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Inequalities for Bochner's subordinates of two-parameter symmetric Markov processes

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ABSTRACT. – We introduce the subordinate process, in Bochner's sense, of a two-parameter symmetric Markov process and exhibite some of its properties. Based on the generalized Dynkin formula, we prove two inequalities satisfied by the subordinate process. These inequalities can be used to give probabilistic characterizations of capacities related to many interesting fractional Sobolev spaces.

Key words: Capacity, two-parameter process, symmetric Markov process, Bochner's subordinate process.

RÉSUMÉ. – Nous introduisons les processus subordonnés, au sens de Bochner, d'un processus de Markov symétrique à deux paramètres et mettons en évidence certaines de leurs propriétés. Nous appuyant sur la formule de Dynkin généralisée, nous montrons deux inégalités satisfaites par les processus subordonnés. Ces inégalités peuvent servir à donner des caractérisations probabilistes de capacités relatives à des espaces du type Sobolev fractionnaire.

A.M.S. Classification : 31C15, 60J25, 60J30, 60J45.

1. INTRODUCTION

In this paper (§ 3) we introduce a class of two-parameter symmetric Markov processes and their subordinates in the sense of Bochner. We prove in § 5, for these processes, two maximal inequalities.

Examples of two-parameter symmetric Markov processes in our sense are the following ones.

a) Let E be a Fréchet space equipped with a centered Gaussian measure m. In [10], Song introduced a two-parameter E-valued Ornstein-Uhlenbeck process, in order to characterize probabilistically the $c_{2,2}$-polar sets appearing in Malliavin calculus. In Hirsch [4], it is shown that such a process, which is a two-parameter E-valued m-symmetric Markov process, may be defined as $(e^{-(s+t)}W_{e^{2s}, e^{2t}})$, where W is a two-parameter E-valued Brownian sheet such that the law of $W_{1,1}$ is m. Notice that this process (in the case $E = R$) was also introduced as a two-parameter Ornstein-Uhlenbeck process in Walsh [13].

b) Let $E = R^d$ and let m be the Lebesgue measure on E. Let X, Y be independent E-valued processes, càdlàg, starting from zero, with homogeneous independent increments, and let ξ be a random variable, independent of (X, Y), whose "law" is m. Then, the process $Z_{s,t} = \xi + X_s + Y_t$ is a two-parameter E-valued m-symmetric Markov process, which is studied in Hirsch [6].

c) In [8], Song characterizes, in a class of diffusion m-symmetric semi-groups on an open interval E of R, those for which there exists a two-parameter E-valued m-symmetric Markov process admitting the given semi-group as transition semi-group with respect to each parameter.

For each of these examples, both maximal inequalities that we prove in general in § 5, are proved in the particular context, and consequences are drawn, as explained in § 6, to interpret probabilistically the corresponding capacities.

Consider again example a). Let σ (resp. τ) be a one-sided stable process of index $0 < \alpha \leq 1$ (resp. $0 < \beta \leq 1$), starting from zero, and suppose that W, σ, τ are independent. Then, the process $(e^{-(\sigma_s+\tau_t)}W_{e^{2\sigma_s}, e^{2\tau_t}})$ is a Bochner subordinate of a two-parameter E-valued m-symmetric Markov process. Consequently the results of this paper are applicable to this process. The corresponding space H (in the sense of § 6) is the image by $(I + (-L)^{\alpha})^{-1/2}(I + (-L)^{\beta})^{-1/2}$ of $L^2(m)$, where L is the Ornstein-Uhlenbeck operator on E. Therefore, H is the space $D_{\alpha+\beta,2} = [(I - L)^{-(\alpha+\beta)/2}](L^2(m))$ with a norm equivalent to the usual $(\alpha + \beta, 2)$-norm. In particular, by § 6, we can estimate probabilistically the $c_{r,2}$-capacity for any $0 < r \leq 2$. Extensions to the case of n-parameter processes can be found in Hirsch-Song [11], [12].

2. PRELIMINARY

We only consider in what follows symmetric Markov processes. We first give the definition that we adopt for one-parameter symmetric Markov processes. In what follows, m denotes a σ-finite measure on a space E.

DEFINITION 1. – *A one-parameter E-valued m-symmetric Markov process is an E-valued measurable process $X = (X_t)_{t \geq 0}$ defined on a measure space (Ω, A, P) such that there exist, a filtration $F = (F_t)_{t \geq 0}$ with $F_\infty \subset A$, and a strongly continuous sub-Markovian symmetric semi-group $(P_t)_{t \geq 0}$ on $L^2(m)$ so that*

1) $\forall\, t \geq 0$, X_t is F_t-measurable and the law of X_t (with respect to P) is m,

2) $\forall\, a, b \geq 0$, $\forall\, f \in L^2(m)$, $E[f(X_{a+b})|F_a] = P_b f(X_a)$.

The above conditional expectations are well defined because, as a consequence of the assumptions, the restriction of P to F_0 is σ-finite.

We shall introduce the subordinates, in Bochner's sense, of such processes. We specially mention the works by Bochner [2], Bouleau [3], Hirsch [5] on this subject.

Let X be a process satisfying the conditions of definition 1. Let $\tau = (\tau_t)_{t \geq 0}$ be a real increasing process with homogeneous independent increments, starting from zero and independent of X (process τ is called a *subordinator*). We denote by N the family of all negligible sets. The following proposition whose proof is omitted, may be considered classical.

PROPOSITION 2. – *Define the following σ-fields:*

$$\forall\, s,\, t \geq 0, \quad G_{s,t} = F_s \vee \sigma(\tau_u;\; u \leq t) \vee N$$

$$\forall\, t \geq 0, \quad H_t = \{A \in G_{\infty,t};\; \forall\, s \geq 0,\; A \cap \{\tau_t \leq s\} \in G_{s,t}\},$$

and set $\forall\, a \geq 0$, $Q_a = \int P_t \nu_a(dt)$, where ν_a denotes the law of τ_a. Then, the process $Y = X_\tau = (X_{\tau_t})_{t \geq 0}$ is an E-valued m-symmetric Markov process with respect to the filtration H and the semi-group $(Q_t)_{t \geq 0}$.

3. BOCHNER'S SUBORDINATION OF A TWO-PARAMETER MARKOV PROCESS

We now consider E a separable metric space equipped with its Borel σ-algebra and m a Borel σ-finite measure on E. Let $D(R_+, E)$ denote the space of càdlàg paths in E equipped with the Skorohod topology, and let (Ω, A, P) be a measure space. We consider a *two-parameter E-valued m-symmetric Markov process*, which means a measurable map $Z = (Z_{s,t}(\omega); s \geq 0, t \geq 0, \omega \in \Omega)$ from $\Omega \times R_+^2$ into E such that

H1

v. The map: $t \to (Z_{s,t}(\omega); s \geq 0)$ belongs to $D(R_+, D(R_+, E))$ for P-almost all ω.

h. The map: $s \to (Z_{s,t}(\omega); t \geq 0)$ belongs to $D(R_+, D(R_+, E))$ for P-almost all ω.

H2 There exist two filtrations $F^v = (F_t^v)_{t\geq 0}$ and $F^h = (F_s^h)_{s\geq 0}$ on (Ω, A) and two strongly continuous sub-Markovian symmetric semi-groups $P^v = (P_t^v)_{t\geq 0}$ and $P^h = (P_s^h)_{s\geq 0}$ on $L^2(m)$ such that

v. For each $s \geq 0$, $(Z_{s,t})_{t\geq 0}$ is a one-parameter E-valued m-symmetric Markov process with respect to the filtration F^v and the semi-group P^v (in the sense of Definition § 2.1).

h. For each $t \geq 0$, $(Z_{s,t})_{s\geq 0}$is a one-parameter E-valued m-symmetric Markov process with respect to the filtration F^h and the semi-group P^h.

H3 We denote by μ^v (resp. μ^h) the law of $(Z_{0,t})_{t\geq 0}$ (resp. $(Z_{s,0})_{s\geq 0}$) on $D(R_+, E)$.

v. The map $t \to (Z_{s,t}; s \geq 0)$ is a one-parameter $D(R_+, E)$-valued μ^h-symmetric Markov process with respect to the filtration F^v.

h. The map $s \to (Z_{s,t}; t \geq 0)$ is a one-parameter $D(R_+, E)$-valued μ^v-symmetric Markov process with respect to the filtration F^h.

Remark 1. – The condition H2 implies automatically that the operators P_t^v and P_s^h commute. We have indeed, if g and f belong to $L^2(m)$,

$$\begin{aligned} E[g(Z_{0,0})P_t^v P_s^h f(Z_{0,0})] &= E[g(Z_{0,0})P_s^h f(Z_{0,t})], \text{ by H2.v,} \\ &= E[g(Z_{0,0})f(Z_{s,t})], \text{ by H2.h and by the fact } Z_{0,0} \in F_0^h, \\ &= E[g(Z_{0,0})P_t^v f(Z_{s,0})] = E[g(Z_{0,0})P_s^h P_t^v f(Z_{0,0})]. \end{aligned}$$

Consider now two independent increasing processes $\sigma = (\sigma_s)$ and $\tau = (\tau_t)$ starting from zero with homogeneous independent increments, defined on Ω, which are independent of Z under P. Bochner's subordinate of Z along the processes (σ, τ) is simply defined as $Y_{s,t}(\omega) = Z_{\sigma_s(\omega), \tau_t(\omega)}(\omega)$. We remark that, in general, process Y is not a two-parameter E-valued m-symmetric Markov process in the sense of the previous definition: Hypotheses H1 and H2 are satisfied, but H3 is not generally true.

In the following text we shall use the notation: $\nu_s^h = \text{law}(\sigma_s)$, $\nu_t^v = \text{law}(\tau_t)$, $Q_t^v = \int \nu_t^v(du) P_u^v$, and $Q_s^h = \int \nu_s^h(du) P_u^h$.

4. DYNKIN'S FORMULA

We introduce the following filtrations (where N denotes the negligible sets):

$$vG_{t',t} = F_{t'}^v \vee \sigma\{\tau_u; u \leq t\} \vee N \vee \sigma\{\sigma_s;\ s \geq 0\},$$

$$vH_t = \{A;\ A \in vG_{\infty,t};\ \forall t',\ A \cap \{\tau_t \leq t'\} \in vG_{t',t}\},$$

$$hG_{s',s} = F_{s'}^h \vee \sigma\{\sigma_u; u \leq s\} \vee N \vee \sigma\{\tau_t;\ t \geq 0\},$$

$$hH_s = \{B;\ B \in hG_{\infty,s};\ \forall s',\ B \cap \{\sigma_t \leq s'\} \in hG_{s',s}\}.$$

We set $H_{s,t} = vH_t \cap hH_s$. By the results of § 2, $Y_{s,t}$ is $H_{s,t}$-measurable.

Let f be a function belonging to $L^2(m)$. We shall calculate $E[f(Z_{\sigma_{s+a}, \tau_{t+b}}) | H_{s,t}]$. For this purpose, let ξ_u be the coordinate process on $D(R_+, E)$. By the results in § 2, we obtain:

$$\begin{aligned} E[f(Z_{\sigma_{s+a}, \tau_{t+b}}) | vH_b] &= E[f(\xi_u(Z_{\bullet, \tau_{t+b}})) | vH_b]_{u=\sigma_{s+a}} \\ &= Q_t^v f(\xi_u(Z_{\bullet, \tau_b}))_{u=\sigma_{s+a}} = Q_t^v f(Z_{\sigma_{s+a}, \tau_b}). \end{aligned}$$

In the same way we can prove

$$E[f(Z_{\sigma_{s+a}, \tau_{t+b}}) | hH_a] = Q_s^h f(Z_{\sigma_a, \tau_{t+b}}).$$

Finally, with help of these formulas, we can prove

LEMMA 1. – *The following equalities hold:*

$$E[f(Z_{\sigma_{s+a}, \tau_{t+b}}) | vH_b | hH_a] = E[Q_t^v f(Z_{\sigma_{s+a}, \tau_b}) | hH_a] = Q_s^h Q_t^v f(Z_{\sigma_a, \tau_b}).$$

$$E[f(Z_{\sigma_{s+a},\tau_{t+b}})|hH_a|vH_b] = E[Q^h_s f(Z_{\sigma_a,\tau_{t+b}})|vH_b] = Q^v_t Q^h_s f(Z_{\sigma_a,\tau_b}).$$

Consequently,

$$E[f(Z_{\sigma_{s+a},\tau_{t+b}})|H_{a,b}] = Q^h_s Q^v_t f(Z_{\sigma_a,\tau_b}) = Q^v_t Q^h_s f(Z_{\sigma_a,\tau_b}).$$

As a corollary of the above lemma we obtain:

COROLLARY 2. – *For any function u in $L^2(m)$, let*

$$M^u_{a,b} = E\left[\int_0^\infty e^{-s-t} u(Y_{s,t}) ds\, dt | H_{a,b}\right],$$

$a \geq 0,\ b \geq 0$. *Then, we have*

$$\begin{aligned} M^u_{a,b} &= E\left[\int_0^\infty e^{-s-t} u(Y_{s,t}) ds\, dt | hH_b | vH_a\right] \\ &= E\left[\int_0^\infty e^{-s-t} u(Y_{s,t}) ds\, dt | vH_a | hH_b\right], \end{aligned}$$

and for any finite subset $B \subset R_+$, the iterated L^2-martingale inequality *holds:*

$$N_2[\sup_{(a,b)\in B\times B} |M^u_{a,b}|] \leq 4N_2\left[\int_0^\infty e^{-s-t} u(Y_{s,t}) ds\, dt\right].$$

Herein and after, N_2 denotes the norms in (various) L^2-spaces.

Proof. – The expression for $M^u_{a,b}$ is a direct consequence of lemma 1. To show the iterated L^2-martingale inequality, let

$$N_a = E\left[\int_0^\infty e^{-s-t} u(Y_{s,t}) ds\, dt | vH_a\right].$$

Then,

$$\begin{aligned} N_2[\sup_{(a,b)\in B\times B} |M^u_{a,b}|] &\leq N_2[\sup_{b\in B} E[\sup_{a\in B} |N_a||hH_b]] \\ &\leq 2N_2[\sup_{a\in B} |N_a|] \leq 4N_2\left[\int_0^\infty e^{-s-t} u(Y_{s,t}) ds\, dt\right]. \quad \square \end{aligned}$$

We introduce the following notation:

$$U^v = \frac{1}{\sqrt{\pi}} \int_0^\infty t^{-1/2} e^{-t} Q^v_t dt, \qquad U^h = \frac{1}{\sqrt{\pi}} \int_0^\infty s^{-1/2} e^{-s} Q^h_s ds.$$

LEMMA 3. – *We have the generalized Dynkin formula:*

$$M^u_{a,b} = \int e^{-s-t} u(Y_{s,t}) 1_{\{0\le s\le a,\, 0\le t\le b\}} ds\, dt$$
$$+ \int e^{-s-b} (U^v)^2 u(Y_{s,b}) 1_{\{0\le s\le a\}} ds$$
$$+ \int e^{-a-t} (U^h)^2 u(Y_{a,t}) 1_{\{0\le t\le b\}} dt$$
$$+ e^{-a-b} (U^v U^h)^2 u(Y_{a,b}).$$

Proof. – It is enough to apply Lemma 1 with the following decomposition:

$$\begin{aligned} 1_{R_+\times R_+}(s,\, t) &= (1_{[0,a]}(s) + 1_{]a,\infty[}(s))(1_{[0,b]}(t) + 1_{]b,\infty[}(t)) \\ &= 1_{[0,a]}(s)1_{[0,b]}(t) + 1_{[0,a]}(s)1_{]b,\infty[}(t) \\ &\quad + 1_{]a,\infty[}(s)1_{[0,b]}(t) + 1_{]a,\infty[}(s)1_{]b,\infty[}(t). \quad \square \end{aligned}$$

LEMMA 4. – $E[(\int_0^\infty e^{-s-t} u(Y_{s,t}) ds dt)^2] = E[(U^v U^h u)^2]$.

Proof. – We have

$$\begin{aligned} &e^{-s-t} u(Y_{s,t}) e^{-a-b} u(Y_{a,b}) \\ &= e^{-s-t} u(Y_{s,t}) e^{-a-b} u(Y_{a,b}) [1_{[0,a]}(s)1_{[0,b]}(t) + 1_{[0,a]}(s)1_{]b,\infty[}(t) \\ &\quad + 1_{]a,\infty[}(s)1_{[0,b]}(t) + 1_{]a,\infty[}(s)1_{]b,\infty[}(t)]. \end{aligned}$$

Integrating separately the terms with respect to the measure $P(d\omega) ds dt da db$, and using Lemma 1, we obtain the claim of the lemma. □

5. INEQUALITIES FOR PROCESS Y

Before beginning, we recall some elementary facts on the potential theory of symmetric sub-Markovian operators.

Facts on potential theory. – Let Ξ be a measurable space equipped with a σ-finite measure μ. Let G be a μ-symmetric sub-Markovian operator defined on $L^2(\mu)$. Then, G is a contraction on $L^2(\mu)$. We assume that G is one to one, and we consider the space $H = G[L^2(\mu)]$ equipped with the scalar product $\|Gg\|^2 = \langle Gg,\, Gg\rangle = \int g^2 d\mu$. The space H is then a Hilbert space.

Fact 1. – We shall say that a function $\varphi \in H$ is a *potential*, if for any $f \in H$ such that $f \geq 0$, we have $\langle f, \varphi \rangle \geq 0$. A potential φ is always positive. In fact, it is enough to see

$$\int u\varphi d\mu = \langle G^2 u, \varphi \rangle \geq 0, \quad \forall u \geq 0, \quad u \in L^2(\mu).$$

Moreover, by the bipolar theorem, any potential φ is the limit in H of a sequence of functions $G^2 u_n$, where u_n are positive functions in $L^2(\mu)$. □

Fact 2. – For any $f = Gg \in H$, we set $c(f) = \inf \{||k||;\, k \in H,\, k \geq |f|$ μ-a.s. $\}$. By the projection theorem, there exists a unique $\varphi \in H$ such that $\varphi \geq |f|$, μ-a.s., and $||\varphi|| = c(f)$. The norm of φ is overestimated by

$$||\varphi|| = c(f) \leq ||G|g|\,|| = \left(\int g^2 d\mu\right)^{1/2} = ||f||.$$

Moreover, by a standard argument, we can prove that the function φ is a potential. We shall call this function φ the *equilibrium potential* of the function f. □

The inequalities on Y will be proved by using two tools: the martingale inequality and the generalized Dynkin formula. For a better understanding of the proof, let us first see what we can do with these two tools for one-parameter Markov processes.

THEOREM 3. – *Let X be a one-parameter Ξ-valued μ-symmetric Markov process with respect to some filtration $(\widehat{F}_t)$. Let $(\widehat{Q}_t)$ be its semi-group in $L^2(\mu)$. Let $\widehat{U} = \frac{1}{\sqrt{\pi}} \int_0^\infty t^{-1/2} e^{-t} \widehat{Q}_t dt$. Then, for any finite subset $B \subset R_+$, for any functions g and u in $L^2(\mu)$, we have*

$$N_2[\sup_{t \in B} e^{-t} |\widehat{U}g|(X_t)] \leq 2N_2(g),$$

$$N_2\left[\sup_{a \in B}\left|\int_0^a e^{-t} u(X_t) dt\right|\right] \leq 4N_2[\widehat{U}u].$$

Proof. – Recall the Dynkin formula:

$$E\left[\int_0^\infty e^{-t} u(X_t) dt | \widehat{F}_a\right] = \int_0^a e^{-t} u(X_t) dt + e^{-a} \widehat{U}^2 u(X_a).$$

Let us prove the first inequality. For this, we apply the potential theory for the symmetric sub-Markovian operator $G = \widehat{U}$ on $L^2(\mu)$. We need to prove that for any $f \in H$,

$$N_2[\sup_{t \in B} e^{-t} |f|(X_t)] \leq 2||f||.$$

Suppose first that $f = \widehat{U}^2 u$ with a function $u \geq 0$. We have by Dynkin's formula

$$0 \leq e^{-a} f(X_a) \leq E\left[\int_0^\infty e^{-t} u(X_t) dt | \widehat{F}_a\right].$$

Consequently, by L^2-martingale inequality,

$$N_2[\sup_{t \in B} e^{-t} f(X_t)] \leq N_2\left[\sup_{a \in B} E\left[\int_0^\infty e^{-t} u(X_t) dt | \widehat{F}_a\right]\right]$$
$$\leq 2N_2\left[\int_0^\infty e^{-t} u(X_t) dt\right] = 2N_2[\widehat{U} u] = 2\|f\|.$$

Now, by Fact 1 and the continuity of $\widehat{U}$, the above inequality remains valid for all potentials. Finally for any $f \in H$, let φ be the equilibrium potential of f. By Fact 2, we have

$$N_2[\sup_{t \in B} e^{-t} |f|(X_t)] \leq N_2[\sup_{t \in B} e^{-t} \varphi(X_t)] \leq 2\|\varphi\| \leq 2\|f\|.$$

This proves the first inequality. For the second inequality, by Dynkin's formula, we have

$$N_2\left[\sup_{a \in B}\left|\int_0^a e^{-t} u(X_t) dt\right|\right]$$
$$\leq N_2[\sup_{a \in B} e^{-a} |\widehat{U}^2 u|(X_a)] + N_2\left[\sup_{a \in B} E\left[\left|\int_0^\infty e^{-t} u(X_t) dt\right| |\widehat{F}_a\right]\right]$$
$$\leq 2N_2[\widehat{U} u] + 2N_2[\widehat{U} u].$$

This proves the second inequality. □

Turn now back to the inequalities for the process Y. We consider the potential theory for the operator $G = U^v U^h = U^h U^v$ on $L^2(m)$.

THEOREM 4. – *The following inequality holds: for any finite subset D of R_+^2, for any $g \in L^2(m)$,*

$$N_2[\sup_{(s,t) \in D} e^{-s-t} |U^v U^h g|(Y_{s,t})] \leq 4N_2[g].$$

Proof. – The idea of the proof is exactly the same as the one of Theorem 3: It is the consequence of the iterated L^2-martingale inequality and the generalized Dynkin formula (Corollary § 4.2, Lemma § 4.3, Lemma § 4.4): For $f = (U^vU^h)^2u$ with $u \geq 0$,

$$N_2[\sup_{(s,t)\in D} e^{-s-t} f(Y_{s,t})] \leq N_2[\sup_{(s,t)\in D} M^u_{s,t}] \\ \leq 4N_2[U^vU^hu] = 4\|f\|.$$

Now, by Fact 1 and continuity, this also remains valid for any potential φ. Finally, by Fact 2, we obtain the general inequality. □

THEOREM 5. – *For any finite subset $D \subset R^2_+$, for any function $u \in L^2(m)$, the following inequality holds:*

$$N_2\left[\left(\sup_{(a,b)\in D}\left|\int_{0\leq s\leq a, 0\leq t\leq b} e^{-s-t}u(Y_{s,t})dsdt\right|\right)\right] \leq 24N_2(U^vU^hu).$$

Proof. – According to the generalized Dynkin formula we have:

$$\int e^{-s-t}u(Y_{s,t})1_{\{0\leq s\leq a, 0\leq t\leq b\}}dsdt \\ = M^u_{a,b} - \int_0^a e^{-s-b}(U^v)^2u(Y_{s,b})ds \\ - \int_0^b e^{-a-t}(U^h)^2u(Y_{a,t})dt - e^{-a-b}(U^vU^h)^2u(Y_{a,b}).$$

Let us consider the right hand side terms. The first term can be controlled by iterated L^2-martingale inequality:

$$N_2[\sup_{(a,b)\in D}|M^u_{a,b}|] \leq 4N_2(U^vU^hu).$$

For the second term, we introduce further notation. By § 2, $t \to Z_{\bullet,\tau_t}$ is a μ^h-symmetric Markov process in $D(R_+, E)$. Let $\widehat{Q}^v_t$ be the corresponding semigroup. We set

$$V = \frac{1}{\sqrt{\pi}}\int_0^\infty t^{-1/2}e^{-t}\widehat{Q}^v_t dt,$$

and, for $s \geq 0$, $\xi \in D(R_+, E)$, $f_s(\xi, \sigma) = U^vu(\xi_{\sigma_s})$. Then, for $r \geq 0$, $\widehat{Q}^v_r[f_s(\bullet, \sigma)](Z_{\bullet,\tau_t}) = Q^v_r(U^vu)(Y_{s,t})$, and therefore $V[f_s(\bullet, \sigma)](Z_{\bullet,\tau_t}) = (U^v)^2u(Y_{s,t})$. Let B be a finite subset of R_+ such that $D \subset B \times B$. We obtain, by applying Theorem 3 first to process $t \to (Z_{\bullet, \tau_t}, \sigma)$ and then to process $s \to Z_{\sigma_s, 0}$,

$$N_2\left[\sup_{(a,b)\in D}\left|\int_0^a e^{-s-b}(U^v)^2 u(Y_{s,b})ds\right|\right]$$
$$\leq N_2\left[\sup_{b\in B} e^{-b} V\left[\sup_{a\in B}\left|\int_0^a e^{-s} f_s(\bullet,\ \sigma)ds\right|\right](Z_{\bullet,\tau_b})\right]$$
$$\leq 2N_2\left[\sup_{a\in B}\left|\int_0^a e^{-s} U^v u(Z_{\sigma_s,0})ds\right|\right] \leq 8N_2(U^h U^v u).$$

The third term is estimated similarly. The last term can be overestimated by the inequality in theorem 4. The proof is complete. □

6. CONSEQUENCES

Define, for any open set O in E, the *capacity* $c(O)$ by

$$\inf\{\|h\|^2;\ h \in H \text{ and } h \geq 1_O \text{ m-a.e.}\},$$

where H is the Hilbert space introduced at the beginning of § 5, with $\Xi = E$, $\mu = m$ and $G = U^v U^h$. A direct consequence of Theorem § 5.4 is: $\forall S,\ T \geq 0$, $\forall O$ open set in E,

$$P[\exists(s,\ t) \in [0, S] \times [0, T],\ Y_{s,t} \in O] \leq 16 e^{2(S+T)} c(O).$$

Now, under some analytic additional assumptions, the potential theory related to the Hilbert space H may be developed like in Kazumi-Shigekawa [7] and Hirsch [6] (*see* Hirsch-Song [11]). In particular, the notion of finite energy measure can be defined and a consequence of Theorem § 5.5 is the possibility to associate, with any finite energy measure, a continuous additive functional of Y. This is done, for the general case of n-parameter processes, in Hirsch-Song [11], [12] (where the considered notion of additive functionals is precised). As a consequence, we obtain as in Hirsch [6], Bauer [1], the following reverse inequality between hitting probabilities and capacities: $\forall S,\ T \geq 0$, $\forall O$ open set in E,

$$P[\exists(s,\ t) \in [0, S] \times [0, T],\ Y_{s,t} \in O] \geq (1 - e^{-S})^2 (1 - e^{-T})^2 c(O).$$

(*see* also Hirsch-Song [11], [12]). Various probabilistic characterizations of the quasi-continuity may also be obtained from the above estimates of capacities in terms of hitting probabilities.

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