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Chung's law of the iterated logarithm for iterated Brownian motion [1]

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ABSTRACT. – Let X and Y be independent, standard Brownian motions. For all $t \geq 0$, we define the *iterated Brownian motion*, Z, by setting $Z_t \doteq X(|Y_t|)$. In this paper we give Chung's form of the law of the iterated logarithm for Z.

Key words: Iterated Brownian motion, the other LIL.

RÉSUMÉ. – Soient X et Y des mouvements browniens indépendants. Pour tout entier $t \geq 0$, on définit le *mouvement brownien itéré*, Z, par $Z(t) \doteq X(|Y(t)|)$. Dans cet article, nous donnons la loi du logarithme itéré de Chung pour Z.

Mots clés : Le mouvement brownien itéré, la loi du logarithme de Chung.

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1. INTRODUCTION

Let X and Y be independent, standard Brownian motions. We define an *iterated Brownian motion* $Z \doteq \{Z_t, t \geq 0\}$ by setting $Z(t) \doteq X(|Y(t)|)$ for all $t \geq 0$. This process and its sundry modifications have been the objects of study in several problems in analysis and mathematical statistics. Indeed, Funaki [F] used a modification of Z to give a probabilistic solution to the partial differential equation:

$$\frac{\partial u}{\partial t} = \frac{1}{8}\frac{\partial^4 u}{\partial x^4} \qquad \text{with} \quad u(0,x) = u_0(x),$$

while Deheuvels and Mason [DH] introduced iterated Brownian motions in their study of the Bahadur–Kiefer process. In [B1] and [B2], Burdzy has studied various properties of the paths of iterated Brownian motion. His work implies the local law of the iterated logarithm:

$$\limsup_{t\to 0} \frac{Z(t)}{t^{1/4}\big(\ln\ln(1/t)\big)^{3/4}} = \frac{2^{5/4}}{3^{3/4}} \qquad \text{a.s.,}$$

which yields the modulus of continuity of Z at the origin. Recently, the authors [KL] determined the corresponding uniform modulus of continuity for iterated Brownian motion. The precise statement is that for all fixed $T > 0$,

$$\lim_{\varepsilon\to 0} \sup_{0\leq s,t\leq T} \sup_{|s-t|\leq\varepsilon} \frac{|Z(s)-Z(t)|}{\varepsilon^{1/4}\big(\ln(1/\varepsilon)\big)^{3/4}} = 1. \qquad \text{a.s.}$$

In 1948, Kai Lai Chung ([C]) proved his celebrated law of the iterated logarithm for the absolute maximum of sums of independent and identically distributed random variables. For Brownian motion, his result takes the following form:

$$\liminf_{t\to\infty} \sqrt{\frac{\ln\ln t}{t}} \sup_{0\leq s\leq t} |X(t)| = \frac{\pi}{8^{1/2}} \qquad \text{a.s.}$$

The above has come to be known as Chung's LIL or the other LIL. In this paper we are concerned with the analogue of Chung's LIL for iterated Brownian motion. Our main result is the following:

THEOREM 1.1. – *With probability one,*

$$\liminf_{t\to\infty} \frac{(\ln\ln t)^{3/4}}{t^{1/4}} \sup_{0\le s\le t} |Z(s)| = \frac{\pi^{3/2}3^{3/4}}{2^{11/4}}.$$

As we shall see, our proof of Theorem 1.1 involves solving two variational problems. The first variational problem arises from the asymptotic estimation of $\mathbb{P}(\sup_{0\le s\le 1}|Z(s)| \le \varepsilon)$ as $\varepsilon \to 0$, from which the lower bound in Theorem 1.1 easily follows. The argument for the upper bound is complicated by the lack of independence amongst the increments of Z. As in [KL], our solution involves an analysis of the lim inf along a random subsequence (the hitting times by $|Y|$ of an increasing sequence of positive real numbers). In essence, this approach reduces the problem to showing that there is an optimal way to balance the tendencies of two independent stochastic processes, which is the second variational problem.

Since the first draft of this paper was submitted, much work has been done on the paths of the samples of iterated Brownian motion. The reader is encouraged to see [CsCsFR] and [HPS] and [S] for details and other applications of iterated Brownian motion, in particular to mathematical statistics. Among other results, Zhan Shi [S] has extended Theorem 1.1. (by way of an integral test) through "hard" methods. In light of this development, our proof is interesting in that it involves only elementary facts about Brownian motion.

2. PRELIMINARY REMARKS AND LOWER BOUND

For convenience, let

$$\mathcal{Z}(t) \doteq \sup_{0\le s\le t}|Z(s)| \qquad \text{for} \quad t\ge 0.$$

and, for $t \ge 0$, set

$$\psi(t) \doteq t^{1/4}(\ln\ln t)^{-3/4} \qquad \text{and} \qquad \xi \doteq \frac{\pi^{3/2}3^{3/4}}{2^{11/4}}.$$

Associated with Y we have the following process:

$$M(t) \doteq \sup_{0\le s\le t}|Y_s| \qquad \text{for} \quad t\ge 0.$$

By way of M we will define

$$Z^*(t) \doteq X(M(t)) \qquad \text{for} \quad t\ge 0.$$

It is an important observation that

$$\mathcal{Z}(t) = \sup_{0 \le u \le M(t)} |X(u)| = \sup_{0 \le s \le t} |Z^*(s)|. \tag{2.1}$$

Therefore it suffices to prove Theorem 1.1 for Z^*.

For all $\varepsilon \ge 0$ and all $t \ge 0$, let

$$\varphi(\varepsilon) \doteq \mathbb{P}\left(\mathcal{Z}(1) \le \varepsilon\right) \qquad \text{and} \qquad H(t) \doteq \mathbb{P}\left(M(1) \le t\right).$$

By the law of total probability, the independence of X and Y and Brownian scaling we obtain:

$$\begin{aligned}\varphi(\varepsilon) &= \int_0^\infty \mathbb{P}\left(\mathcal{Z}(1) \le \varepsilon \,|\, M(1) = t\right) dH(t) \\ &= \int_0^\infty \mathbb{P}\left(\sup_{0 \le u \le 1} |X(u)| \le \varepsilon/\sqrt{t}\right) dH(t) \\ &= \int_0^\infty H(\varepsilon/\sqrt{t}) dH(t)\end{aligned}$$

The essential ingredient in the proof of the lower bound is the following small-ball estimate for $\varphi(\varepsilon)$.

LEMMA 2.1. – $\lim_{\varepsilon \to 0} \varepsilon^{4/3} \ln \varphi(\varepsilon) = -\xi^{4/3}$.

Proof. – We will need the classical result (*see* [C]):

$$H(\varepsilon) = \frac{4}{\pi} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)} \exp\left[\frac{-\pi^2(2n+1)^2}{8\varepsilon^2}\right].$$

From this it follows that

$$\lim_{a \to 0} a^2 \ln H(a) = -\frac{\pi^2}{8}. \tag{2.2}$$

For $x > 0$, let $f(x) \doteq x + x^{-2}$ and let $x_0 \doteq 2^{1/3}$, which is where f achieves its minimum value on $0 < x < \infty$.

To obtain the upper bound, let $\delta > 0$, and take points $0 < \alpha < x_0 < \beta < \infty$ so that

$$f(x_0) < \alpha^{-2} \wedge \beta. \tag{2.3}$$

Let

$$\alpha = a_0 < a_1 < s < a_{n-1} < a_n = \beta$$

be a partition of the interval $[\alpha, \beta]$ such that $|a_i - a_{i-1}| \leq \delta$ for all $1 \leq i \leq n$ and $x_0 = a_i$ for some $1 \leq i \leq n$. Let $s_i = a_i \varepsilon^{2/3}$ for all $0 \leq i \leq n$ and observe that

$$\int_0^{s_0} H(\varepsilon/\sqrt{t}) dH(t) \leq H(s_0) = H(a_0 \varepsilon^{2/3})$$
$$\int_{s_n}^{\infty} H(\varepsilon/\sqrt{t}) dH(t) \leq H(\varepsilon/\sqrt{s_n}) = H(a_n^{-1/2} \varepsilon^{2/3}).$$

Moreover,

$$\begin{aligned} \int_{s_0}^{s_n} H(\varepsilon/\sqrt{t}) dH(t) &= \sum_{k=1}^{n} \int_{s_{k-1}}^{s_k} H(\varepsilon/\sqrt{t}) dH(t) \\ &\leq \sum_{k=1}^{n} H(\varepsilon/\sqrt{s_{k-1}}) H(s_k) \\ &= \sum_{k=1}^{n} H(a_{k-1}^{-1/2} \varepsilon^{2/3}) H(a_k \varepsilon^{2/3}) \end{aligned}$$

Thus

$$\varphi(\varepsilon) \leq H(a_0 \varepsilon^{2/3}) + H(a_n^{-1/2} \varepsilon^{2/3}) + \sum_{k=1}^{n} H(a_{k-1}^{-1/2} \varepsilon^{2/3}) H(a_k \varepsilon^{2/3}).$$

From (2.2) it follows that

$$\lim_{\varepsilon \to 0} \varepsilon^{4/3} \ln H(a_0 \varepsilon^{2/3}) = -\frac{\pi^2}{8a_0^2} = -\frac{\pi^2}{8\alpha^2}$$
$$\lim_{\varepsilon \to 0} \varepsilon^{4/3} \ln H(a_n^{-1/2} \varepsilon^{2/3}) = -\frac{\pi^2}{8} a_n = -\frac{\pi^2}{8} \beta$$

Likewise,

$$\begin{aligned} \lim_{\varepsilon \to 0} \varepsilon^{4/3} \ln \left[H(a_{k-1}^{-1/2} \varepsilon^{2/3}) H(a_k \varepsilon^{2/3}) \right] &= -\frac{\pi^2}{8} \left(a_{k-1} + 1/a_k^2 \right) \\ &\leq -\frac{\pi^2}{8} \left(f(a_k) - \delta \right) \end{aligned}$$

It follows that

$$\begin{aligned} \limsup_{\varepsilon \to 0} \varepsilon^{4/3} \ln \varphi(\varepsilon) &\leq -\frac{\pi^2}{8} \left[\alpha^{-2} \wedge \beta \wedge \min_{1 \leq k \leq n} \left(f(a_k) - \delta \right) \right] \\ &= -\frac{\pi^2}{8} \left(f(x_0) - \delta \right), \end{aligned}$$

where we have used (2.3) and the fact that f is minimized at x_0. Finally, let δ go to zero and observe that $\pi^2 f(x_0)/8 = \xi^{4/3}$.

To obtain the lower bound, observe that

$$\begin{aligned}\varphi(\varepsilon) &\geq \int_0^{x_0\varepsilon^{2/3}} H(\varepsilon/\sqrt{t})dH(t) \\ &\geq H(\varepsilon^{2/3}x_0^{-1/2})H(x_0\varepsilon^{2/3})\end{aligned}$$

By (2.2) we obtain

$$\liminf_{\varepsilon\to 0} \varepsilon^{4/3}\ln\varphi(\varepsilon) \geq -\pi^2 f(x_0)/8 = -\xi^{4/3}. \qquad \square$$

The proof of the lower bound in Theorem 1.1 is an immediate consequence of Lemma 2.1 and a standard blocking argument.

Proof of the lower bound. – Let $B > 1$ and, for all $n \geq 1$, let $t_n \doteq B^n$. Let $\eta > 1$ be given and define the following events: for all $n \geq 1$, let

$$A_n \doteq \{\omega : \mathcal{Z}(t_n) \leq \eta^{-3/4}\xi\psi(t_n)\}$$

By scaling, we have

$$\mathbb{P}(A_n) = \mathbb{P}\left(\mathcal{Z}(1) \leq \eta^{-3/4}\xi(\ln\ln t_n)^{-3/4}\right).$$

Fix $1 < p < \eta$. By Lemma 2.1, there exists a constant, $C = C(p)$ such that for all n, $\mathbb{P}(A_n) \leq Cn^{-p}$. Consequently, $\mathbb{P}(A_n)$ is summable and, by the Borel–Cantelli lemma, $\mathbb{P}(A_n,\ \text{i.o.}) = 0$. Thus

$$\liminf_{n\to\infty} \frac{\mathcal{Z}(t_n)}{\psi(t_n)} \geq \eta^{-3/4}\xi \qquad \text{a.s.}$$

If $t_n \leq t \leq t_{n+1}$, then

$$\frac{\mathcal{Z}(t)}{\psi(t)} \geq \frac{\mathcal{Z}(t_n)}{\psi(t_n)}\frac{\psi(t_n)}{\psi(t_{n+1})}$$

Thus

$$\liminf_{t\to\infty} \frac{\mathcal{Z}(t)}{\psi(t)} \geq \eta^{-3/4}\xi B^{-1/4} \qquad \text{a.s.}$$

Now let $\eta \downarrow 1$ and $B \downarrow 1$ to obtain the desired lower bound. $\square$

3. THE UPPER BOUND

Given $a > 0$, let $\tau(a) \doteq \inf\{t > 0 : |Y(t)| = a\}$. By an eigenfunction expansion (e.g., see p. 52 of Port and Stone [PS]), for $x \in [0, \varepsilon)$,

$$(3.1)\quad \mathbb{P}\left(\tau(\varepsilon) \geq 1 \,|\, Y(0) = x\right) = \frac{4}{\pi}\sum_{n=0}^{\infty} \frac{1}{(2n+1)} \exp\left[-\frac{(2n+1)^2\pi^2}{8\varepsilon^2}\right] \sin\left[\frac{(2n+1)(x+\varepsilon)\pi}{2\varepsilon}\right].$$

The next lemma is an easy corollary of this expansion.

LEMMA 3.1. – *There exists an $\varepsilon_0 > 0$ and constants c_1 and c_2 (depending only on ε_0) such that for all $\varepsilon \in (0, \varepsilon_0)$ and all $x \in \left(0, \varepsilon/2\right)$,*

$$c_1 \exp\left[-\frac{\pi^2}{8\varepsilon^2}\right] \leq \mathbb{P}\left(\tau(\varepsilon) \geq 1 \,|\, Y(0) = x\right) \leq c_2 \exp\left[-\frac{\pi^2}{8\varepsilon^2}\right].$$

Proof. – Let $\varepsilon_0 > 0$ and suppose that $0 < \varepsilon < \varepsilon_0$. We have

$$\begin{aligned} &\left|\frac{\pi}{4}\exp\left[\frac{\pi^2}{8\varepsilon^2}\right]\mathbb{P}\left(\tau(\varepsilon) \geq 1 \,|\, Y(0) = 0\right) - \sin\left[\frac{\pi(x+\varepsilon)}{2\varepsilon}\right]\right| \\ &\leq \sum_{n=1}^{\infty} \frac{1}{2n+1}\exp\left[-\frac{n(n+1)\pi^2}{2\varepsilon_0^2}\right] \\ &= c(\varepsilon_0), \end{aligned}$$

with obvious notation. Choose ε_0 small enough so that $c(\varepsilon_0) < 1/\sqrt{2}$. Since $0 \leq x \leq \varepsilon/2$, it follows that

$$\frac{1}{\sqrt{2}} \leq \sin\left[\frac{(x+\varepsilon)\pi}{2\varepsilon}\right] \leq 1.$$

The conclusion follows with $c_1 \doteq \pi(1/\sqrt{2} - c(\varepsilon_0))/4$ and $c_2 \doteq \pi(1 + c(\varepsilon_0))/4$. □

Fix $1 < p < \infty$ and, for $n \geq 1$, let $t_n \doteq \exp\left(n^p\right)$. Define $T_0 \doteq 0$ and, for $n \geq 1$, let $T_n \doteq \tau(t_n)$ and $\Delta T_n \doteq T_n - T_{n-1}$. In the sequel, let $\rho(t) \doteq t^{1/4}(\ln\ln t)^{3/4}$. This function is the correct normalization for the law of the iterated logarithm for iterated Brownian motion. (See [B1] for a related result.) We will need the following technical lemma.

LEMMA 3.2. – *With probability one, $\rho(T_{k-1}) = o\left(\psi(T_k)\right)$ as $k \to \infty$.*

Proof. – Observe that

$$\frac{\rho^4(T_{k-1})}{\psi^4(T_k)} \le \frac{T_{k-1}}{T_k}(\ln\ln T_k)^6.$$

Therefore it suffices to show that the right hand side tends to zero with probability one. To this end, let $\varepsilon > 0$ be given, and let $a_k \doteq t_k^2(\ln k)^6$. Then, for k sufficiently large,

$$\begin{aligned}\mathbb{P}\left(T_{k-1} \ge \varepsilon T_k(\ln\ln T_k)^{-6}\right) &\le \mathbb{P}\left(T_k \ge a_k\right) + \mathbb{P}\left(T_{k-1} \ge \varepsilon a_k(\ln\ln a_k)^{-6}\right)\\ &\doteq P_k + Q_k,\end{aligned}$$

with obvious notation. By Lemma 3.1 and scaling, and for all k sufficiently large,

$$\begin{aligned}P_k &= \mathbb{P}\left(\tau(1) \ge (\ln k)^6\right)\\ &\le c_2 \exp\left[-\frac{\pi^2(\ln k)^6}{8}\right].\end{aligned}$$

Thus it is clear that $\sum_k P_k < \infty$.

Observe that as $k \to \infty$,

$$(\ln\ln a_k)^6 \sim p^6(\ln k)^6.$$

Let $0 < \delta < \varepsilon p^{-6}$. Then for all k sufficiently large,

$$\frac{\varepsilon a_k}{(\ln\ln a_k)^6} \ge \delta t_k^2.$$

Consequently by Lemma 3.1 and scaling, and for all k sufficiently large,

$$\begin{aligned}Q_k &\le \mathbb{P}\left(T_{k-1} \ge \delta t_k^2\right)\\ &= \mathbb{P}\left(\tau(1) \ge \delta t_k^2/t_{k-1}^2\right)\\ &\le c_2 \exp\left[-\frac{\delta\pi^2}{8}(t_k/t_{k-1})^2\right].\end{aligned}$$

Since

$$\frac{t_k}{t_{k-1}} \ge \exp\left[p(k-1)^{p-1}\right],$$

certainly $\sum_k Q_k < \infty$. The proof is completed by an application of the Borel–Cantelli lemma. □

Having developed the requisite lemmas, we proceed to the proof of the upper bound in Theorem 1.1.

Proof of the upper bound. – For all $t > 0$ let

$$F(t) \doteq \sqrt{\frac{t}{\ln\ln t}} \qquad \text{and} \qquad G(t) \doteq t^2 \ln\ln t.$$

We note that $G(\cdot)$ is invertible, and we will denote its inverse by $G^{-1}(\cdot)$. Also observe that $G^{-1}(t) \sim F(t)$ as $t \to \infty$ and that $F(F(t)) \sim \psi(t)$ as $t \to \infty$; consequently,

$$F(G^{-1}(t)) \sim \psi(t) \qquad \text{as} \quad t \to \infty. \tag{3.2}$$

Given $0 < p < \infty$, choose $\alpha > \left(p\pi^2/8\right)^{1/2}$ and $\gamma \doteq 8/(p\pi^2) - \alpha^{-2}$. As a consequence,

$$\frac{p\pi^2}{8}(\alpha^{-2} + \gamma) = 1, \tag{3.3}$$

which will be an important observation in the sequel.

For $k \geq 1$, let

$$\begin{aligned} A_k &\doteq \left\{\omega : \sup_{t_{k-1} \leq t \leq t_k} |X(t) - X(t_{k-1})| \leq \alpha F(t_k)\right\} \\ B_k &\doteq \{\omega : \Delta T_k \geq \gamma G(t_k)\} \\ C_k &\doteq A_k \cap B_k. \end{aligned}$$

First we will show $\mathbb{P}(C_k \text{ i.o.}) = 1$. Since the events $\{C_k, k \geq 1\}$ are independent, it suffices to show $\sum_k \mathbb{P}(C_k) = \infty$. Since X and Y are independent processes, A_k and B_k are independent and $\mathbb{P}(C_k) = \mathbb{P}(A_k)\mathbb{P}(B_k)$. However,

$$\begin{aligned} \mathbb{P}(A_k) &\geq \mathbb{P}\Big(\sup_{0 \leq s \leq \Delta t_k} |X(s)| \leq \alpha\sqrt{\Delta t_k}(\ln\ln t_k)^{-1/2}\Big) \\ &= \mathbb{P}\Big(\sup_{0 \leq s \leq 1} |X(s)| \leq \alpha(\ln\ln t_k)^{-1/2}\Big) \quad \text{(by scaling)} \\ &= \mathbb{P}(\tau(\beta_k) \geq 1), \end{aligned}$$

where $\beta_k = \alpha(\ln\ln t_k)^{-1/2}$. By Lemma 3.1, we see that for all k sufficiently large,

$$\mathbb{P}(A_k) \geq c_1 \exp\left[-\frac{\pi^2}{8\beta_k^2}\right].$$

We also have

$$\begin{aligned} \mathbb{P}(B_k) &= \mathbb{P}\left(\tau(t_k) \geq \gamma t_k^2 \ln\ln t_k \,|\, Y(0) = t_{k-1}\right) \\ &= \mathbb{P}\left(\tau(\varepsilon_k) \geq 1 \,|\, Y(0) = x_k\right) \quad \text{(by scaling)} \end{aligned}$$

where

$$\varepsilon_k \doteq \frac{1}{\sqrt{\gamma \ln\ln t_k}} \qquad \text{and} \qquad x_k \doteq \frac{t_{k-1}}{t_k\sqrt{\gamma \ln\ln t_k}}$$

However, $x_k/\varepsilon_k = t_{k-1}/t_k \to 0$ and $\varepsilon_k \to 0$ as $k \to \infty$; consequently, by Lemma 3.1, for all k large enough,

$$\mathbb{P}(B_k) \geq c_1 \exp\left[-\frac{\pi^2}{8\varepsilon_k^2}\right].$$

Thus, by (3.3) for all k sufficiently large we have,

$$\mathbb{P}(C_k) \geq c_1^2 \exp\left[-\frac{p\pi^2}{8}(\alpha^{-2}+\gamma)\ln k\right] = c_1^2 k^{-1}.$$

By the definition of $\{C_k, k \geq 1\}$ and the above, infinitely often we have

$$\sup_{t_{k-1}\leq s\leq t_k} |X(s) - X(t_{k-1})| \leq \alpha F(t_k) \quad \text{and} \quad T_k \geq \gamma G(t_k),$$

where we have used the obvious fact that $\Delta T_k \leq T_k$. Since $T_k \geq \gamma G(t_k)$ implies $t_k \leq G^{-1}(T_k/\gamma)$, it follows that infinitely often $D_k \leq \alpha F(G^{-1}(T_k/\gamma))$, where

$$D_k \doteq \sup_{T_{k-1}\leq s\leq T_k} |Z^*(s) - Z^*(T_{k-1})|$$

Given $\varepsilon > 0$ and (3.2), we see that infinitely often $D_k \leq (1+\varepsilon)\alpha\gamma^{-1/4}\psi(T_k)$, which is to say

$$\liminf_{k\to\infty} \frac{D_k}{\psi(T_k)} \leq (1+\varepsilon)\frac{\alpha}{\gamma^{1/4}}. \tag{3.4}$$

Finally, since $\mathcal{Z}(T_k) \leq \mathcal{Z}(T_{k-1}) + D_k$, we have

$$\frac{\sup_{0\leq s\leq T_k}|Z^*(s)|}{\psi(T_k)} \leq \frac{\sup_{0\leq s\leq T_{k-1}}|Z^*(s)|}{\rho(T_{k-1})}\frac{\rho(T_{k-1})}{\psi(T_k)} + \frac{D_k}{\psi(T_k)}.$$

By the law of the iterated logarithm for iterated Brownian motion,

$$\limsup_{k\to\infty} \frac{\sup_{0\leq s\leq T_{k-1}}|Z^*(s)|}{\rho(T_{k-1})} \leq 1 \qquad \text{a.s.}$$

Combining this with Lemma 3.2, we obtain

$$\lim_{k\to\infty} \frac{\sup_{0\leq s\leq T_{k-1}}|Z^*(s)|}{\rho(T_{k-1})}\frac{\rho(T_{k-1})}{\psi(T_k)} = 0 \qquad \text{a.s.}$$

Thus, by the definition of lim inf and $\mathcal{Z}$ and (3.4), we obtain

$$\liminf_{t\to\infty} \frac{\mathcal{Z}(t)}{\psi(t)} \leq (1+\varepsilon)\frac{\alpha}{\gamma^{1/4}} \qquad \text{a.s.}$$

Now let $\varepsilon \to 0$ and $p \to 1$ to obtain:

$$\liminf_{t\to\infty} \frac{\mathcal{Z}(t)}{\psi(t)} \leq \alpha\left[\frac{8}{\pi^2} - \frac{1}{\alpha^2}\right]^{-1/4} \qquad \text{a.s.}$$

The optimal choice for α is $\sqrt{3}\pi/4$, which yields the desired upper bound. □

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