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Tail probabilities of Gaussian suprema and Laplace transform

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ABSTRACT. – Let $\{\xi_t,\ t\in T\}$ be a bounded Gaussian random function, $\zeta \equiv \sup_T \xi_t$. We investigate the large deviations by means of the Laplace transform $\Psi_p(\lambda) \equiv E \exp\{\lambda \zeta^p\}$, $1 \leqq p < 2$. We derive for large r the asymptotical equivalence

$$P\{\xi \geqq r\} \sim (2-p)^{1/2}\,\Psi_p\left(\frac{r^{2-p} - dr^{1-p}}{p\,\sigma^2}\right) \times \exp\left\{-\frac{(2-p)\,r^2}{2p\,\sigma^2} - \frac{d(p-1)\,r}{p\,\sigma^2} + \frac{d^2}{2\,\sigma^2}\right\}\left(1-\Phi\left(\frac{r-d}{\sigma}\right)\right)$$

where d and σ are some important numeric parameters of the random function ξ, and Φ is the distribution function of the standard normal distribution law.

We apply this relation to the investigation of the Gaussian measure of large balls in the space l^p in order to generalize some recent results due to Linde and author. The broad range of possible types of behavior of large deviations is under consideration and some of them turn out to be unusual.

Key words : Gaussian processes, supremum, large deviations, l^p-spaces.

RÉSUMÉ. – Soit $\{\xi_t,\ t\in T\}$ une fonction aléatoire gaussienne bornée, $\xi \equiv \sup_T \xi_t$. Nous étudions ces grandes déviations à l'aide de la transformée de Laplace $\Psi_p(\lambda) \equiv E \exp\{\lambda \zeta^p\}$, $1 \leqq p < 2$. Nous trouvons pour les grandes valeurs de r, l'équivalence asymptotique

$$\mathrm{P}\{\xi \geqq r\} \sim (2-p)^{1/2} \Psi_p\left(\frac{r^{2-p}-d r^{1-p}}{p \sigma^2}\right) \times \exp\left\{-\frac{(2-p) r^2}{2 p \sigma^2}-\frac{d(p-1) r}{p \sigma^2}+\frac{d^2}{2 \sigma^2}\right\}\left(1-\Phi\left(\frac{r-d}{\sigma}\right)\right)$$

où d et σ sont des paramètres numériques importants de la fonction aléatoire ξ et, Φ est la fonction de répartition de la loi gaussienne standard. Cette relation est appliquée pour les études des mesures gaussiennes des grandes boules dans l'espace l^p et pour la généralisation des résultats récents obtenue par W. Linde et l'auteur. Un large spectre du comportement des grandes déviations est considéré et des exemples originaux sont analysés.

I. NOTATIONS AND KNOWN RESULTS

Let $\{\xi_t, t \in \mathrm{T}\}$ be a bounded Gaussian random function with an arbitrary parametric set T. We do not suppose that ξ_t is centered. The boundedness of ξ implies the relative compactness of T with respect to the natural semi-metric $\rho(s, t) \equiv (\mathbf{E}|\xi_s-\xi_t|^2)^{1/2}$. It implies also the existence of the separable version of ξ. We can consider for our purpose only this separable version. The definition of separability implies that $\zeta \equiv \sup_{\mathrm{T}} \xi_t$ is a random variable. The main subject of our consideration is the behavior of the following probabilities of large deviations:

$$\mathrm{P}\{\zeta \geqq r\}, \qquad r \rightarrow \infty. \tag{1.1}$$

Due to its great importance in probability theory and statistics, this problem has been intensively treated in the numerous papers by Landau and Shepp, Sudakov and Tsyrelson, Dudley, Borell, Fernique, Ehrhard, Piterbarg, Dmitrowskii, Talagrand, Adler, Samorodnitsky and many others. We refer to R. Adler [1] as complete survey of the subject and to the book of M. Ledoux and M. Talagrand [8] presenting deep exposition of a general frame including Gaussian case.

We introduce now some notations in order to describe the basic known results.

Let F and f be the distribution function and the density of r.v. ζ.

Let Φ be the distribution function of the standard normal law $N(0, 1)$.

We need the following notation for the asymptotic relations. For each pair of functions h_1 and h_2 defined on $\mathbf{R}^+$ we shall write

$$h_1 \sim h_2 \text{ or } h_1 < \sim h_2 \quad \text{if} \quad \lim_{r \to \infty} h_1(r)/h_2(r) = 1$$

or

$$\limsup_{r \to \infty} h_1(r)/h_2(r) \leqq 1, \text{ respectively.}$$

Particularly, the direction of our investigation is a search of some computable function h such that $h(r) \sim 1 - \mathrm{F}(r) = \mathbf{P}\{\zeta \geqq r\}$.

We denote

$$\sigma \equiv \sup [\mathbf{D}\xi_t]^{1/2}$$

and eliminate from the following consideration the degenerated and trivial case $\sigma = 0$. The boundedness of ξ implies, certainly, that σ is finite.

We denote also

$$d \equiv \lim_{\varepsilon \to 0} \lim_{\delta \to 0} \sup_{\{s, t : \mathbf{D}\xi_t > \sigma^2 - \varepsilon, \rho(s, t) < \delta\}} (\mathbf{E}\,\xi_t + (\xi_s - \xi_t)/2). \tag{1.3}$$

One can see that the limit in the right-hand side of this equality is, in fact, *nonrandom* and finite. Hence, we can consider d as a real number. Under the additional assumption of completeness of the space (T, ρ), one can easily express d by means of Ito-Nisio oscillation function $\alpha(t)$ defined in [7] as

$$\alpha(t) \equiv 2 \lim_{\varepsilon \to 0} \sup_{\{s : \rho(s, t) < \varepsilon\}} (\xi_s - \xi_t).$$

Namely,

$$d = \sup_{\{t : \mathbf{D}\xi_t = \sigma^2\}} (\mathbf{E}\,\xi_t + \alpha(t)/2),$$

but we shall not use this representation.

It is well known for a long time that σ is responsible for the main term of the asymptotics of large deviations. In fact, d is responsible for the second term (see the formula (1.5) below).

We introduce now the useful function

$$\hat{\mathrm{F}}(r) \equiv \Phi^{-1}(\mathrm{F}(r)), \qquad r \in \mathbf{R}^1.$$

This function is *concave* due to Ehrhard's [5] powerful inequality. This property makes $\hat{\mathrm{F}}$ more convenient for investigation than the initial function F. In fact, the introduced objects $\hat{\mathrm{F}}$, σ, d are connected by the simple relation

$$\lim_{r \to \infty} [\hat{\mathrm{F}}(r) - (r - d)/\sigma] = 0, \tag{1.4}$$

i.e. σ and d define the asymptote of $\hat{\mathrm{F}}$ for $r \to \infty$. One can find the proof of (1.4) for centered random functions in [2].

A one-sided estimate of this type was earlier obtained by M. Talagrand in [14].

The following relation is the immediate consequence of (1.4):

$$\lim_{r \to \infty} [\log (1 - \mathrm{F}(r)) + (r-d)^2/2\sigma^2]/r = 0. \tag{1.5}$$

This relation was the main achievement of the first phase of development of the theory of Gaussian large deviations described in the surveys [4, 6] (see also [1,8]). It shows, particularly, that there exists a *single* type of behavior of the logarithm of probabilities of large deviations for all bounded Gaussian random functions. However, we shall see below that more precise asymptotic may differ essentially.

We denote now $\chi(r) \equiv (r-d)/\sigma - \hat{\mathrm{F}}(r)$ and see from (1.4) that χ is a convex nonnegative function, decreasing to zero when r tends to infinity. The differentiation of the equality

$$\mathrm{F}(r) = \Phi(\hat{\mathrm{F}}(r)) = \Phi((r-d)/\sigma - \chi(r))$$

leads us to the relation

$$f(r) = (2\pi)^{-1/2} \exp\left\{ -\frac{(r-d-\sigma\chi(r))^2}{2\sigma^2} \right\} (\sigma^{-1} - \chi'(r)).$$

It seems quite natural to extract the "normal part" from f. That's why we define the function g by the relation

$$f(r) = (2\pi)^{-1/2} \sigma^{-1} \exp\left\{ -\frac{(r-d)^2}{2\sigma^2} \right\} g(r), \tag{1.6}$$

i.e.

$$g(r) \equiv \exp\left\{ \frac{(r-d)\chi(r)}{\sigma} - \frac{\chi(r)^2}{2} \right\} (1 - \chi'(r)\sigma).$$

It follows directly from the above mentioned properties of χ that

$$g(r) \sim \exp\left\{ \frac{r\chi(r)}{\sigma} \right\}. \tag{1.7}$$

We can also derive easily from (1.6) and (1.7) the following relation for the distribution function (see the details in [9]):

$$1 - \mathrm{F}(r) \sim (1 - \Phi((r-d)/\sigma))\, g(r). \tag{1.8}$$

In the following we shall concentrate our efforts on the investigation of the function g.

II. LAPLACE TRANSFORM AND LARGE DEVIATIONS

Let $p \in [1, 2)$. Define the Laplace transform of the r.v. ζ by the following formula: let for $\lambda \geqq 0$

$$\Psi_p(\lambda) \equiv \mathbf{E}\exp\{\lambda\zeta^p\}\mathbf{1}_{\{\zeta>0\}} = (2\pi)^{-1/2}\sigma^{-1}\int_0^\infty \exp\left\{\lambda r^p - \frac{(r-d)^2}{2\sigma^2}\right\} g(r)\,dr. \quad (2.1)$$

The following theorem connects the probabilities of large deviations with Laplace transform. There exist many results of this type for different classes of random variables. These results called usually Tauber theorems, use for the evaluation of large deviations the values of $\Psi_1(\lambda)$ with *small negative* λ. However, we use *large positive* λ. The extremely fast decreasing of the Gaussian tail probabilities is the reason of this difference.

THEOREM 1. — *Let* F, f *and* Ψ_p *be the distribution function, the density and the Laplace transform of r.v.* $\zeta \equiv \sup_{\mathrm{T}} \xi_t$, *respectively. Then*

$$f(r) \sim \frac{(2-p)^{1/2}}{\sigma(2\pi)^{1/2}} \Psi_p\left(\frac{r^{2-p} - dr^{1-p}}{p\sigma^2}\right) \times \exp\left\{\frac{-(2-p)}{2p\sigma^2}r^2 - \frac{d(p-1)}{p\sigma^2}r + \frac{d^2}{2\sigma^2} - \frac{(r-d)^2}{2\sigma^2}\right\}, \quad (2.2)$$

$$1 - \mathrm{F}(r) \sim (2-p)^{1/2}\Psi_p\left(\frac{r^{2-p} - dr^{1-p}}{p\sigma^2}\right) \times \exp\left\{\frac{-(2-p)}{2p\sigma^2}r^2 - \frac{d(p-1)}{p\sigma^2}r + \frac{d^2}{2\sigma^2}\right\}\left(1 - \Phi\left(\frac{r-d}{\sigma}\right)\right). \quad (2.3)$$

Proof. — It is more convenient to inverse the direction of our research and express Ψ_p by means of f. We start from (2.1) and denote by r_0 the real number providing the maximum of the exponent $\lambda r^p - \frac{(r-d)^2}{2\sigma^2}$. There exists an obvious equation for r_0:

$$\lambda p r_0^{p-1} - (r_0 - d)/\sigma^2 = 0, \quad (2.4)$$

and we are able to express λ as

$$\lambda = [r_0^{2-p} - dr_0^{1-p}]/p\sigma^2. \quad (2.5)$$

It would be nice to replace the exponent in (2.1) by some square form. For this purpose we use the Taylor expansion

$$r^p = r_0^p + p r_0^{p-1}(r-r_0) + \frac{p(p-1)}{2} r_0^{p-2}(r-r_0)^2 + \frac{p(p-1)(p-2)}{6} r_1^{p-3}(r-r_0)^3 \quad (2.6)$$

with some r_1 situated between r_0 and r.

Using the expressions (2.5) and (2.6), one can obtain the following identity

$$\lambda r^p - \frac{(r-d)^2}{2\sigma^2} = \frac{-(2-p)}{2\sigma^2}(r-r_0)^2 + \frac{(2-p)}{2p\sigma^2} r_0^2 + \frac{d(p-1)}{p\sigma^2} r_0 - \frac{d^2}{2\sigma^2} + \mathscr{R}(r, r_0), \quad (2.7)$$

in which the remainder term

$$\mathscr{R}(r, r_0) \equiv \left[\frac{(p-1)}{2r_0} d(r-r_0)^2 + \frac{(p-1)(p-2)}{6}(r_0^{2-p} - d r_0^{1-p}) r_1^{p-3}(r-r_0)^3 \right] / \sigma^2$$

tends to zero when r, r_0 tend to infinity, but $r-r_0$ remains bounded or increases slowly.

We substitute the identity (2.7) in (2.1) and obtain the expression

$$\Psi_p(\lambda) = (2\pi)^{-1/2}\sigma^{-1}\left[\int_0^\infty \exp\left\{\frac{-(2-p)}{2\sigma^2}(r-r_0)^2 + \mathscr{R}(r, r_0)\right\} g(r)\,dr\right] \times \exp\left\{\frac{(2-p)}{2p\sigma^2} r_0^2 + \frac{d(p-1)}{p\sigma^2} r_0 - \frac{d^2}{2\sigma^2}\right\}.$$

We need now the following natural relation

$$\int_0^\infty \exp\left\{\frac{-(2-p)}{2\sigma^2}(r-r_0)^2 + \mathscr{R}(r, r_0)\right\} g(r)\,dr \sim \frac{(2\pi)^{1/2}\sigma}{(2-p)^{1/2}} g(r_0). \quad (2.8)$$

We omit the considerably long elementary caclulations proving (2.8), because the proof of almost the same relation (with a slightly different remainder term $\mathscr{R}$) was exposed in [9, p. 194-196].

The application of (2.8) leads to the relation

$$\Psi_p(\lambda) \sim (2-p)^{-1/2} g(r_0) \exp\left\{\frac{(2-p)}{2p\sigma^2} r_0^2 + \frac{d(p-1)}{p\sigma^2} r_0 - \frac{d^2}{2\sigma^2}\right\}.$$

We substitute (2.5), make the notation change $r_0 \to r$ and obtain the desired expression for the function g:

$$g(r) \sim (2-p)^{1/2} \Psi_p\left(\frac{r^{2-p} - d r^{1-p}}{\sigma^2 p}\right) \exp\left\{\frac{-(2-p)}{2p\sigma^2} r^2 - \frac{d(p-1)}{p\sigma^2} r + \frac{d^2}{2\sigma^2}\right\}.$$

Now the relations (2.2) and (2.3) follow, respectively, from (1.6) and (1.8). ■

COROLLARY. — *If* $d=0$, *we obtain the following simple expressions instead of* (2.2) *and* (2.3):

$$f(r) \sim \frac{(2-p)^{1/2}}{\sigma(2\pi)^{1/2}} \Psi_p\left(\frac{r^{2-p}}{\sigma^2 p}\right) \exp\left\{\frac{-(2-p)}{2p\sigma^2} r^2 - \frac{r^2}{2\sigma^2}\right\},$$

$$1-\mathrm{F}(r) \sim (2-p)^{1/2} \Psi_p\left(\frac{r^{2-p}}{\sigma^2 p}\right) \exp\left\{\frac{-(2-p)}{2p\sigma^2} r^2\right\} (1-\Phi(r/\sigma)).$$

Particularly, if ξ is centered and continuous, we see from (1.3) that $d=0$, and the previous expressions hold. This case was under consideration in [9]. The most instructive case $d=0$, $p=1$ was also partially considered in [10].

III. MEASURE OF LARGE BALLS IN l^p

In this section we apply the general theorem 1 to the calculation of the asymptotic behavior of the measure of large balls in the space l^p, $1 \leqq p < 2$, with respect to Gaussian product-measure. Let $p \in [1, 2)$, let $\{a_i\}$ and $\{\sigma_i\}$, $i=1, 2, \ldots$ be the sequences of real numbers such that $\sigma_i \geqq 0$ and

$$\sum_{i=1}^{\infty} |a_i|^p < \infty, \qquad \sum_{i=1}^{\infty} \sigma_i^p < \infty.$$

Let $\{\xi_i\}$ be a sequence of independent standard normal random variables. We are going to study the tail probabilities of r.v. ζ defined as follows:

$$\zeta = \left(\sum_{i=1}^{\infty} |\sigma_i \xi_i + a_i|^p\right)^{1/p},$$

i.e. ζ is the norm of the random Gaussian l^p-valued vector with independent coordinates.

We need some additional notations for several constants. Let $q \equiv p/(p-1)$, $\nu \equiv (2-p)^{-1}$, $m = 2p/(2-p)$,

$$\sigma \equiv \left(\sum_{i=1}^{\infty} \sigma_i^m\right)^{1/m}, \qquad d \equiv \sum_{i=1}^{\infty} (\sigma_i/\sigma)^{2(p-1)/(2-p)} |a_i|. \tag{3.1}$$

For the case $p=1$ we understand (3.1) as $d \equiv \sum_{i=1}^{\infty} |a_i|$, avoiding problems with $\sigma_i = 0$.

We define also a function

$$\mathscr{E}(\lambda, a) \equiv \mathbf{E} \exp\{\lambda |\xi_1 + a|^p\}, \qquad (\lambda \geqq 0, a \in \mathbf{R}^1)$$

and its normalized modification

$$\hat{\mathscr{E}}(\lambda, a) \equiv \mathscr{E}(\lambda, a) \exp\left\{\frac{-(2-p)}{2} p^{p\nu} \lambda^{2\nu} - p^{\nu} |a| \lambda^{\nu}\right\}.$$

One can obtain (after some complicated but elementary calculations) that this normalized function possesses a limit for $\lambda \to \infty$, namely

$$\hat{\mathscr{E}}(\infty, a) \equiv \lim_{\lambda \to \infty} \hat{\mathscr{E}}(\lambda, a) = \nu^{1/2} (1 + 1_{\{\alpha=0\}}) \exp\left\{\frac{(p-1)\nu}{2} a^2\right\}. \quad (3.2)$$

Remark. — For the case $p=1$ we can rewrite the function $\hat{\mathscr{E}}$ in a more explicit form. Namely, we can write, according to the definition of $\mathscr{E}$,

$$\begin{aligned}\mathscr{E}(\lambda\ a) &= (2\pi)^{-1/2} \int_0^{\infty} \exp\{\lambda r\} [\exp\{-(r-a)^2/2\} + \exp\{-(r+a)^2/2\}]\, dr \\ &= (2\pi)^{-1/2} \exp\{a\lambda + \lambda^2/2\} \int_0^{\infty} \left[\exp\left\{\frac{-(r-(\lambda+a))^2}{2}\right\}\right. \\ &\qquad \left. + \exp\{-2a\lambda\} \exp\left\{\frac{-(r-(\lambda-a))^2}{2}\right\}\right] dr \\ &= \exp\{a\lambda + \lambda^2/2\} [\Phi(\lambda+a) + \exp\{-2a\lambda\} \Phi(\lambda-a)].\end{aligned}$$

Respectively,

$$\begin{aligned}\hat{\mathscr{E}}(\lambda, a) &= \mathscr{E}(\lambda, a) \exp\{-\lambda^2/2 - |a|\lambda\} \\ &= \Phi(\lambda+|a|) + \exp\{-2|a|\lambda\} \Phi(\lambda-|a|). \quad (3.3)\end{aligned}$$

THEOREM 2. — *Let $\sigma_i > 0$ for each $i=1, 2, \ldots$ Then the following relation holds for the deviations of the r.v. ζ generated by sequences $\{a_i\}$ and $\{\sigma_i\}$,*

$$\begin{aligned}\mathrm{P}\{\zeta \geqq r\} \sim (2-p)^{1/2} &\left[\prod_{i=1}^{\infty} \hat{\mathscr{E}}\left(\frac{\sigma_i^p (r^{2-p} - d r^{1-p})}{p \sigma^2}, \frac{a_i}{\sigma_i}\right)\right] \\ &\times \exp\left\{\frac{(1-p) d^2}{2(2-p)\sigma^2}\right\} \left(1 - \Phi\left(\frac{r-d}{\sigma}\right)\right). \quad (3.4)\end{aligned}$$

Proof. — First of all, if we want to apply theorem 1, we have to represent ζ as a functional of supremum type. It is easy to do via the standard representation of l^p-norm as a supremum of values of linear functionals which belong to the unit ball of the conjugated space l^q. Let, for simplicity, $p > 1$. Then

$$\zeta = \sup_t \left\{ \sum_{i=1}^{\infty} (\sigma_i \xi_i + a_i) t_i \,\middle|\, \sum_{i=1}^{\infty} |t_i|^q = 1 \right\} = \sup_{\mathrm{T}} \xi_t$$

with the Gaussian random field ξ_t, $t \in \mathrm{T}$,

$$\xi_t \equiv \sum_{i=1}^{\infty} (\sigma_i \xi_i + a_i) t_i. \qquad \mathrm{T} \equiv \left\{ t \in l^q \,\middle|\, \sum_{i=1}^{\infty} |t_i|^q = 1 \right\}.$$

It is necessary to compute the parameters σ and d defined by (1.2) and (1.3) for this specific function ξ. In fact, we have to check the formula (3.1) for these parameters. The definition (1.2) leads in our case to the optimization problem

$$\sigma^2 = \sup\left\{ \sum_{i=1}^{\infty} \sigma_i^2 t_i^2 \,\middle|\, \sum_{i=1}^{\infty} |t_i|^q = 1 \right\}$$

This problem is easily solvable by the Lagrange method and we can see that really $\sigma = \left(\sum_{i=1}^{\infty} \sigma_i^m \right)^{1/m}$ and this maximal value is achieved on each sequence in the set

$$T_\sigma \equiv \{ t \in T \mid t_i = (\sigma_i/\sigma)^{2(p-1)\nu} \alpha_i;\ \alpha_i = 1 \quad \text{or} \ \alpha_i = -1 \}.$$

In the case $p=1$ we have the same formula (3.1) for σ and the following set of solutions of the optimization problem:

$$T_\sigma = \left\{ t \in l^\infty \,\middle|\, \begin{array}{ll} t_i = 1 \text{ or } t_i = -1, & \text{if} \quad \sigma_i > 0 \\ t_i \in [-1, 1], & \text{if} \quad \sigma_i = 0 \end{array} \right\}.$$

It is also easy to see that ξ possesses a continuous version, hence we don't observe any oscillation effects and (1.3) gives us the expression mentioned in (3.1):

$$d = \lim_{\varepsilon \to 0} \sup_{t : \mathbf{D}\xi_t > \sigma^2 - \varepsilon} \mathbf{E}\xi_t = \sup_{t \in T_\sigma} \mathbf{E}\, \xi_t = \sum_{i=1}^{\infty} (\sigma_i/\sigma)^{2(p-1)\nu} |a_i|.$$

Now we have to write down the Laplace transform of ζ, providing the most interesting factor in the expression given by Theorem 1.

$$\Psi_p(\lambda) \equiv \mathbf{E} \exp\left(\lambda \sum_{i=1}^{\infty} |\sigma_i \xi_i + a_i|^p \right) = \prod_{i=1}^{\infty} \mathbf{E} \exp(\lambda \sigma_i^p |\xi_i + a_i/\sigma_i|^p)$$

$$= \prod_{i=1}^{\infty} \mathscr{E}(\lambda \sigma_i^p, a_i/\sigma_i) = \prod_{i=1}^{\infty} \hat{\mathscr{E}}(\lambda \sigma_i^p, a_i/\sigma_i)$$

$$\times \exp\left\{ \frac{(2-p) p^{p\nu} \lambda^{2\nu} \sigma_i^{2p\nu}}{2} + p^\nu \lambda^\nu \sigma_i^{p\nu - 1} |a_i| \right\}.$$

We substitute in the right-hand side of this inequality the value of λ recommended by theorem 1 $\left(\lambda = \frac{r^{2-p} - d r^{1-p}}{p\,\sigma^2} \right)$ and obtain the following expression for each exponent.

$$(2-p) p^{\nu(p-2)} \sigma^{-4\nu} [r^{2-p} - d r^{1-p}]^{2\nu} \sigma_i^m/2 + \sigma^{-2\nu} \sigma_i^{p\nu-1} [r^{2-p} - d r^{1-p}]^\nu |a_i|$$
$$= (2-p) p^{-1} \sigma^{-4\nu} r^2 [1 - d/r]^{2\nu} \sigma_i^m/2 + \sigma^{-2\nu} \sigma_i^{2(p-1)/(2-p)} r [1 - d/r]^\nu |a_i|.$$

Taking into account the expressions (3.1) we observe that the sum of these exponents is finite and may be rewritten in the form

$$(2-p)p^{-1}\sigma^{(2p-4)\nu}r^2[1-d/r]^{2\nu}/2+\sigma^{-2\nu+2(p-1)\nu}r[1-d/r]^{\nu}.d$$
$$=m^{-1}\sigma^{-2}r^2[1-d/r]^{2\nu}+\sigma^{-2}r[1-d/r]^{\nu}d.$$

As the last step of our calculations, we use Taylor expansion for both binomial factors to obtain the expression

$$\begin{aligned}
&m^{-1}\sigma^{-2}r^2[1-2\nu d/r+\nu(2\nu-1)d^2/r^2]+\sigma^{-2}r[1-\nu d/r]d+\bar{\bar{o}}(1)\\
&\quad=m^{-1}\sigma^{-2}r^2-2\nu m^{-1}\sigma^{-2}dr\\
&\qquad\qquad+\nu(2\nu-1)m^{-1}\sigma^{-2}d^2+\sigma^{-2}dr-\nu\sigma^{-2}d^2+\bar{\bar{o}}(1)\\
&\quad=m^{-1}\sigma^{-2}r^2+(-2\nu m^{-1}+1)\sigma^{-2}dr+((2\nu-1)m^{-1}-1)\nu\sigma^{-2}d^2+\bar{\bar{o}}(1)\\
&\quad=m^{-1}\sigma^{-2}r^2+(1-1/p)\sigma^{-2}dr-\nu\sigma^{-2}d^2/2+\bar{\bar{o}}(1).
\end{aligned}$$

The following relation is a resume of our calculations:

$$\Psi_p\left(\frac{r^{2-p}-dr^{1-p}}{p\sigma^2}\right)\sim\prod_{i=1}^{\infty}\hat{\mathscr{E}}\left(\frac{r^{2-p}-dr^{1-p}}{p\sigma^2}\sigma_i^p,\ a_i/\sigma_i\right)$$
$$\times\exp\left\{\frac{m^{-1}r^2+(1-1/p)dr-\nu d^2/2}{\sigma^2}\right\}.$$

We substitute this expression in (2.3), reduce two pairs of exponents and obtain the equivalence

$$1-\mathrm{F}(r)\sim(2-p)^{1/2}\prod_{i=1}^{\infty}\hat{\mathscr{E}}\left(\frac{r^{2-p}-dr^{1-p}}{p\sigma^2}\sigma_i^p,\ a_i/\sigma_i\right)$$
$$\times\exp\left\{\frac{(1-\nu)d^2}{2\sigma^2}\right\}\left(1-\Phi\left(\frac{r-d}{\sigma}\right)\right).$$

This is exactly the same as (3.4). ■

Remark 1. – If some members of the sequence $\{\sigma_i\}$ are equal to zero, the formula (3.4) should be slightly changed.

Namely, for $p>1$ the factor

$$\prod_{i=1}^{\infty}\hat{\mathscr{E}}\left(\frac{r^{2-p}-dr^{1-p}}{p\sigma^2}\sigma_i^p,\ a_i/\sigma_i\right)$$

should be replaced by

$$\left[\prod_{i:\sigma_i>0}\hat{\mathscr{E}}\left(\frac{r^{2-p}-dr^{1-p}}{p\sigma^2}\sigma_i^p,\ a_i/\sigma_i\right)\right]\exp\left\{\frac{r^{2-p}}{p\sigma^2}\sum_{i:\sigma_i=0}|a_i|^p\right\}$$

and for $p=1$ the same factor should be replaced by

$$\prod_{i:\sigma_i>0}\hat{\mathscr{E}}\left(\frac{r-d}{\sigma^2}\sigma_i,\ a_i/\sigma_i\right).$$

Remark 2. – Let us consider the finite-dimensional case, *i.e.* let $\sigma_i = a_i = 0$, $i = n+1, n+2, \ldots$ We have under this assumption a finite number of factors in (3.4); each of them tends to the finite limit (3.2). Hence, we obtain from (3.4) the formula

$$\mathbf{P}\{\zeta \geqq r\} \sim (2-p)^{(1-n)/2}\, 2^{\mu} \exp\left\{\frac{(p-1)\nu}{2(2-p)} \sum_{i=1}^{n} \frac{a_i^2}{\sigma_i^2}\right\} \times \exp\left\{\frac{(1-p)\, d^2}{2(2-p)\,\sigma^2}\right\}\left(1-\Phi\left(\frac{r-d}{\sigma}\right)\right),$$

where μ denotes the number of indexes i with $a_i = 0$.

This result was earlier obtained by Linde [13].

Remark 3. – The behavior of the measure of large balls in l^p, $p \geqq 2$ is much more simple than one has in our case. Let us, for completeness, briefly recall the known results. The Hilbert case, $p = 2$, was initially treated by V. M. Zolotarev in [16]. The following recent and complete result, including non-centered case is due to W. Linde [12]

Let $p = 2$ and $\sigma_1 = \ldots = \sigma_n > \sigma_{n+1} \geqq \ldots \geqq 0$.

Let

$$\mathrm{A} \equiv \left(\sum_{i=1}^{n} a_i^2\right)^{1/2}, \qquad \mathrm{C} \equiv \prod_{i=n+1}^{\infty} (1 - \sigma_i^2/\sigma_1^2)^{-1/2}.$$

If $\mathrm{A} > 0$, then

$$\mathbf{P}\{\zeta \geqq r\} \sim \mathrm{A}^{-(n-1)/2}\,\mathrm{C} \times \exp\left\{\sum_{i=n+1}^{\infty} a_i^2/2(\sigma_1^2 - \sigma_i^2)\right\} r^{(n-1)/2} (1 - \Phi((r - \mathrm{A})/\sigma_1)).$$

If $\mathrm{A} = 0$, then

$$\mathbf{P}\{\zeta \geqq r\} \sim 2^{3/2 - n/2}\, \sigma_1^{1-n}\, \pi^{1/2}\, \Gamma(n/2)^{-1}\, \mathrm{C} \times \exp\left\{\sum_{i=n+1}^{\infty} a_i^2/2(\sigma_1^2 - \sigma_i^2)\right\} r^{n-1} (1 - \Phi(r/\sigma_1))$$

As for the case $p > 2$, the asymptotics is as simple as possible. The following result is due to V. Dobrič, M. Marcus, M. Weber [3] and was deduced from a general result of M. Talagrand [15].

Let $p > 2$ and $\sigma_1 = \ldots = \sigma_n > \sigma_{n+1} \geqq \ldots \geqq 0$. Then

$$\mathbf{P}\{\zeta \geqq r\} \sim 2n(1 - \Phi(r/\sigma_1)).$$

Thus, each maximal eigenvalue generates a contribution equivalent to the tail of standard normal distribution. We refer to author's article [11] for detailed discussion and generalizations.

IV. SOME EXAMPLES. LARGE OCTAHEDRONS

In this section, we apply the theorem 2 to some particular sequences $\{a_i\}$ and $\{\sigma_i\}$, in order to show that the exact behavior of the deviations may vary in a very broad range. Particularly, we show that the asymptotics of the probability of large deviations may contain the *periodical* component. We restrict our consideration by the case $p=1$ and $d\neq 0$. One can find similar examples for $p=1$, $d=0$ and $p\in[1, 2)$, $d=0$ in [10] and [9], respectively.

Recall, that in the case $p=1$ we deal with the behavior of Gaussian measure of the large l^1-balls. These balls sometimes are called *octahedrons*, since the l^1-ball in $\mathbf{R}^3$ is octahedron.

In what follows, we use all the notations of the theorem 2, but assume that $p=1$. In view of (3.3) we shall often use the function

$$Q(\lambda, a)\equiv\Phi(\lambda+a)+\exp\{-2a\lambda\}\Phi(\lambda-a).$$

We can rewrite the formula (3.4) in the form

$$\mathbf{P}\{\zeta\geqq r\}\sim\left[\prod_{i=1}^{\infty} Q\left(\frac{\sigma_i(r-d)}{\sigma^2}, a_i/\sigma_i\right)\right]\left(1-\Phi\left(\frac{r-d}{\sigma}\right)\right]. \quad (4.1)$$

The following example provides the most curious behavior of the large deviations.

Example 1. – Let $B>1$, $A>0$ be some constants and $\sigma_i\equiv B^{-i}$, $a_i\equiv AB^{-i}$. We show that

$$\mathbf{P}\{\zeta\geqq r\}\sim\chi\left(\frac{r-d}{\sigma^2}\right)\left(1-\Phi\left(\frac{r-d}{\sigma}\right)\right) \quad (4.2)$$

with the function

$$\chi(\lambda)\equiv\prod_{i=-\infty}^{\infty} Q(\lambda B^i, A),$$

which is logarithmically periodic with period B, *i.e.* for any λ we have $\chi(\lambda B)=\chi(\lambda)$.

Proof. – We have the following expression for the product from (4.1):

$$\prod_{i=1}^{\infty} Q\left(\frac{B^{-i}(r-d)}{\sigma^2}, A\right)=\chi\left(\frac{r-d}{\sigma^2}\right)\Big/\prod_{i=0}^{\infty} Q\left(\frac{B^{i}(r-d)}{\sigma^2}, A\right).$$

In order to obtain the estimate of the remainder term, we note that

$$\sum_{i=0}^{\infty}\log Q\left(\frac{B^i(r-d)}{\sigma^2}, A\right)\leqq\sum_{i=0}^{\infty}\left[Q\left(\frac{B^i(r-d)}{\sigma^2}, A\right)-1\right]$$
$$\leqq\sum_{i=0}^{\infty}\exp\left\{-2A\frac{B^i(r-d)}{\sigma^2}\right\}\to 0.\,(r\to\infty)$$

Hence,

$$\prod_{i=0}^{\infty} Q\left(\frac{B^i (r-d)}{\sigma^2}, A\right) <\sim 1.$$

The opposite estimate is the consequence of the following lemma.

LEMMA 1. – *For any* λ, $a \geqq 0$ *the inequality* $Q(\lambda, a) \geqq 1$ *holds.*

Proof. – The simple variable change gives

$$1-\Phi(\lambda+a)=(2\pi)^{-1/2}\int_{\lambda+a}^{\infty} \exp\{-u^2/2\}\,du=(2\pi)^{-1/2}$$
$$\times \int_{\lambda-a}^{\infty} \exp\{-v^2/2-2av-2a^2\}\,dv$$
$$\leqq (1-\Phi(\lambda-a))\exp\{-2a(\lambda-a)-2a^2\}=(1-\Phi(\lambda-a))\exp\{-2a\lambda\}.$$

Hence,

$$1 \leqq \Phi(\lambda+a)+(1-\Phi(\lambda-a))\exp\{-2a\lambda\}=Q(\lambda, a). \quad \blacksquare$$

We obtain from this lemma the relation

$$1 <\sim \prod_{i=0}^{\infty} Q\left(\frac{B^i (r-d)}{\sigma^2}, A\right)$$

that finishes the proof of (4.2). ■

The next example is close to the first one, but the polynomial factor appears in the asymptotics, and we have to deal with two periodical functions with different periods.

Example 2. – Let $B>1$, $\alpha>0$ be some constants and $\sigma_i \equiv B^{-i}$, $a_i \equiv \sigma_i^{1+\alpha}$. We show that

$$\mathbf{P}\{\zeta \geqq r\} \sim \chi_1\left(\frac{r-d}{\sigma^2}\right)\chi_2\left(\frac{r-d}{\sigma^2}\right)$$
$$\times \left(\frac{r-d}{\sigma^2}\right)^{(\alpha \log 2)/(1+\alpha)\log B}\left(1-\Phi\left(\frac{r-d}{\sigma}\right)\right) \quad (4.3)$$

with the function

$$\chi_1(\lambda) \equiv \prod_{i=1}^{\infty} [2\Phi(\lambda B^{-i})] \,.\, \prod_{i=0}^{\infty} [\Phi(\lambda B^i)]\, \lambda^{-(\log 2/\log B)}$$

which possesses logarithmical period B, and the function

$$\chi_2(\lambda) \equiv \prod_{i=1}^{\infty} [(1+\exp\{-2\lambda B^{-i(1+\alpha)}\})/2]$$
$$\times \prod_{i=0}^{\infty} [1+\exp\{-2\lambda B^{i(1+\alpha)}\}]\, \lambda^{\log 2/(1+\alpha)\log B},$$

which possesses logarithmical period $B^{1+\alpha}$.

Proof. – We have the following expression for the product from (4.1):

$$\prod_{i=1}^{\infty} Q(\lambda\sigma_i, \sigma_i^\alpha), \qquad \lambda \equiv \frac{r-d}{\sigma^2} \to \infty.$$

We fix some small β, $\varepsilon > 0$ and define two boards of indexes splitting the product into three factors. One of them turns out to be negligible, the second leads to χ_1 and the third one leads to χ_2. Let

$$i_1 \equiv \inf\{i : \sigma_i < \lambda^{-(1+\varepsilon)/(1+\alpha)}\}, \; i_2 \equiv \inf\{i : \sigma_i < \beta\lambda^{-1}\}.$$

It is easy to see that there exists small $\delta = \delta(\beta)$ such that $\delta \to 0$ when $\beta \to 0$ and

$$1 \leqq \prod_{i > i_2}^{\infty} Q(\lambda\sigma_i, \sigma_i^\alpha) \leqq \prod_{i > i_2}^{\infty} 2\Phi(\lambda\sigma_i + \sigma_i^\alpha) \leqq 1 + \delta.$$

Hence, the behavior of this part of our product does not influence on the asymptotics of the deviations. For the next couple, $i \in [i_1, i_2]$, we shall use the approximation $Q(\lambda\sigma_i, \sigma_i^\alpha) \cong 2\Phi(\lambda\sigma_i)$ and give the following estimate which is valid for large λ.

$$\sum_{i=i_1+1}^{i_2} |\log Q(\lambda\sigma_i, \sigma_i^\alpha) - \log(2\Phi(\lambda\sigma_i))| \leqq \sum_{i=i_1+1}^{i_2} |Q(\lambda\sigma_i, \sigma_i^\alpha) - 2\Phi(\lambda\sigma_i)|$$
$$\leqq 3\lambda^{-\varepsilon}(\log\lambda - \log\beta)/\log B \to 0$$

for $\lambda \to \infty$, β and ε fixed. Hence,

$$\prod_{i=i_1+1}^{i_2} [Q(\lambda\sigma_i, \sigma_i^\alpha)/2\Phi(\lambda\sigma_i)] \sim 1,$$

and we derive that

$$\prod_{i=i_1+1}^{\infty} Q(\lambda\sigma_i, \sigma_i^\alpha) \sim \prod_{i=i_1+1}^{\infty} [2\Phi(\lambda\sigma_i)] \sim 2^{-i_1}\chi_1(\lambda)\lambda^{\log 2/\log B}.$$

For the last couple of factors, $i \leqq i_1$, we use the approximation $Q(\lambda\sigma_i, \sigma_i^\alpha) \cong 1 + \exp\{-2\lambda\sigma_i^{1+\alpha}\}$ with the following estimate of its accuracy:

$$\sum_{i=1}^{i_1} |\log Q(\lambda\sigma_i, \sigma_i^\alpha) - \log(1 + \exp\{-2\lambda\sigma_i^{1+\alpha}\})|$$
$$\leqq 2 i_1 \exp\{-(\lambda^{(\alpha-\varepsilon)/(1+\alpha)} - 1)^2/2\}$$
$$\leqq 4\log\lambda(\log B)^{-1}\exp\{-(\lambda^{(\alpha-\varepsilon)/(1+\alpha)} - 1)^2/2\} \to 0$$

for $\lambda \to \infty$, ε fixed. This estimate leads us to the asymptotics

$$\prod_{i=1}^{i_1} Q(\lambda\sigma_i, \sigma_i^\alpha) \sim \prod_{i=1}^{i_1} (1 + \exp\{-2\lambda\sigma_i^{1+\alpha}\}) \sim \chi_2(\lambda) 2^{i_1} \lambda^{-\log 2/(1+\alpha)\log B}.$$

We join the obtained estimates, reduce the annihilating factors and see that

$$\prod_{i=1}^{\infty} Q(\lambda\sigma_i, \sigma_i^\alpha) \sim \chi_1(\lambda)\chi_2(\lambda)\lambda^{\alpha \log 2/(1+\alpha)\log B}.$$

Now (4.3) follows directly from (4.1). ■

This example finishes the demonstration of curious "periodical" effects. We mention now, omitting the proofs, some cases, where the asymptotic behavior is more regular.

Example 3. – Let $B>1$, $A>0$ be some constants, and $\sigma_i \equiv i^{-B}$, $a_i \equiv A\,\sigma_i$. We prove that in this case

$$P\{\xi \geqq r\} \sim \exp\left\{\beta I\left(\frac{r-d}{\sigma^2}\right)^\beta\right\}\left(1-\Phi\left(\frac{r-d}{\sigma}\right)\right) \tag{4.4}$$

with constants β and I defined as

$$\beta \equiv B^{-1}, \qquad I \equiv \int_0^\infty \frac{\log Q(z, A)}{z^{1+\beta}}\,dz.$$

Example 4. – Let $B>1$, $A>0$ be some constants and $\sigma_i \equiv i^{-B}$, $a_i \equiv i^{-B-A}$. Let n be the integral part of A^{-1}. Then there exist constants $c_0, c_1, \ldots, c_n$ such that

$$P\{\zeta \geqq r\} \sim \exp\left\{c_0(r-d)^{1/B} + c_1(r-d)^{1/(A+B)} + \sum_{k=2}^{n} c_k(r-d)^{(1-kA)/B}\right\} \left(1-\Phi\left(\frac{r-d}{\sigma}\right)\right). \tag{4.5}$$

We finish our examples by the proof of the statement, which shows, that the asymptotics of the large deviations is more simple, if it is mainly defined not by the covariance but by the shift of measure.

THEOREM 3. – *Let* $p=1$, $\sigma_i>0$ *for each* i *and* $\sum_{i=1}^{\infty}(1-\Phi(|a_i|/\sigma_i))<\infty$. *Then*

$$P\{\zeta \geqq r\} \sim 1-\Phi\left(\frac{r-d}{\sigma}\right). \tag{4.6}$$

Proof. – According to (4.1), it is enough to show that

$$\prod_{i=1}^{\infty} Q(\lambda\sigma_i, |a_i|/\sigma_i) \sim 1. \tag{4.7}$$

We can use the following estimates in order to check this relation.

$$0 \leqq \log Q(\lambda\sigma_i, |a_i|/\sigma_i) \leqq Q(\lambda\sigma_i, |a_i|/\sigma_i) - 1$$
$$\leqq \exp\{-2\lambda|a_i|\}\Phi(\lambda\sigma_i - |a_i|/\sigma_i)$$
$$= \exp\{-2\lambda|a_i|\}(1 - \Phi(|a_i|/\sigma_i - \lambda\sigma_i)).$$

In what follows, we consider separately two cases.

(a) If $\lambda\sigma_i \geqq |a_i|/\sigma_i$, we obtain the estimate

$$\log Q(\lambda\sigma_i, |a_i|/\sigma_i) \leqq \exp\{-2\lambda|a_i|\}/2 \leqq \exp\{-2|a_i|^2/\sigma_i^2\}/2$$
$$\leqq (1 - \Phi(3/2))(1 - \Phi(|a_i|/\sigma_i))/2.$$

(b) If $\lambda\sigma_i \leqq |a_i|/\sigma_i$ and $1 \leqq |a_i|/\sigma_i$, then

$$\exp\{-2\lambda|a_i|\}(1 - \Phi(|a_i|/\sigma_i - \lambda\sigma_i))$$
$$\leqq \exp\{-2\lambda|a_i|\}(1 - \Phi(|a_i|/\sigma_i) + (2\pi)^{-1/2}\lambda\sigma_i$$
$$\times \exp\{-(|a_i|/\sigma_i - \lambda\sigma_i)^2/2\})$$
$$\leqq 1 - \Phi(|a_i|/\sigma_i) + \exp\{-\lambda|a_i|\}(2\pi)^{-1/2}\lambda\sigma_i \exp\{-(|a_i|/\sigma_i)^2/2\}$$
$$\leqq (1 + 2\sup_y\{y\exp\{-y\}\}(1 - \Phi(|a_i|/\sigma_i)) \leqq 2(1 - \Phi(|a_i|/\sigma_i)).$$

It follows from the assumption of our theorem that for large indexes i the inequality $1 \leqq |a_i|/\sigma_i$ holds.

Hence, either (a) or (b) is valid, and we obtain the estimate

$$1 \leqq \prod_{i=n}^{\infty} Q(\lambda\sigma_i, |a_i|/\sigma_i) \leqq \exp\left\{ c \sum_{i=n}^{\infty} (1 - \Phi(|a_i|/\sigma_i)) \right\} \to 1 \qquad (n \to \infty).$$

It is sufficient for our purpose, since each finite product in (4.7) evidently tends to 1. ■

FINAL REMARK

The author is far from consideration of theorem 1 as a final word on the problem of Gaussian tail probabilities, in spite of its full generality and accuracy. In fact, the domain of its useful application is essentially limited by our capability to calculate the corresponding Laplace transform. Besides 1^p-norms discussed here, another domain would be mentioned – suprema of some processes related to the Brownian motion. In the latter case Laplace transform of the suprema can be found as solution of a partial differential equation. This subject will be considered elsewhere.

Therefore, the author sees the direct goal and the key-point of the article in the spectrum of *new types* of tail probabilities behavior (included, surely, in the general frame of theorem 1).