S(y_2)) \ldots \bar{g}(y_{k-1} - S(y_k))\, g_1(y_k)\, dy_1, \ldots, dy_{k-1} > 0$$

in a certain neighborhood of y_k^0. Choose $(x_1, y_1, \ldots, y_k) \in G$ for which $y_k = x_0$,

$$y_{i-1} < \frac{y_i + S(y_i)}{2}, \qquad i = 2, 3, \ldots, k$$

and

$$x_1 < \frac{y_1 + S(y_1)}{2}$$

Hence from (5.1) and (5.2) it follows that there exists a certain neighborhood of x_1 in which $g_1(x) > 0$. Additionally

$$x_1 < \frac{x_0 + S(x_0)}{2}$$

Proceeding this reasoning we find a set $A = \bigcup_{i=1}^{+\infty} (a_i, b_i)$ which satisfies

(i) $b_i > a_i > 0$, $b_i \to 0$, $i \to +\infty$

(ii) if $x \in A$ then $g_1(x) > 0$.

Let $x \in [0, \delta]$ then we have such a $y_k \in A$ that $y_k < x$. Finding

$$y_k > y_{k-1} > S(y_k), \ldots, y_2 > y_1 > S(y_2)$$

we deduce that

$$\bar{g}(x - S(y_1)) \ldots \bar{g}(y_{k-1} - S(y_k))\, g_1(y_k) > 0.$$

From (5.1) $g_1(x) > 0$, $x \in [0, \delta]$. Because g_1 was arbitrarily chosen from the set $\{g_1, \ldots, g_k\}$ we see that $d = 1$ in (P1). □

Example. – Consider a Markov operator $P: L^1[0, +\infty) \to L^1[0, +\infty)$ given by

$$P f(x) = \int_0^{\lambda(x)} K(x, y) . f(y)\, dy$$

with the kernel

$$k(x, y) = -\frac{\partial}{\partial x} \exp\{Q(y) - Q(\lambda(x))\}$$

where Q and λ are nondecreasing continuously differentiable functions satisfying

(i) $Q'(x)>0$, $\lambda'(x)>0$ for $x>0$
(ii) $Q, \lambda : \mathbb{R}^+ \to \mathbb{R}^+$.

Such a class of operators was studied by J. Tyrcha in [11]. These operators play an essential role in the mathematical model of the cell cycle given there. In [4] it is shown that P is the transition operator for the following Markov chain

$$X_{n+1}=\lambda^{-1}\{Q^{-1}[Q(X_n)+\xi_n]\}, \qquad n=0,1,\ldots$$

where $\{\xi_n\}_{n\geqq 0}$ is a sequence of independent random variables with the common density $g(x)=e^{-x}$, $x>0$. Denoting $\bar{X}_n=Q\lambda(X_n)$ we obtain

$$\bar{X}_{n+1}=S(\bar{X}_n)+\xi_n, \qquad n\geqq 0,$$

where $S=Q\lambda^{-1}Q^{-1}$. Now if $\liminf_{x\to+\infty}[Q\lambda(x)-Q(x)]>1=E\xi_0$ then $\liminf_{x\to+\infty}[x-S(x)]=\liminf_{x\to+\infty}[Q\lambda(x)-Q(x)]>1=E\xi_0$. From this we see that the transition operator for $\{\bar{X}_n\}_{n\geqq 0}$, as well as the one for $\{X_n\}_{n\geqq 0}$ are asymptotically stable. This result was obtained in a different way by J. Tyrcha in [11]. □

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