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Average properties of random walks on Galton-Watson trees

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ABSTRACT. – We study the λ-biased random walk on Galton-Watson trees by the Dirichlet principle and a formula of mean exit time of a Markov chain. We prove that the average of escaping probability and mean exit time are bounded by the counterparts of the corresponding random walks on $\{0, 1, 2, \cdots\cdots\}$. In particular we partially verified the recent conjecture of Lyons, Pemantle and Peres on the upper bound of the speed of λ-biased random walk on Galton-Watson trees.

RÉSUMÉ. – Nous étudions la marche aléatoire de biais λ sur un arbre de Galton-Watson. Nous démontrons que la probabilité de fuite et le temps de sortie en moyenne sont bornés par ceux de la marche aléatoire correspondante sur $\{0, 1, 2, \cdots\cdots\}$. En particulier nous confirmons partiellement une conjecture de Lyons, Pemantle et Peres sur la limite supérieure de vitesse de la marche aléatoire de biais λ sur un arbre de Galton-Watson

1. INTRODUCTION

For a given tree T, a vertex is selected as the *root* and is denoted by o.

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The distance from vertex v to o is the minimum number of edges linking o and v, and is denoted by $|v|$. It is called the *level* or *generation* of v. For vertex v other than root o ($i.e$, $|v| > 0$), there is a unique adjacent vertex which is of level $|v| - 1$. This unique adjacent vertex is called the *parent* of v, and is denoted by v_*. Other adjacent vertices of v are all of level $|v| + 1$, and are called *children* of v. Let k_v be the number of children of v. It is also known as the *branching number* of v. Children of v are denoted by v_i, $i = 1, 2, \cdots k_v$.

For positive number λ, *λ-biased random walk* on T is a Markov chain $\{X_n\}$ on the vertices of T with transition probability

$$p(v, v_*) = \frac{\lambda}{\lambda + k_v}, \qquad p(v, v_i) = \frac{1}{\lambda + k_v}, \quad v \neq o. \tag{1}$$

The transition probability at o is different slightly in accordance with the lack of o_*. Let k_o be the branching number of o and o_i a child of o. We define $p(o, o_i) = 1/k_o$ in addition to (1). Note that (1) is also well defined for $\lambda = 0$ if $k_v \geq 1$ for all vertices v's of T. Let

$$\tau_s = \min\{n \geq 0; |X_n| = s\}; \tag{2}$$
$$\tau_o = \min\{n \geq 1; X_n = o\};$$
$$\gamma(T) = \lim_{s \to \infty} P(\tau_s < \tau_o | X_0 = o). \tag{3}$$

Tree T is called a Galton-Watson tree if it is a realization of a Galton-Watson process. Namely, k_v's are $i.i.d.$ random variables. Assume that the offspring distribution satisfies that

$$P(k = 0) = 0; \quad P(k = i) \geq 0, \quad \sum_{i=1}^{\infty} P(k = i) = 1. \tag{4}$$

The offspring distribution induces naturally a probability measure in the collection $\mathbf{T}$ of all Galton-Watson trees. Let $E_{\mathbf{T}}$ be the expectation according to that probability measure on $\mathbf{T}$. Define

$$m = \sum_i iP(k = i); \qquad \frac{1}{m'} = \sum_i \frac{1}{i} P(k = i). \tag{5}$$

Certainly $m \geq m' \geq 1$. *λ-biased random walk on random trees* is defined in two steps. First, take a Galton-Watson tree T according to the probability measure in $\mathbf{T}$. Then, define a random walk X_n on T according to (1) starting at root o. Thus a point in the big probability space has two components: a random tree and a random path. The offspring distribution and parameter λ determine a unique probability measure in this big space. In the following Theorem 2, the double expectation $E_{\mathbf{T}}E$ is the average first over all random walks on a fixed tree starting at root o, then over all Galton-Watson trees.

THEOREM 1. – *If $P(k = 0) = 0$ and $\lambda \leq m < \infty$, then*

$$1 - \frac{\lambda}{m} \geq E_{\mathbf{T}}\gamma(T) \geq 1 - \frac{\lambda}{m'}.$$

The equalities hold if and only if $m = m'$, i.e., *m is an integer and $P(k = m) = 1$.*

THEOREM 2. – *Assume that $P(k = 0) = 0$. Then*

$$\lim_{s\to\infty} E_T E\frac{\tau_s}{s} \geq \frac{m+\lambda}{m-\lambda} \quad \text{if } \lambda < m < \infty;$$
$$\lim_{s\to\infty} E_T E\frac{\tau_s}{s} \leq \frac{m'+\lambda}{m'-\lambda} \quad \text{if } \lambda < m'.$$

The equalities hold if and only if $m = m'$, i.e., *m is an integer and $P(k = m) = 1$.*

Random walk on random trees has been an active subject in recent years. It is shown in [4] that the random walk on random trees is transient $a.s.$ in the *big* space if $\lambda < m$. The *speed,* or the *rate of escape,* of the random walk is defined to be $\liminf_{n\to\infty} |X_n|/n$. Lyons, Pemantle and Peres proved recently in [5] that for a fixed λ ($\lambda < m$) and for $a.e.$ Galton-Watson tree T,

$$\lim_{n\to\infty} \frac{|X_n|}{n} \tag{6}$$

exists and is a positive constant, denoted by $speed(\lambda)$. $speed(\lambda)$ depends only on λ and the offspring distribution. For the case $\lambda = 1$, they computed the speed explicitly in [6].

$$speed(1) = \sum_i P(k = i)\frac{i-1}{i+1}. \tag{7}$$

On the other hand, consider the random walk on $\{0, 1, 2, 3, \cdots\}$ (which is the simplest tree) with the following transition probabilities.

$$p(0,1) = 1; \quad p(j, j-1) = \frac{\lambda}{\lambda+m}, \quad p(j, j+1) = \frac{m}{\lambda+m}, j \geq 1. \tag{8}$$

One can easily verify that $speed(\lambda) = (m - \lambda)/(m + \lambda)$ in this case. Comparing with (7) we see that when $\lambda = 1$ the random walk on random trees is slower than the corresponding random walk on $\{0, 1, 2, 3, \cdots\}$. It is often observed that a random walk is slowed down in random environments. A related example can be found in [8]. It is conjectured in [7] that

$$\lim_{n\to\infty} \frac{|X_n|}{n} \leq \frac{m - \lambda}{m + \lambda} \qquad a.s. \text{ if } \lambda < m.$$

We are motivated by this conjecture, and verify it partially.

COROLLARY 3. – *If* $P(k = 0) = 0$, $\lambda \leq 1$ *and* $m < \infty$, *then*

$$\frac{m' - \lambda}{m' + \lambda} \leq \lim_{n\to\infty} \frac{|X_n|}{n} \leq \frac{m - \lambda}{m + \lambda} \qquad a.s.$$

The equality holds if and only if $m = m'$, i.e., $P(k = m) = 1$ *for some integer* m.

By (7) and the convexity of function $(x - 1)/(x + 1)$, Corollary 3 holds for $\lambda = 1$. For $\lambda < 1$, one can show by coupling that τ_s is bounded above by that of a random walk on $\{0, 1, 2, 3, \cdots\}$ with transition probabilities

$$p(0, 1) = 1; \qquad p(j, j - 1) = \frac{\lambda}{\lambda + 1}, \quad p(j, j + 1) = \frac{1}{\lambda + 1}, j \geq 1.$$

Hence τ_s/s is uniformly integrable in the *big space*. By Proposition 5.112 of [1], we can exchange the integration and the limit, *i.e.*, the last equality, in the following derivation.

$$\frac{1}{speed(\lambda)} = \lim_{s\to\infty} \frac{\tau_s}{s} = E_T E \lim_{s\to\infty} \frac{\tau_s}{s} = \lim_{s\to\infty} E_T E \frac{\tau_s}{s}.$$

The corollary now follows from Theorem 2. The next two sections are devoted to the proof of Theorems 1 and 2 respectively.

2. PROOF OF THEOREM 1

For computing $P(\tau_s < \tau_o | X_0 = o)$ on a fixed Galton-Watson tree T, it suffices to consider $T_{[s]}$, the subtree of generations $0, 1, 2, \cdots, s$ of T. On $T_{[s]}$ define a random walk $\{X_n\}$ according to

$$\begin{aligned} p(v, v_*) &= \frac{\lambda}{\lambda + k_v}, \qquad p(v, v_i) = \frac{1}{\lambda + k_v}, \quad \text{if } 1 \leq |v| < s; \\ p(o, o_i) &= \frac{1}{k_o}; \qquad p(v, v_*) = 1 \quad \text{if } |v| = s. \end{aligned} \tag{9}$$

Then the random walk so defined is reversible in the sense $\pi_x p(x,y) = \pi_y p(y,x)$ for any vertices x, y (not necessarily adjacent) of T, and

$$\pi_o = k_o; \qquad \pi_x = \frac{\lambda + k_x}{\lambda^{|x|}} \quad \text{if } 1 \leq |x| < s; \qquad \pi_v = \frac{1}{\lambda^{s-1}} \quad \text{if } |v| = s.$$

Let H be the collection of all functions h on the vertices of $T_{[s]}$ such that

$$0 \leq h(x) \leq 1; \quad h(o) = 1; \quad h(y) = 0 \text{ if } |y| = s.$$

Then, by the *Dirichlet principle* (page 99 of [3]),

$$\pi_o P(\tau_s < \tau_o | X_0 = o) = \inf_{h \in H} \sum_{x,y} \frac{1}{2} \pi_x p(x,y) [h(x) - h(y)]^2.$$

Consequently,

$$P(\tau_s < \tau_o | X_0 = o) = \inf_{h \in H} \frac{1}{k_o} \sum_{|x|<s} \frac{1}{\lambda^{|x|}} \sum_{i=1}^{k_x} [h(x) - h(x_i)]^2. \qquad (10)$$

Upper bound. Define the decreasing sequence

$$c_n = \frac{\sum_{l=n}^{s-1} (\frac{\lambda}{m})^l}{\sum_{l=0}^{s-1} (\frac{\lambda}{m})^l} \quad n = 0, 1, 2, \cdots, s-1; \quad \text{and } c_s = 0.$$

Take $h \in H$ such that $h(x) = c_{|x|}$. Then

$$\begin{aligned} &P(\tau_s < \tau_o | X_0 = o) \\ &\leq \frac{1}{k_o} \sum_{|x|<s} \frac{1}{\lambda^{|x|}} \sum_{i=1}^{k_x} [c_{|x|} - c_{|x|+1}]^2 \\ &= \frac{1}{k_o} \sum_{l=0}^{s-1} \frac{\text{number of vertices of level } (l+1)}{\lambda^l} [c_l - c_{l+1}]^2. \\ &E_{\mathbf{T}} P(\tau_s < \tau_o | X_0 = o) \\ &\leq E_{\mathbf{T}} \frac{1}{k_o} \sum_{l=0}^{s-1} \frac{\text{number of vertices of level } (l+1)}{\lambda^l} [c_l - c_{l+1}]^2 \\ &= \sum_{l=0}^{s-1} \frac{m^l}{\lambda^l} [c_l - c_{l+1}]^2 = \frac{1}{\sum_{l=0}^{s-1} (\frac{\lambda}{m})^l} = \frac{1 - \frac{\lambda}{m}}{1 - (\frac{\lambda}{m})^s}. \end{aligned}$$

Since $P(\tau_s < \tau_o | X_0 = o)$ is decreasing in s, converges to $\gamma(T)$, and is bounded,

$$\begin{aligned} E_{\mathbf{T}}\gamma(T) &= E_{\mathbf{T}} \lim_{s\to\infty} P(\tau_s < \tau_o | X_0 = o) \\ &= \lim_{s\to\infty} E_T P(\tau_s < \tau_o | X_0 = o) \leq 1 - \frac{\lambda}{m}. \end{aligned}$$

Lower bound. Given a tree T, consider the simple *forward* random walk which chooses randomly (uniformly) among the children of the present vertex as the next vertex. Let $\mu(x)$ be the probability that the random walk starting at root o will visit vertex x. If k_{ix}'s are the branching numbers of the vertices along the shortest path from root o to x, then $\mu(x) = (k_o k_{1x} k_{2x} \cdots k_{x_*})^{-1}$. This is the *visibility measure* of the set of *rays* emanating from root o and passing vertex x. See §2 of [6] for the details.

By the Cauchy-Schwarz inequality, for any $h \in H$,

$$\left(\sum_{|x|<s}\sum_{i=1}^{k_x} \frac{1}{\lambda^{|x|}}[h(x) - h(x_i)]^2\right)^{\frac{1}{2}} \left(\sum_{|x|<s}\sum_{i=1}^{k_x} \lambda^{|x|}(\mu(x_i))^2\right)^{\frac{1}{2}}$$

$$\geq \sum_{|x|<s}\sum_{i=1}^{k_x} \mu(x_i)[h(x) - h(x_i)]$$

Since $\sum_{i=1}^{k_x} \mu(x_i) = \mu(x)$, the right hand side of the above inequality actually is equal to

$$\begin{aligned} &\sum_{l=0}^{s-1}\sum_{|x|=l}\left[\mu(x)h(x) - \sum_{i=1}^{k_x}\mu(x_i)h(x_i)\right] \\ &= \sum_{l=0}^{s-1}\left[\sum_{|x|=l}\mu(x)h(x) - \sum_{|y|=l+1}\mu(y)h(y)\right] = 1. \end{aligned}$$

Thus by (10),

$$\begin{aligned} P(\tau_s < \tau_o | X_0 = o) &\geq \frac{1}{k_o \sum_{|x|<s}\sum_{i=1}^{k_x} \lambda^{|x|}(\mu(x_i))^2} \\ &= \left[k_o \sum_{|x|<s}\sum_{i=1}^{k_x} \frac{\lambda^{|x|}}{(k_o k_{1x} k_{2x} \cdots k_x)^2}\right]^{-1}; \end{aligned}$$

and

$$E_{\mathbf{T}}P(\tau_s < \tau_o | X_0 = o) \geq \left[E_{\mathbf{T}} k_o \sum_{|x|<s} \sum_{i=1}^{k_x} \frac{\lambda^{|x|}}{(k_o k_{1x} k_{2x} \cdots k_x)^2} \right]^{-1}$$

$$= \left[1 + \frac{\lambda}{m'} + \left(\frac{\lambda}{m'}\right)^2 + \cdots + \left(\frac{\lambda}{m'}\right)^{s-1} \right]^{-1} = \frac{1 - \frac{\lambda}{m'}}{1 - (\frac{\lambda}{m'})^s}.$$

Letting $s \to \infty$ we obtain the other half of Theorem 1. □

It is shown in the proof of Corollary 3.5 of [5] that

$$E_{\mathbf{T}}\gamma(T) \geq \frac{\lambda - 1}{2\lambda}(1 - q_\lambda)$$

where q_λ is the smallest nonnegative number satisfying

$$\sum_{j=0}^{\infty} P(k = j)(1 - \lambda^{-1}(1 - q_\lambda))^j = q_\lambda.$$

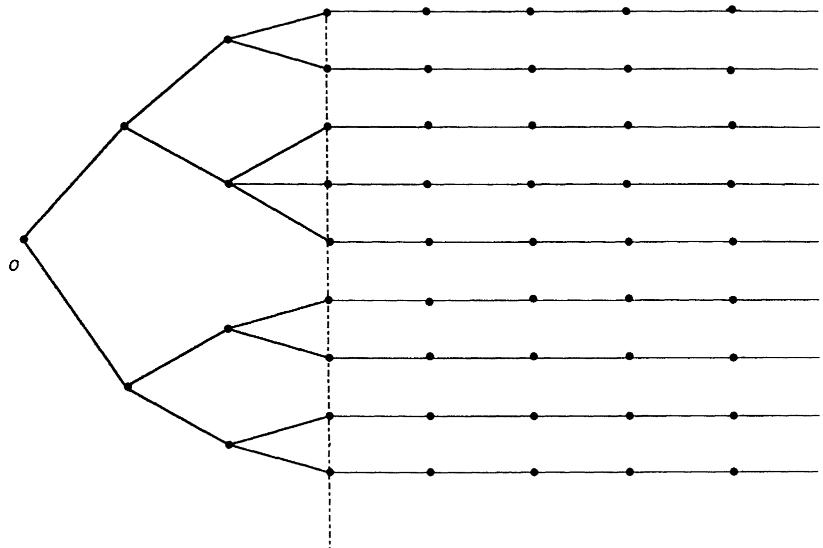
The lower bound of Theorem 1 is simpler and works better when $\lambda < 1$. $\gamma(T)$ is called the escaping probability. If tree T is thought as an electrical network, and if the resistance of an edge linking vertices of level l and $(l+1)$ is λ^l, then the total resistance between vertex o and the infinity is $1/\gamma(T)$. In deriving the lower bound we actually proved a stronger statement.

COROLLARY 4. – *If $P(k = 0) = 0$ and $\lambda \leq m' < \infty$, then the total resistance between root o and the infinity has a finite mean over all Galton-Watson trees. Namely,*

$$E_{\mathbf{T}} \frac{1}{\gamma(T)} \leq \frac{m'}{m' - \lambda}.$$

3. PROOF OF THEOREM 2

Choose $l \in [0, s]$. Take the subtree $T_{[l]}$ of the first l levels of a Galton-Watson tree and extend it by *pipes* (*see* Figure). In our earlier notation the tree is characterized by $k_v = 1$ for $|v| \geq l$. The collection of all such infinite trees with pipes at level l is denoted by $\mathbf{T}(l)$. The offspring distribution induces a probability measure on $\mathbf{T}(l)$ for every l. In the following Lemma 5, $E_{\mathbf{T}(l)}$ is the expectation taken with respect to this



induced measure on $\mathbf{T}(l)$. Restricting attention only to the first l levels, a subset of $\mathbf{T}(l)$ can be regarded also as a subset of $\mathbf{T}(l+1)$ and it has the same probability measure in both $\mathbf{T}(l)$ and $\mathbf{T}(l+1)$. This consistence of induced measures on $\mathbf{T}(l)$'s is used in the proofs of Lemma 5 and Theorem 2 below.

Run a random walk $\{X_n\}$ on $T \in \mathbf{T}(l)$ with transition probabilities

$$\begin{aligned} p(v, v_*) &= \frac{\lambda}{\lambda + k_v}, \quad p(v, v_i) = \frac{1}{\lambda + k_v} \quad \text{if } 0 < |v| < l; \\ p(v, v_*) &= \frac{\lambda}{\lambda + m}, \quad p(v, v_1) = \frac{m}{\lambda + m} \quad \text{if } 1 \leq l \leq |v|. \end{aligned} \tag{11}$$

Some obvious change is needed if $l = 0$ or $v = o$. Let $E_x \tau_s$ be the mean of the first hitting time of level s by the random walk defined by (11) starting at vertex x.

LEMMA 5. – $E_{\mathbf{T}(l+1)} E_o \tau_s \geq E_{\mathbf{T}(l)} E_o \tau_s$ *for* $0 \leq l \leq s - 1$.

Proof 1.1. – Suppose that tree $T' \in \mathbf{T}(l+1)$. That is, from level $(l+1)$ on there is only one child for each vertex. Suppose that u is a vertex of T', $|u| = l$ and k_u is the branching number of u. Notice that there are k_u pipes emanating from u and the transition probabilities along these pipes are identical. So we combine these pipes together as one *combined* *pipe*. Let u_1 be the only child of u after this combination, and change the transition probability at u as

$$p(u, u_*) = \frac{\lambda}{\lambda + k_u}, \quad p(u, u_1) = \frac{k_u}{\lambda + k_u}. \tag{12}$$

The randomness of the branching number of u is converted to the randomness of transition probability at u. The distribution of τ_s is preserved after this modification. In particular, we have

$$E_u\tau_s = 1 + \frac{\lambda}{\lambda + k_u} E_{u_*}\tau_s + \frac{k_u}{\lambda + k_u} E_{u_1}\tau_s. \tag{13}$$

In general

$$E_x\tau_s = 1 + \frac{\lambda}{\lambda + k_x} E_{x_*}\tau_s + \sum_{i=1}^{k_x} \frac{1}{\lambda + k_x} E_{x_i}\tau_s \qquad \text{if } 1 \le |x| \le l, x \ne u;$$

$$E_x\tau_s = 1 + \frac{\lambda}{\lambda + m} E_{x_*}\tau_s + \frac{m}{\lambda + m} E_{x_1}\tau_s \qquad \text{if } l + 1 \le |x| \le s - 1;$$

$$E_o\tau_s = 1 + \sum_{i=1}^{k_o} \frac{1}{k_o} E_{o_i}\tau_s; \qquad \text{and} \quad E_x\tau_s = 0 \quad \text{if } |x| = s.$$

Replacing (13) by

$$(\lambda + k_u)E_u\tau_s = (\lambda + k_u) + \lambda E_{u_*}\tau_s + k_u E_{u_1}\tau_s$$

and solving the system of linear equations by the Cramer rule, we see that $E_o\tau_s$ is the quotient of two determinants. Notice that k_u appears only in the last equation. Thus each determinant is a linear function of k_u and

$$E_o\tau_s = \frac{ak_u + b}{ck_u + d} \tag{14}$$

where a, b, c and d are independent of k_u.

Function $f(x) = (ax+b)/(cx+d)$ is convex if and only if $f(0) \ge f(\infty)$. However, $f(0)$ is $E_o\tau_s$ when $k_u = 0$, or in other words, $p(u, u_1) = 0$, $p(u, u_*) = 1$; and $f(\infty)$ is $E_o\tau_s$ when $p(u, u_1) = 1$, $p(u, u_*) = 0$. Define two random walks $\{Y_n\}$ and $\{Z_n\}$, both starting at root o, with the same transition probability everywhere except at u. For $\{Y_n\}$, $p(u, u_1) = 0$, $p(u, u_*) = 1$; for $\{Z_n\}$, $p(u, u_1) = 1$, $p(u, u_*) = 0$. Notice that the combined pipe and other pipes of the tree are symmetric beyond level

$(l+1)$, including level $(l+1)$. So $|Y_n| \leq |Z_n|$ by the method of coupling. It is follows from this fact that $f(0) > f(\infty)$ (unless $s = 1$).

We have demonstrated that $E_o\tau_s$ is a convex function of k_u. By the Jensen's inequality, the average of $E_o\tau_s$ over all possible k_u is greater than or equal to $(am+b)/(cm+d)$. This is exactly the mean hitting time of level s by the random walk with deterministic transition probability at u,

$$p(u, u_*) = \frac{\lambda}{\lambda + m}, \quad p(u, u_1) = \frac{m}{\lambda + m}.$$

The above argument can be applied to other vertices of level l one by one to decrease the mean hitting time of level s. What we have proved is that for $T \in \mathbf{T}(l)$, $E_o\tau_s$ is less than or equal to the average of $E_o\tau_s$ over those trees of $\mathbf{T}(l+1)$ whose subtree of first l levels is T. The equality holds if and only if $P(k = m) = 1$ for some integer m. The statement of this lemma then follows by taking the average of random trees of $\mathbf{T}(l)$. Namely, take $E_{\mathbf{T}(l)}$. $\square$

Remark. – This simplied proof is kindly suggested to the author by Professor R. Lyons. The original proof is lengthy and uses a cumbersome formula of the mean exit time from [2].

Proof of Theorem 2. – The distribution of first hitting time τ_s of level s is determined by the subtree of first s levels. By the consistence of induced measures on $\mathbf{T}(s)$ and $\mathbf{T}$, and by Lemma 5, we have that

$$E_{\mathbf{T}} E\tau_s = E_{\mathbf{T}(s)} E_o\tau_s \geq E_{\mathbf{T}(0)} E_o\tau_s. \tag{15}$$

However, there is only one member of $\mathbf{T}(0)$. The right hand side of (15) further reduces to $E_0\tau_s$, the mean of the first hitting time τ_s of s by the random walk on $\{0, 1, 2, 3, \cdots\}$ starting at 0 with transition probabilities given by (8). This can be calculated by solving a system of linear equations.

$$E_0\tau_s = s\frac{m+\lambda}{m-\lambda} - \frac{2m\lambda}{(m-\lambda)^2} + \left(\frac{\lambda}{m}\right)^{s-1} \frac{2\lambda^2}{(m-\lambda)^2}. \tag{16}$$

The first half of Theorem 2 is now an easy consequence of (15) and (16).

For the second half, rewrite (14) as

$$E_o\tau_s = \frac{a + b/k_u}{c + d/k_u}$$

which is a concave function of $1/k_u$. Taking the average over k_u we get

$$E_{k_u} E_o\tau_s \leq \frac{a + bE(1/k_u)}{c + dE(1/k_u)} = \frac{a + b/m'}{c + d/m'} = \frac{am' + b}{cm' + d}.$$

The remaining argument is identical with that of the first half. $\square$

Remark. – It is for simplicity that we assume throughout this paper that $P(k = 0) = 0$. This assumption is needed in the half involving m' of both theorems; but is not required for the other half (involving m).

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