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Multiplicative chaos and random translation

by

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ABSTRACT. — Let $\mathbf{G} = \{G_k\}_{k \geq 1}$ be a standard Gaussian sequence and $\mathbf{Y} = \{Y_k\}_{k \geq 1}$ an independent non-negative random sequence which is also independent of \mathbf{G} . We shall analyse the conditions on \mathbf{Y} for the equivalence (=mutual absolute continuity) of the measures $\mu_{\mathbf{G}}$ and $\mu_{\mathbf{G}+\mathbf{Y}}$ on \mathbf{R}^∞ induced by \mathbf{G} and $\mathbf{G}+\mathbf{Y}$, respectively. This problem implies a typical example of the multiplicative chaos. In particular we shall analyse in detail the case where Y_k 's are two valued in view of the regularity of the multiplicative chaos and, as an application, give a negative answer to a conjecture of J.-P. Kahane on the regularity of the multiplicative chaos.

Key words : Multiplicative chaos, absolute continuity.

RÉSUMÉ. — Soit $\mathbf{G} = \{G_k\}_{k \geq 1}$ une suite de variables aléatoires gaussiennes et $\mathbf{Y} = \{Y_k\}_{k \geq 1}$ une suite de variables aléatoires non-négatives indépendantes qui soit aussi indépendante de \mathbf{G} . On donne les conditions sur \mathbf{Y} pour l'équivalence des mesures $\mu_{\mathbf{G}}$ et $\mu_{\mathbf{G}+\mathbf{Y}}$ sur \mathbf{R}^∞ induites par \mathbf{G} et $\mathbf{G}+\mathbf{Y}$, respectivement. Ce problème fournit un exemple typique du chaos multiplicatif. En particulier, on analyse en détail le cas où Y_k sont deux-valorées au point de vue de la régularité du chaos multiplicatif et, comme application, on donne une réponse négative à une des conjectures de J.-P. Kahane sur la régularité du chaos multiplicatif.

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1. INTRODUCTION

Let T be a compact metric space and $\{X_k(t, \omega)\}_{t \in T} (k \geq 1)$ an independent family of centered Gaussian processes on a probability space (Ω, \mathcal{F}, P) . For every $k \in \mathbb{N}$ we assume that

$$p_k(s, t) = E[X_k(s, \omega)X_k(t, \omega)] \geq 0 (s, t \in T)$$

and $X_k(t, \omega)$ is $(\Sigma \otimes \mathcal{F})$ -measurable, where Σ is the Borel field of T , and define

$$M_n(t, \omega) = \exp \left[\sum_1^n \left\{ X_k(t, \omega) - \frac{1}{2} p_k(t, t) \right\} \right]. \quad (1.1)$$

Then, for every fixed $t \in T$, $\{M_n(t, \omega)\}_{n \geq 1}$ is naturally a positive martingale.

For every $\sigma \in \mathcal{M}(T)$, the collection of all finite measures on (T, Σ) , define a random measure $M \sigma$ by

$$\int_T \phi(t) (M \sigma)(\omega, dt) = \lim_n \int_T M_n(t, \omega) \phi(t) \sigma(dt) \quad \text{a.s.},$$

for every continuous function ϕ on T . After Kahane [2] the above map $M; \sigma \in \mathcal{M}(T) \rightarrow M \sigma$ is called a *multiplicative chaos*, and $\sigma \in \mathcal{M}(T)$ is said to be *M-regular* or *M-singular* according as $E[(M \sigma)(T)] = \sigma(T)$ or 0.

When T is the d -dimensional torus, σ is the Lebesgue measure and the covariance functions satisfy

$$\sum_k p_k(s, t) = \xi \log \frac{1}{\|s - t\|} + O(1)$$

for some $\xi > 0$, Kahane [2] proved that σ is *M-regular* if $\xi < 2d$ and *M-singular* if $\xi \geq 2d$. In other words σ is *M-regular* if and only if

$$\int_T \int_T \exp \left[\frac{1}{2} \sum_k p_k(s, t) \right] \sigma(ds) \sigma(dt) < \infty.$$

We should remark that this result implies the complete solution to the problem of absolute continuity of measures in the 2-space time dimensional Høegh-Krohn's model of quantum fields, which has been investigated by many authors (Høegh-Krohn [1], Kusuoka [7] and its references).

For $\sigma \in \mathcal{M}(T)$ and $u \geq 0$ define

$$I(u; \sigma) = \int_T \int_T \exp \left[\frac{u}{2} \sum_k p_k(s, t) \right] \sigma(ds) \sigma(dt) \leq \infty.$$

Then Kahane posed the following conjecture.

Conjecture (Kahane [4]). — Let M be a multiplicative chaos. Then $\sigma \in \mathcal{M}(T)$ is M -regular if and only if σ is expressed as a sum $\sum_n \sigma_n$ (convergence in total variation) of $\sigma_n \in \mathcal{M}(T)$ such that $I(1; \sigma_n) < \infty$.

On the other hand let $X = \{X_k(\omega)\}_{k \geq 1}$ be an *i.i.d.* random sequence defined on (Ω, \mathcal{F}, P) , $\sigma \in \mathcal{M}(T)$ a probability measure on (T, Σ) and $Y = \{Y_k(t)\}_{k \geq 1}$ an independent random sequence defined on (T, Σ, σ) . Then $X + Y = \{X_k(\omega) + Y_k(t)\}_{k \geq 1}$ is defined on the product probability space $(\Omega \times T, \mathcal{F} \otimes \Sigma, P \otimes \sigma)$ and X and Y are independent. The authors [6, 8, 9] investigated the problem of the equivalence of the probability measures μ_X and μ_{X+Y} on \mathbf{R}^∞ induced by X and $X + Y$, respectively.

In particular let $G = \{G_k(\omega)\}_{k \geq 1}$ be a standard Gaussian sequence on (Ω, \mathcal{F}, P) , $Y = \{Y_k(t)\}_{k \geq 1}$ an independent non-negative random sequence on (T, Σ, σ) and

$$M_n(t, \omega) = \exp \left[\sum_1^n \left\{ G_k(\omega) Y_k(t) - \frac{1}{2} Y_k(t)^2 \right\} \right]. \tag{1.2}$$

Then (1.2) defines a multiplicative chaos M for $X_k(t, \omega) = Y_k(t) G_k(\omega)$ and we have

$$I(u; \sigma) = \int_T \int_T \exp \left[\frac{u}{2} \sum_k Y_k(s) Y_k(t) \right] \sigma(ds) \sigma(dt).$$

On the other hand μ_G and μ_{G+Y} are equivalent ($\mu_G \sim \mu_{G+Y}$) or singular ($\mu_G \perp \mu_{G+Y}$) according as σ is M -regular or M -singular. Owed to the well known Kakutani dichotomy [5] we have either $\mu_G \sim \mu_{G+Y}$ or $\mu_G \perp \mu_{G+Y}$. To characterize the equivalence of μ_G and μ_{G+Y} is our first aim. In Section 2 we shall prove the following theorem.

THEOREM 1.

$$\sum_k \sigma(Y_k > \varepsilon) < \infty$$

and

$$\sum_k \mathbf{E}_\sigma[Y_k; Y_k \leq \varepsilon]^2 < \infty \tag{1.3}$$

for some $\varepsilon > 0$ imply $\mu_G \sim \mu_{G+Y}$.

Conversely $\mu_G \sim \mu_{G+Y}$ implies

$$\sum_k \sigma(Y_k > \varepsilon)^2 < \infty$$

and (1.3) for all $\varepsilon > 0$.

As a corollary we obtain a positive answer to the conjecture in the case where $\sup_k Y_k \leq L$ σ -a.s. for some $L \geq 0$ (Proposition 1).

In particular we analyse, in Section 3, when $Y = \{ \varepsilon(a_k, p_k) \}_{k \geq 1}$ is an independent random sequence with distributions

$$\sigma(\varepsilon(a_k, p_k) = a_k) = p_k, \quad \sigma(\varepsilon(a_k, p_k) = 0) = 1 - p_k, \quad (1.4)$$

where $a_k > 0$ and $0 < p_k < 1$ for every $k \in \mathbb{N}$. Define

$$\alpha_k = \frac{1}{a_k^2} \log \frac{1 + p_k}{p_k}.$$

Then, relating to the Kahane's conjecture, we shall prove:

THEOREM 2. - (a) Assume $\sup_k a_k < \infty$. Then $I(u; \sigma) < \infty$ for some $u > 0$ implies $\mu_G \sim \mu_{G+Y}$. Conversely, $\mu_G \sim \mu_{G+Y}$ implies $I(u; \sigma) < \infty$ for every $u > 0$. Consequently we have $\mu_G \sim \mu_{G+Y}$ if and only if $I(1; \sigma) < \infty$.

(b) Assume $\sup_k a_k = \infty$, and define $\underline{\alpha} = \liminf_{\{k; a_k > 1\}} \alpha_k$ and $\bar{\alpha} = \limsup_{\{k; a_k > 1\}} \alpha_k$.

(b-i) Assume $\underline{\alpha} > \frac{3}{2}$. Then $\mu_G \sim \mu_{G+Y}$ if and only if $I(2; \sigma) < \infty$.

(b-ii) Assume $\frac{1}{2} \leq \underline{\alpha} \leq \bar{\alpha} \leq \frac{3}{2}$. Then $I(2; \sigma) < \infty$ implies $\mu_G \sim \mu_{G+Y}$. Conversely, $\mu_G \sim \mu_{G+Y}$ implies $I(1 - \varepsilon; \sigma) < \infty$ for every $0 < \varepsilon \leq 1$.

(b-iii) Assume $\bar{\alpha} < \frac{1}{2}$. Then $I(2\bar{\alpha} + \varepsilon; \sigma) < \infty$ for some $\varepsilon > 0$ implies $\mu_G \sim \mu_{G+Y}$. In particular $I(1; \sigma) < \infty$ implies $\mu_G \sim \mu_{G+Y}$. Conversely $\mu_G \sim \mu_{G+Y}$ implies $I((2\underline{\alpha} - \varepsilon)_+; \sigma) < \infty$ for every $\varepsilon > 0$, where a_+ denotes $\max(a, 0)$.

More precisely we shall analyse the case (b-ii). Define

$$\theta = \sup \{ u \geq 0; I(u; \sigma) < \infty \}$$

and

$$\lambda(x) = 2 - \left(\frac{3}{2} - x \right)^2, \quad x \geq 0.$$

Then we shall prove:

THEOREM 3. - Assume $\sup_k a_k = \infty$ and $\frac{1}{2} \leq \underline{\alpha} = \bar{\alpha} = \alpha \leq \frac{3}{2}$. Then $\lambda(\alpha) < \theta$ implies $\mu_G \sim \mu_{G+Y}$. Conversely $\lambda(\alpha) > \theta$ implies $\mu_G \perp \mu_{G+Y}$.

As an application we shall give a negative answer to the Kahane's conjecture by giving examples in Section 4.

2. GENERAL CASE

Let $\mathbf{G} = \{G_k(\omega)\}_{k \geq 1}$ be a standard Gaussian sequence defined on (Ω, \mathcal{F}, P) , $\sigma \in \mathcal{M}(T)$ a probability measure and $\mathbf{Y} = \{Y_k(t)\}_{k \geq 1}$ an independent non-negative random sequence defined on (T, Σ, σ) . Define

$$g(x) = \frac{1}{\sqrt{2\pi}} \exp\left[-\frac{x^2}{2}\right], \quad x \in \mathbf{R}$$

and

$$Z_k(x) = \int_T \exp\left[x Y_k(t) - \frac{1}{2} Y_k(t)^2\right] \sigma(dt) - 1, \quad x \in \mathbf{R}, k \in \mathbf{N}. \quad (2.1)$$

Then the following theorem is our starting point.

THEOREM 4. [6, Theorem 2]. — *The next four statements are equivalent.*

(a) σ is M -regular, where M is the multiplicative chaos defined by (1.2).

(b) $\mu_{\mathbf{G}} \sim \mu_{\mathbf{G}+\mathbf{Y}}$.

(c) $\sum_k Z_k(\mathbf{G}_k)$ converges almost surely.

(d) For some, so that any, $K (\geq 1)$

(d-1) $\sum_k \mathbf{E}[Z_k(\mathbf{G}_k); Z_k(\mathbf{G}_k) > K] < \infty$

and

(d-2) $\sum_k \mathbf{E}[Z_k(\mathbf{G}_k)^2; Z_k(\mathbf{G}_k) \leq K] < \infty$.

Proof of Theorem 1. — For any $\varepsilon > 0$ decompose $Z_k(\mathbf{G}_k)$ into

$$\begin{aligned} Z_k(\mathbf{G}_k) &= \mathbf{E}_\sigma \left[\exp \left[G_k Y_k - \frac{1}{2} Y_k^2 \right] - 1; Y_k > \varepsilon \right] \\ &\quad + \mathbf{E}_\sigma \left[\exp \left[G_k Y_k - \frac{1}{2} Y_k^2 \right] - 1; Y_k \leq \varepsilon \right] \\ &= V_\varepsilon(\mathbf{G}_k) + W_\varepsilon(\mathbf{G}_k), \end{aligned}$$

where \mathbf{E}_σ denotes the expectation with respect to σ . Then, from the same arguments as in [6, Lemma 1, Theorem 4] and [8, Theorem 3.2 (B)], $\sum_k \sigma(Y_k > \varepsilon) < \infty$ implies the almost sure absolute convergence of $\sum_k V_\varepsilon(\mathbf{G}_k)$, and (1.3) implies the L^2 -convergence, therefore the almost sure convergence, of $\sum_k W_\varepsilon(\mathbf{G}_k)$. Thus we obtain the sufficiency.

Conversely assume the almost sure convergence of $\sum_k Z_k(\mathbf{G}_k)$. Then, since $Y_k \geq 0$, $Z_k(x)$ is increasing and continuous in x , and

$Z_k(1) < \exp\left[\frac{1}{2}\right]$, Theorem 4 (d-2) implies

$$\begin{aligned} \infty &> \sum_k \mathbf{E} \left[Z_k(G_k)^2; Z_k(G_k) \leq \exp\left[\frac{1}{2}\right] \right] \\ &\geq \sum_k \int_0^1 Z_k(x)^2 g(x) dx \geq \int_0^1 g(x) dx \sum_k Z_k(0)^2. \end{aligned}$$

Therefore we have $\sum_k Z_k(0)^2 = \sum_k \mathbf{E}_\sigma \left[1 - \exp\left[-\frac{1}{2} Y_k^2\right] \right]^2 < \infty$, which implies $\sum_k \sigma(Y_k > \varepsilon)^2 < \infty$ for every $\varepsilon > 0$.

Next we shall prove (1.3). For any $\varepsilon > 0$ we have $Z_k(x) < \exp\left[\frac{9}{2}\varepsilon^2\right]$ for every $x \in [0, 3\varepsilon]$. Then Theorem 4 (d-2) implies

$$\begin{aligned} \infty &> \sum_k \mathbf{E} \left[Z_k(G_k)^2; Z_k(G_k) \leq \exp\left[\frac{9}{2}\varepsilon^2\right] \right] \\ &\geq \sum_k \int_{2\varepsilon}^{3\varepsilon} Z_k(x)^2 g(x) dx \\ &= \frac{1}{\sqrt{2\pi}} \sum_k \int_{2\varepsilon}^{3\varepsilon} \mathbf{E}_\sigma \left[\exp\left[-\frac{(x-Y_k)^2}{2}\right] - \exp\left[-\frac{x^2}{2}\right] \right]^2 \exp\left[\frac{1}{2}x^2\right] dx. \end{aligned}$$

Since $\sum_k \sigma(Y_k > \varepsilon)^2 < \infty$, we have

$$\begin{aligned} \infty &> \sum_k \int_{2\varepsilon}^{3\varepsilon} \mathbf{E}_\sigma \left[\exp\left[-\frac{(x-Y_k)^2}{2}\right] - \exp\left[-\frac{x^2}{2}\right]; Y_k \leq \varepsilon \right]^2 \exp\left[\frac{1}{2}x^2\right] dx \\ &= \sum_k \int_{2\varepsilon}^{3\varepsilon} \mathbf{E}_\sigma \left[\int_0^1 (x-tY_k) \exp\left[-\frac{(x-tY_k)^2}{2}\right] dt; Y_k \leq \varepsilon \right]^2 \exp\left[\frac{1}{2}x^2\right] dx \\ &\geq \varepsilon^2 \exp[-9\varepsilon^2] \int_{2\varepsilon}^{3\varepsilon} \exp\left[\frac{1}{2}x^2\right] dx \sum_k \mathbf{E}_\sigma[Y_k; Y_k \leq \varepsilon]^2, \end{aligned}$$

which completes the proof of Theorem 1. \square

PROPOSITION 1. — (a) Assume $\sup_k Y_k < \infty$, σ -a.s.. Then $I(1; \sigma) < \infty$ implies $\mu_{\mathbf{G}} \sim \mu_{\mathbf{G}+\mathbf{Y}}$.

(b) Assume $\sup_k Y_k < L$, σ -a.s. for some $L > 0$. Then $\mu_{\mathbf{G}} \sim \mu_{\mathbf{G}+\mathbf{Y}}$ if and only if $I(1; \sigma) < \infty$.

Proof. — Since $\{Y_k(t)\}_{k \geq 1}$ is an independent random sequence, we have

$$I(1; \sigma) = \prod_k \int_T \int_T \exp \left[\frac{1}{2} Y_k(t) Y_k(s) \right] \sigma(ds) \sigma(dt),$$

and thus we have $I(1; \sigma) < \infty$ if and only if

$$\begin{aligned} & \sum_k \left\{ \int_T \int_T \exp \left[\frac{1}{2} Y_k(t) Y_k(s) \right] \sigma(ds) \sigma(dt) - 1 \right\} \\ &= \frac{1}{2} \sum_k \int_T Y_k(t) \sigma(dt) \int_T Y_k(s) \sigma(ds) \int_0^1 \exp \left[\frac{1}{2} Y_k(t) Y_k(s) x \right] dx < \infty. \end{aligned}$$

(a) Assume $\sup_k Y_k < \infty$, σ -a.s. and $I(1; \sigma) < \infty$. Then we have $\sum_k \sigma(Y_k > L) < \infty$ for some $L > 0$ so that $\sum_k \mathbf{E}_\sigma[Y_k : Y_k \leq L]^2 < \infty$ implies the almost sure convergence of $\sum_k Z_k(G_k)$ by Theorem 1. In fact, since $Y_k \geq 0$, $I(1; \sigma) < \infty$ implies

$$\begin{aligned} \infty > \sum_k \int_T Y_k(t) \sigma(dt) \int_T Y_k(s) \sigma(ds) \int_0^1 \\ \times \exp \left[\frac{1}{2} Y_k(t) Y_k(s) x \right] dx \geq \sum_k \mathbf{E}_\sigma[Y_k : Y_k \leq L]^2. \end{aligned}$$

(b) Assume $\sup_k Y_k < L$, σ -a.s. for some $L > 0$. Then we obtain “if” part by (a). Conversely assume the almost sure convergence of $\sum_k Z_k(G_k)$. Then, by Theorem 1, we have $\sum_k \mathbf{E}_\sigma[Y_k]^2 = \sum_k \mathbf{E}_\sigma[Y_k : Y_k \leq L]^2 < \infty$, so that

$$\begin{aligned} \sum_k \int_T Y_k(t) \sigma(dt) \int_T Y_k(s) \sigma(ds) \int_0^1 \exp \left[\frac{1}{2} Y_k(t) Y_k(s) x \right] dx \\ \leq \exp \left[\frac{1}{2} L^2 \right] \sum_k \mathbf{E}_\sigma[Y_k]^2 < \infty, \end{aligned}$$

which proves (b). \square

In Section 4 (4.3) we shall give an example that $\sup_k Y_k < \infty$, σ -a.s. and $\mu_G \sim \mu_{G+Y}$ do not imply $I(1; \sigma) < \infty$.

3. TWO-VALUED CASE

In this section we consider the case $\mathbf{Y} = \{\varepsilon(a_k, p_k)\}$. By definition (1.4) and (2.1) we have

$$Z_k(x) = p_k \left(\exp \left[a_k x - \frac{1}{2} a_k^2 \right] - 1 \right), \quad k \in \mathbf{N}, x \in \mathbf{R}.$$

By Theorem 4, M-regularity of σ , the equivalence of $\mu_{\mathbf{G}}$ and $\mu_{\mathbf{G}+\mathbf{Y}}$ and the almost sure convergence of $\sum_k Z_k(\mathbf{G}_k)$ are equivalent. Relating to the conjecture, we shall characterize them in terms of

$$I(u; \sigma) = \prod_k \left\{ 1 + p_k^2 \left(\exp \left[\frac{u}{2} a_k^2 \right] - 1 \right) \right\}.$$

First we shall prove the following.

PROPOSITION 2. — (a) $\sum_k p_k < \infty$ implies the almost sure absolute convergence of $\sum_k Z_k(\mathbf{G}_k)$.

(b) $I(2; \sigma) < \infty$ implies the almost sure convergence of $\sum_k Z_k(\mathbf{G}_k)$.

Proof. — (a) Assume $\sum_k p_k < \infty$. Then

$$\mathbf{E} \left[\sum_k |Z_k(\mathbf{G}_k)| \right] = \sum_k p_k \int_{-\infty}^{\infty} \left| \exp \left[a_k x - \frac{1}{2} a_k^2 \right] - 1 \right| g(x) dx \leq 2 \sum_k p_k < \infty,$$

which proves (a).

(b) Assume $I(2; \sigma) < \infty$. Since $\{Z_k(\mathbf{G}_k)\}_{k \geq 1}$ is a sequence of independent random variables with mean 0, $\sum_k \mathbf{E}[Z_k(\mathbf{G}_k)^2] < \infty$ implies the L^2 -convergence, consequently the almost sure convergence, of $\sum_k Z_k(\mathbf{G}_k)$. In fact we have

$$\begin{aligned} \sum_k \mathbf{E}[Z_k(\mathbf{G}_k)^2] &= \sum_k p_k^2 \int_{-\infty}^{\infty} \left(\exp \left[a_k x - \frac{1}{2} a_k^2 \right] - 1 \right)^2 g(x) dx \\ &= \sum_k p_k^2 (\exp[a_k^2] - 1) < \infty. \quad \square \end{aligned}$$

Decompose \mathbf{N} into

$$\mathbf{N} = \{k \geq 1; a_k \leq 1\} \cup \{k \geq 1; a_k > 1\} = \mathcal{N}_1 \cup \mathcal{N}_2.$$

Remark 1. – We have $\sum_{k \in \mathcal{N}_1} p_k^2 a_k^2 < \infty$ if and only if

$$\sum_{k \in \mathcal{N}_1} p_k^2 \left(\exp \left[\frac{u}{2} a_k^2 \right] - 1 \right) < \infty$$

for some, so that any, $u > 0$.

The next lemma is immediately derived from Proposition 2 and Remark 1.

LEMMA 1. – (a) $\sum_{k \in \mathcal{N}_2} p_k < \infty$ implies the almost sure convergence of $\sum_{k \in \mathcal{N}_2} Z_k(G_k)$.

(b) $\sum_{k \in \mathcal{N}_1} p_k^2 a_k^2 < \infty$ implies the almost sure convergence of $\sum_{k \in \mathcal{N}_1} Z_k(G_k)$.

The following lemma plays a central role in our discussion.

LEMMA 2. – $\sum_k Z_k(G_k)$ converges almost surely if and only if

$$\sum_{k \in \mathcal{N}_1} p_k^2 a_k^2 < \infty, \tag{3.1}$$

$$\sum_{k \in \mathcal{N}_2} p_k^2 < \infty, \tag{3.2}$$

$$\sum_{k \in \mathcal{N}_2} p_k \int_{a_k(\alpha_k - (1/2))}^{a_k(\alpha_k + (1/2))} g(x) dx < \infty \tag{3.3}$$

and

$$\sum_{k \in \mathcal{N}_2} p_k^2 \exp[a_k^2] \int_{a_k((3/2) - \alpha_k)}^{\infty} g(x) dx < \infty. \tag{3.4}$$

Proof. – Assume the almost sure convergence of $\sum_k Z_k(G_k)$. Then, by Theorem 1, we have

$$\infty > \sum_k \mathbf{E}_\sigma [\varepsilon(a_k, p_k) : \varepsilon(a_k, p_k) \leq 1]^2 = \sum_{k \in \mathcal{N}_1} p_k^2 a_k^2,$$

$$\infty > \sum_k \sigma(\varepsilon(a_k, p_k) > 1)^2 = \sum_{k \in \mathcal{N}_2} p_k^2,$$

which proves (3.1) and (3.2).

Since $Z_k(x)$ is strictly increasing and $Z_k\left(a_k\left(\alpha_k + \frac{1}{2}\right)\right) = 1$, we have by Theorem 4 (d-1) and (d-2)

$$\begin{aligned} \infty > \sum_k \mathbf{E}[Z_k(G_k); Z_k(G_k) > 1] &= \sum_{k \in \mathcal{N}_2} p_k \int_{a_k(\alpha_k + (1/2))}^{\infty} \\ &\times \left(\exp\left[a_k x - \frac{1}{2} a_k^2 \right] - 1 \right) g(x) dx \\ &= \sum_{k \in \mathcal{N}_2} p_k \int_{a_k(\alpha_k - (1/2))}^{a_k(\alpha_k + (1/2))} g(x) dx, \end{aligned}$$

which proves (3.3), and

$$\begin{aligned} \infty > \sum_k \mathbf{E}[Z_k(G_k)^2; Z_k(G_k) \leq 1] &= \sum_{k \in \mathcal{N}_2} p_k^2 \int_{-\infty}^{a_k(\alpha_k + (1/2))} \\ &\times \left(\exp\left[a_k x - \frac{1}{2} a_k^2 \right] - 1 \right)^2 g(x) dx \\ &= \sum_{k \in \mathcal{N}_2} p_k^2 \left\{ \exp[a_k^2] \int_{a_k((3/2) - \alpha_k)}^{\infty} g(x) dx - 2 \right. \\ &\times \left. \int_{a_k((1/2) - \alpha_k)}^{\infty} g(x) dx + \int_{-a_k((1/2) + \alpha_k)}^{\infty} g(x) dx \right\} \\ &\geq \sum_{k \in \mathcal{N}_2} p_k^2 \left\{ \exp[a_k^2] \int_{a_k((3/2) - \alpha_k)}^{\infty} g(x) dx - 2 \right\}, \end{aligned}$$

thus, by (3.2), this proves (3.4).

Conversely (3.1) implies the almost sure convergence of $\sum_{k \in \mathcal{N}_1} Z_k(G_k)$ by Lemma 1 (b). On the other hand, by Theorem 4, (3.2), (3.3) and (3.4) implies the almost sure convergence of $\sum_{k \in \mathcal{N}_2} Z_k(G_k)$, which completes the proof. \square

Remark 2. – We have proved Lemma 2 for a decomposition of \mathbf{N} according as $a_k \leq 1$ or $a_k > 1$. But it is not difficult to show that Lemma 2 is true for any decomposition of \mathbf{N} according as $a_k \leq \varepsilon$ or $a_k > \varepsilon$, where ε is an arbitrary positive number.

Remark 3. – Since the series (3.1)~(3.4) are of positive terms, for any decomposition $N = \bigcup_{j=1}^n \mathcal{X}_j$, $\sum_k Z_k(G_k)$ converges almost surely if and only if $\sum_{k \in \mathcal{X}_j} Z_k(G_k)$, $j=1, 2, \dots, n$, separately converge almost surely.

PROPOSITION 3. – (a) If $\alpha_k > -\frac{1}{a_k} + \frac{3}{2}$ for every $k \in \mathcal{N}_2$, then $\sum_k Z_k(G_k)$ converges almost surely if and only if $I(2; \sigma) < \infty$.

(b) If $\alpha_k \leq \frac{1}{2}$ for every $k \in \mathcal{N}_2$, then $\sum_k Z_k(G_k)$ converges almost surely if and only if (3.1) and $\sum_{k \in \mathcal{N}_2} p_k < \infty$.

Proof. – (a) $I(2; \sigma) < \infty$ implies the almost sure convergence of $\sum_k Z_k(G_k)$ by Proposition 2(b).

Conversely assume $\alpha_k > -\frac{1}{a_k} + \frac{3}{2}$ for every $k \in \mathcal{N}_2$ and the almost sure convergence of $\sum_k Z_k(G_k)$. Then we have by Lemma 2

$$\infty > \sum_{k \in \mathcal{N}_2} p_k^2 \exp[a_k^2] \int_{a_k((3/2)-\alpha_k)}^{\infty} g(x) dx \geq \int_1^{\infty} g(x) dx \sum_{k \in \mathcal{N}_2} p_k^2 \exp[a_k^2].$$

On the other hand we have by Lemma 2(3.1) and Remark 1

$$\sum_{k \in \mathcal{N}_1} p_k^2 (\exp[a_k^2] - 1) < \infty.$$

Therefore we have $\sum_k p_k^2 (\exp[a_k^2] - 1) < \infty$ and, consequently, $I(2; \sigma) < \infty$.

(b) By Lemma 1, (3.1) and $\sum_{k \in \mathcal{N}_2} p_k < \infty$ imply the almost sure convergence of $\sum_k Z_k(G_k)$.

Conversely assume the almost sure convergence of $\sum_k Z_k(G_k)$. Since

$\alpha_k - \frac{1}{2} \leq 0$ and $a_k \left(\alpha_k + \frac{1}{2} \right) \geq \frac{1}{2}$ for every $k \in \mathcal{N}_2$, we have, by Lemma 2

$$\infty > \sum_{k \in \mathcal{N}_2} p_k \int_{a_k(\alpha_k - (1/2))}^{a_k(\alpha_k + (1/2))} g(x) dx \geq \int_0^{1/2} g(x) dx \sum_{k \in \mathcal{N}_2} p_k,$$

consequently $\sum_{k \in \mathcal{N}_2} p_k < \infty$. Then Lemma 2 completes the proof of (b). \square

Proof of theorem 2. – (a) is proved by Proposition 1 (b).

(b) Assume $\sup_k a_k = \infty$.

(b-i) Assume $\underline{\alpha} > \frac{3}{2}$. Then $\alpha_k > \frac{3}{2}$ for large $k \in \mathbb{N}$, so that Proposition 3 (a) and Theorem 4 prove (b-i).

(b-ii) Assume $\frac{1}{2} \leq \underline{\alpha} \leq \bar{\alpha} \leq \frac{3}{2}$.

$I(2; \sigma) < \infty$ implies $\mu_{\mathbf{G}} \sim \mu_{\mathbf{G}+\mathbf{Y}}$ by Proposition 2 (b) and Theorem 4.

Conversely assume $\mu_{\mathbf{G}} \sim \mu_{\mathbf{G}+\mathbf{Y}}$ and fix any $0 < \varepsilon \leq 1$. Then, by Theorem 4, $\sum_k Z_k(\mathbf{G}_k)$ converges almost surely and we have (3.2) by Lemma 2. Choose $\tau > 0$ such that $\tau < \sqrt{1+\varepsilon} - 1$ and also choose $k_0 \in \mathcal{N}_2$ such that $k \geq k_0, k \in \mathcal{N}_2$ implies $\alpha_k \geq \frac{1}{2} - \tau$. Then we have by Theorem 4 and Lemma 2

$$\begin{aligned} \infty &> \sum_{k \geq k_0, k \in \mathcal{N}_2} p_k^2 \exp[a_k^2] \int_{a_k((3/2)-\alpha_k)}^{\infty} g(x) dx \\ &\geq \sum_{k \geq k_0, k \in \mathcal{N}_2} p_k^2 \exp[a_k^2] \int_{(1+\tau)a_k}^{\sqrt{1+\varepsilon}a_k} g(x) dx \\ &\geq \frac{\sqrt{1+\varepsilon} - (1+\tau)}{\sqrt{2\pi}} \sum_{k \geq k_0, k \in \mathcal{N}_2} p_k^2 \exp\left[\frac{1-\varepsilon}{2} a_k^2\right], \end{aligned}$$

which proves (b-ii).

(b-iii) Assume $\bar{\alpha} < \frac{1}{2}$. Then $\alpha_k < \frac{1}{2}$ for large $k \in \mathbb{N}$, thus we have $\mu_{\mathbf{G}} \sim \mu_{\mathbf{G}+\mathbf{Y}}$ if and only if (3.1) and $\sum_{k \in \mathcal{N}_2} p_k < \infty$ by Proposition 3 (b) and

Theorem 4. On the other hand, for any $\varepsilon > 0$, we may choose $k_0 \in \mathcal{N}_2$, by definition, such that $k \geq k_0, k \in \mathcal{N}_2$ implies

$$\left(\underline{\alpha} - \frac{1}{2}\varepsilon\right)_+ \leq \frac{1}{a_k^2} \log \frac{1+p_k}{p_k} \leq \bar{\alpha} + \frac{1}{2}\varepsilon.$$

It is easy to check

$$\frac{1}{2} p_k^2 \exp\left[\left(\underline{\alpha} - \frac{1}{2}\varepsilon\right)_+ a_k^2\right] \leq p_k \leq p_k^2 \exp\left[\left(\bar{\alpha} + \frac{1}{2}\varepsilon\right) a_k^2\right]$$

for $k \geq k_0, k \in \mathcal{N}_2$, which proves (b-iii). \square

For the proof of Theorem 3 we shall give the next lemma. Define

$$\lambda'(x, y) = 2x - \left(y - \frac{1}{2}\right)^2, \quad x, y \geq 0,$$

and note that $\lambda'(x, x) = \lambda(x)$ for every $x \geq 0$.

LEMMA 3. — Assume $\limsup_k a_k > 1$, (3.1), (3.2) and $\frac{1}{2} < \alpha_k \leq -\frac{1}{a_k} + \frac{3}{2}$

for every $k \in \mathcal{N}_2$.

- (a) (i) If $\lambda'(\bar{\alpha}, \underline{\alpha}) < \theta$, then we have (3.3).
 (ii) If $\lambda'(\underline{\alpha}, \bar{\alpha}) > \theta$, then (3.3) does not hold.
 (b) (i) If $\lambda(\bar{\alpha}) < \theta$, then we have (3.4).
 (ii) If $\lambda(\underline{\alpha}) > \theta$, then (3.4) does not hold.

Proof. — Before proving the lemma we shall remark that $\frac{1}{2} \leq \alpha \leq \bar{\alpha} \leq \frac{3}{2}$, $0 \leq \lambda'(\underline{\alpha}, \bar{\alpha}) \leq \lambda'(\bar{\alpha}, \underline{\alpha}) \leq 2$ and $1 \leq \lambda(\underline{\alpha}) \leq \lambda(\bar{\alpha}) \leq 2$, and that (3.2) implies $\lim_{k \in \mathcal{N}_2} p_k = 0$, consequently,

$$\underline{\alpha} = \liminf_{k \in \mathcal{N}_2} \alpha_k = \liminf_{k \in \mathcal{N}_2} \frac{1}{a_k^2} (-\log p_k).$$

Therefore, for any $0 < \delta < \frac{1}{2}$, we may choose $k(\delta) \in \mathcal{N}_2$ such that

$$\frac{1}{\bar{\alpha} + \delta} \log \frac{1 + p_k}{p_k} \leq a_k^2 \leq \frac{1}{\underline{\alpha} - \delta} (-\log p_k) \quad (3.5)$$

for every $k \geq k(\delta)$, $k \in \mathcal{N}_2$.

On the other hand (3.1) implies

$$\sum_{k \in \mathcal{N}_1} p_k^2 \left(\exp \left[\frac{u}{2} a_k^2 \right] - 1 \right) < \infty$$

for every $u > 0$ and we have by (3.2)

$$\theta = \sup \left\{ u \geq 0; \sum_{k \in \mathcal{N}_2} p_k^2 \exp \left[\frac{u}{2} a_k^2 \right] < \infty \right\}. \quad (3.6)$$

(a-i) Assume $\lambda'(\bar{\alpha}, \underline{\alpha}) < \theta$ and fix any $u, u_0 \geq 0$ such that $\lambda'(\bar{\alpha}, \underline{\alpha}) < u_0 < u < \theta$. Then, by (3.6), we have $\sum_{k \in \mathcal{N}_2} p_k^2 \exp\left[\frac{u}{2} a_k^2\right] < \infty$.

According as $\underline{\alpha} > \frac{1}{2}$ or $\underline{\alpha} = \frac{1}{2}$, choose $0 < \delta < \frac{1}{2}$ such that $\lambda'(\bar{\alpha} + \delta, \underline{\alpha} - \delta) \leq u_0$

and $\underline{\alpha} \geq \frac{1}{2} + \delta$ or $\lambda'(\bar{\alpha} + \delta, \frac{1}{2}) \leq u_0$. Then we have

$$\frac{(\underline{\alpha} - \delta - (1/2))_+^2 + u_0}{2(\bar{\alpha} + \delta)} - 1 \geq 0,$$

$$a_k\left(\alpha_k - \frac{1}{2}\right) = \left(\frac{1}{a_k^2} \log \frac{1+p_k}{p_k} - \frac{1}{2}\right) a_k \geq \left(\underline{\alpha} - \delta - \frac{1}{2}\right)_+ a_k$$

and

$$\frac{1}{2} a_k^2 \geq \frac{1}{2(\bar{\alpha} + \delta)} (-\log p_k)$$

for every $k \geq k(\delta), k \in \mathcal{N}_2$. Consequently we have

$$\begin{aligned} & \sum_{k \geq k(\delta), k \in \mathcal{N}_2} p_k \int_{a_k(\alpha_k - (1/2))}^{a_k(\alpha_k + (1/2))} g(x) dx \\ & \leq \sum_{k \geq k(\delta), k \in \mathcal{N}_2} a_k p_k g\left(a_k\left(\alpha_k - \frac{1}{2}\right)\right) \\ & \leq \frac{1}{\sqrt{2\pi}} \sum_{k \geq k(\delta), k \in \mathcal{N}_2} a_k p_k \exp\left[-\frac{a_k^2}{2} \left(\underline{\alpha} - \delta - \frac{1}{2}\right)_+^2\right] \\ & \leq \frac{1}{\sqrt{2\pi}} \sum_{k \geq k(\delta), k \in \mathcal{N}_2} a_k p_k^2 \exp\left[\frac{u_0}{2} a_k^2\right] p_k^{u_0 + (\underline{\alpha} - \delta - (1/2))_+^2 / 2 (\bar{\alpha} + \delta) - 1} \\ & \leq \frac{1}{\sqrt{2\pi}} \sum_{k \geq k(\delta), k \in \mathcal{N}_2} p_k^2 \exp\left[\frac{u}{2} a_k^2\right] a_k \exp\left[-\frac{u - u_0}{2} a_k^2\right] < \infty, \end{aligned}$$

which proves (a-i).

(a-ii) Assume $\lambda'(\underline{\alpha}, \bar{\alpha}) > \theta$ and fix any $u \geq 0$ such that $\lambda'(\underline{\alpha}, \bar{\alpha}) > u > \theta$. Then, by (3.6), we have $\sum_{k \in \mathcal{N}_2} p_k^2 \exp\left[\frac{u}{2} a_k^2\right] = \infty$. Choose $0 < \delta < \frac{1}{2}$ such that $\lambda'(\underline{\alpha} - \delta, \bar{\alpha} + 2\delta) \geq u$. Then we have

$$\frac{(\bar{\alpha} + 2\delta - (1/2))^2 + u}{2(\underline{\alpha} - \delta)} - 1 \leq 0,$$

so that

$$\begin{aligned}
 & \sum_{k \geq k(\delta), k \in \mathcal{N}_2} p_k \int_{a_k(\alpha_k - (1/2))}^{a_k(\alpha_k + (1/2))} g(x) dx \\
 \cong & \sum_{k \geq k(\delta), k \in \mathcal{N}_2} p_k \int_{a_k(\alpha_k - (1/2))}^{a_k(\alpha_k - (1/2)) + \delta a_k} g(x) dx \\
 \cong & \delta \sum_{k \geq k(\delta), k \in \mathcal{N}_2} p_k a_k g\left(a_k\left(\alpha_k - \frac{1}{2}\right) + \delta a_k\right) \\
 \cong & \frac{\delta}{\sqrt{2} \pi} \sum_{k \geq k(\delta), k \in \mathcal{N}_2} p_k \exp\left[-\frac{a_k^2}{2} \left\{\bar{\alpha} + 2\delta - \frac{1}{2}\right\}^2\right] \\
 \cong & \frac{\delta}{\sqrt{2} \pi} \sum_{k \geq k(\delta), k \in \mathcal{N}_2} p_k^2 \exp\left[\frac{u}{2} a_k^2\right] = \infty,
 \end{aligned}$$

which proves (a-ii).

(b-i) Assume $\lambda(\bar{\alpha}) < \theta$.

If $\bar{\alpha} = \frac{3}{2}$, then $\sum_{k \in \mathcal{N}_2} p_k^2 \exp[a_k^2] < \infty$ by $\lambda\left(\frac{3}{2}\right) = 2$ and (3.6), thus we obtain the conclusion.

Next we assume $\bar{\alpha} < \frac{3}{2}$ and fix any $u > 0$ such that $\lambda(\bar{\alpha}) < u < \theta$. Then,

by (3.6), $\sum_{k \in \mathcal{N}_2} p_k^2 \exp\left[\frac{u}{2} a_k^2\right] < \infty$. Choose $0 < \delta < \frac{1}{2}$ such that

$$\bar{\alpha} + \delta < \frac{3}{2} \quad \text{and} \quad u \geq \lambda(\bar{\alpha} + \delta) = 2 - \left(\frac{3}{2} - (\bar{\alpha} + \delta)\right)^2.$$

Then from (3.5) we have

$$a_k\left(\frac{3}{2} - \alpha_k\right) \geq \left(\frac{3}{2} - (\bar{\alpha} + \delta)\right) a_k > 0$$

for every $k \geq k(\delta)$, $k \in \mathcal{N}_2$. Thus

$$\begin{aligned}
 & \sum_{k \geq k(\delta), k \in \mathcal{N}_2} p_k^2 \exp[a_k^2] \int_{a_k((3/2) - \alpha_k)}^{\infty} g(x) dx \\
 \leq & \sum_{k \geq k(\delta), k \in \mathcal{N}_2} \frac{p_k^2 \exp[a_k^2]}{a_k(3/2 - \alpha_k)} g\left(a_k\left(\frac{3}{2} - \alpha_k\right)\right) \\
 \leq & \frac{1}{\sqrt{2} \pi} \sum_{k \geq k(\delta), k \in \mathcal{N}_2} p_k^2 \exp[a_k^2] \exp\left[-\frac{(3/2 - (\bar{\alpha} + \delta))^2}{2} a_k^2\right] \\
 \leq & \frac{1}{\sqrt{2} \pi} \sum_{k \geq k(\delta), k \in \mathcal{N}_2} p_k^2 \exp\left[\frac{u}{2} a_k^2\right] < \infty,
 \end{aligned}$$

which proves (b-i).

(b-ii) Assume $\lambda(\underline{\alpha}) > \theta$ and fix any $u \geq 0$ such that $\lambda(\underline{\alpha}) > u > \theta$. Then, by (3.6), we have $\sum_{k \in \mathcal{N}_2} p_k^2 \exp\left[\frac{u}{2} a_k^2\right] = \infty$. Choose $0 < \delta < \frac{1}{2}$ such that $\lambda(\underline{\alpha} - 2\delta) \geq u$. Then we have from (3.5)

$$0 < a_k \left(\frac{3}{2} - \alpha_k\right) \leq \left(\frac{3}{2} - (\underline{\alpha} - \delta)\right) a_k$$

for every $k \geq k(\delta)$, $k \in \mathcal{N}_2$, and

$$\begin{aligned} & \sum_{k \geq k(\delta), k \in \mathcal{N}_2} p_k^2 \exp[a_k^2] \int_{a_k((3/2) - \alpha_k)}^{\infty} g(x) dx \\ & \geq \sum_{k \geq k(\delta), k \in \mathcal{N}_2} p_k^2 \exp[a_k^2] \int_{((3/2) - (\underline{\alpha} - \delta)) a_k}^{((3/2) - (\underline{\alpha} - \delta)) a_k + \delta a_k} g(x) dx \\ & \geq \frac{\delta}{\sqrt{2\pi}} \sum_{k \geq k(\delta), k \in \mathcal{N}_2} p_k^2 \exp[a_k^2] \exp\left[-\frac{((3/2) - (\underline{\alpha} - 2\delta))^2}{2} a_k^2\right] \\ & \geq \frac{\delta}{\sqrt{2\pi}} \sum_{k \geq k(\delta), k \in \mathcal{N}_2} p_k^2 \exp\left[\frac{u}{2} a_k^2\right] = \infty, \end{aligned}$$

which proves (b-ii). \square

Proof of theorem 3. – First we consider the case $\alpha = \frac{3}{2} \cdot \lambda\left(\frac{3}{2}\right) = 2 < \theta$ implies $I(2; \sigma) < \infty$, thus we have $\mu_G \sim \mu_{G+Y}$ by Proposition 2(b) and Theorem 4.

Conversely $\lambda\left(\frac{3}{2}\right) > \theta$ implies $I(2 - 2\tau; \sigma) = \infty$ for some $0 < \tau < 1$, thus we have

$$\sum_k p_k^2 (\exp[(1 - \tau) a_k^2] - 1) = \infty.$$

Without loss of generality, by Lemma 2 and Theorem 4, we may assume (3.1) and (3.2), and consequently we have

$$\sum_{k \in \mathcal{N}_2} p_k^2 \exp[(1 - \tau) a_k^2] = \infty.$$

Choose $\delta > 0$ such that $\delta^2 < 2\tau$ and $k_0 \in \mathcal{N}_2$ such that $k \geq k_0$, $k \in \mathcal{N}_2$ implies $\alpha_k + \delta \geq \frac{3}{2}$. Then we have

$$\sum_{k \geq k_0, k \in \mathcal{N}_2} p_k^2 \exp[a_k^2] \int_{a_k((3/2) - \alpha_k)}^{\infty} g(x) dx$$

$$\begin{aligned} &\geq \sum_{k \geq k_0, k \in \mathcal{N}_2} p_k^2 \exp[a_k^2] \int_{\delta a_k}^{\sqrt{2\tau} a_k} g(x) dx \\ &\geq \frac{\sqrt{2\tau} - \delta}{\sqrt{2\pi}} \sum_{k \geq k_0, k \in \mathcal{N}_2} p_k^2 \exp[(1-\tau)a_k^2] = \infty, \end{aligned}$$

which implies $\mu_G \perp \mu_{G+Y}$ by Lemma 2 and Theorem 4.

Next we consider the case $\frac{1}{2} < \alpha < \frac{3}{2}$. Choose $0 < \eta < 1$ and $k_0 \in \mathcal{N}_2$ such that

$$\frac{1}{2} < \alpha_k < -\eta + \frac{3}{2} \tag{3.7}$$

for every $k \geq k_0, k \in \mathcal{N}_2$. $\lambda(\alpha) < \theta$ implies (3.1) and (3.2) since $\lambda(\alpha) > 1$ for $\frac{1}{2} < \alpha < \frac{3}{2}$. On the other hand, by (3.2) and (3.7), we have

$\lim_{k \in \mathcal{N}_2} p_k = 0$, so that $\lim_{k \in \mathcal{N}_2} a_k = \infty$. Therefore we may choose $k_1 \geq k_0, k_1 \in \mathcal{N}_2$ such that

$$\frac{1}{2} < \alpha_k < -\frac{1}{a_k} + \frac{3}{2}$$

for every $k \geq k_1$. Thus we have $\mu_G \sim \mu_{G+Y}$ by Lemma 3, Lemma 2 and Theorem 4.

Conversely assume $\lambda(\alpha) > \theta$. By Lemma 2 and Theorem 4, we may assume (3.1) and (3.2), consequently we have $\mu_G \perp \mu_{G+Y}$ by the same argument from above.

Finally we consider the case $\alpha = \frac{1}{2} \cdot \lambda\left(\frac{1}{2}\right) = 1 < \theta$ implies $I(1+\tau; \sigma) < \infty$ for some $0 < \tau < 1$, then we have (3.1), (3.2) and

$$\sum_{k \in \mathcal{N}_2} p_k^2 \exp\left[\frac{1+\tau}{2} a_k^2\right] < \infty.$$

Choose $k_0 \in \mathcal{N}_2$ such that $\alpha_k < \frac{1+\tau}{2}$ for every $k \geq k_0, k \in \mathcal{N}_2$. Then we have

$$\sum_{k \geq k_0, k \in \mathcal{N}_2} p_k \leq \sum_{k \geq k_0, k \in \mathcal{N}_2} p_k^2 \exp\left[\frac{1+\tau}{2} a_k^2\right] < \infty,$$

so that $\mu_G \sim \mu_{G+Y}$ by Lemma 1 and Theorem 4.

Conversely $\lambda\left(\frac{1}{2}\right) > \theta$ implies $I(1-\tau; \sigma) = \infty$ for some $0 < \tau < 1$, so that

$$\sum_k p_k^2 \left(\exp \left[\frac{1-\tau}{2} a_k^2 \right] - 1 \right) = \infty.$$

By Lemma 2 and Theorem 4, we may assume (3.1) and (3.2), and consequently we have

$$\sum_{k \in \mathcal{N}_2} p_k^2 \exp \left[\frac{1-\tau}{2} a_k^2 \right] = \infty.$$

Choose $\delta > 0$ such that $\delta < \sqrt{1+\tau} - 1$ and $k_0 \in \mathcal{N}_2$ such that $k \geq k_0, k \in \mathcal{N}_2$ implies $\alpha_k + \delta \geq \frac{1}{2}$. Then we have

$$\begin{aligned} & \sum_{k \geq k_0, k \in \mathcal{N}_2} p_k^2 \exp [a_k^2] \int_{a_k((3/2)-a_k)}^{\infty} g(x) dx \\ \cong & \sum_{k \geq k_0, k \in \mathcal{N}_2} p_k^2 \exp [a_k^2] \int_{(1+\delta)a_k}^{\sqrt{1+\tau} a_k} g(x) dx \\ \cong & \frac{\sqrt{1+\tau} - (1+\delta)}{\sqrt{2\pi}} \sum_{k \geq k_0, k \in \mathcal{N}_2} p_k^2 \exp \left[\frac{1-\tau}{2} a_k^2 \right] = \infty, \end{aligned}$$

which implies $\mu_G \perp \mu_{G+Y}$ by Lemma 2 and Theorem 4. \square

4. EXAMPLES

In this section we shall give negative answers to Kahane’s conjecture. For the two-valued sequence $Y = \{ \varepsilon(a_k, p_k) \}_{k \geq 1}$ on (T, Σ, σ) , define

$$a_k = \sqrt{\beta \log(k^\gamma + 1)}, \quad p_k = k^{-\gamma}, \quad k \in \mathbf{N},$$

where β is positive constant and $\gamma > \frac{1}{2}$. Then we have

$$\alpha_k = \frac{1}{a_k^2} \log \frac{1+p_k}{p_k} = \frac{1}{\beta}, \quad k \in \mathbf{N},$$

and $\alpha = \lim_k \alpha_k = \frac{1}{\beta}$. Moreover we have

$$\sum_1^n p_k^2 \exp \left[\frac{u}{2} a_k^2 \right] = \sum_1^n k^{-2\gamma} (k^\gamma + 1)^{(u/2)\beta} = O \left(\sum_1^n k^{-2\gamma + (u/2)\beta\gamma} \right), \quad n \rightarrow \infty,$$

for every $u > 0$, so that $\theta = \frac{2(2\gamma - 1)}{\beta\gamma}$ and $I(\theta; \sigma) = \infty$. The multiplicative chaos M defined in (1.2) is given by

$$M_n(t, \omega) = \exp \left[\sum_1^n \left\{ G_k(\omega) \varepsilon(a_k, p_k)(t) - \frac{1}{2} \varepsilon(a_k, p_k)(t)^2 \right\} \right].$$

$$(4.1) \quad \beta = \frac{1}{2} \text{ and } \gamma = \frac{2}{3}.$$

In this case $I(1; \sigma) < \infty$ but σ is not M -regular. In fact we have $\sup_k a_k = \infty$, $\alpha = 2 > \frac{3}{2}$ and $\theta = 2$, so that $I(2; \sigma) = \infty$, $I(1; \sigma) < \infty$ and σ is not M -regular by Theorem 2(b-i) and Theorem 4.

$$(4.2) \quad \beta = \frac{5}{6} \text{ and } \gamma = \frac{2}{3}.$$

In this case also $I(1; \sigma) < \infty$ but σ is not M -regular. In fact we have $\sup_k a_k = \infty$, $\frac{1}{2} < \alpha = \frac{6}{5} < \frac{3}{2}$, $\theta = \frac{6}{5} > 1$ and $\lambda(\alpha) = \frac{191}{100} > \theta$, so that $I(1; \sigma) < \infty$ and σ is not M -regular by Theorem 3 and Theorem 4.

$$(4.3) \quad \beta = 3 \text{ and } \gamma = 2.$$

In this case $I(1; \sigma) = \infty$ but σ is M -regular. In fact we have

$$\sum_k p_k = \sum_k k^{-2} < \infty,$$

so that $\sup_k \varepsilon(a_k, p_k) < \infty$ σ -a.s. and σ is M -regular by Proposition 1(a) and Theorem 4.

On the other hand $\theta = 1$, hence $I(1; \sigma) = \infty$.

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