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Shortest excursion lengths

by

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ABSTRACT. – An interval is composed of excursions of Brownian motion. We determine the asymptotic contributions from the shortest lengths of these excursions. © Elsevier, Paris

Key words: Excursion length, Brownian motion.

RÉSUMÉ. – On étudie la somme des longueurs des excursions browniennes les plus courtes. © Elsevier, Paris

1. INTRODUCTION

Let $\{W(t); t \geq 0\}$ be real-valued Brownian motion starting from 0, and let

$$V_1(t) \geq V_2(t) \geq \dots \geq V_n(t) \geq \dots$$

be the ordered lengths of its excursions away from 0 before time t , the last zero-free interval being considered as a possibly incomplete excursion

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interval. The study of ranked excursion lengths of Brownian motion and other stochastic processes has received much research interest, concerning both distributional and sample paths properties. See for example, Csáki et al. [6], Hu and Shi [8], Pitman [10], Pitman and Yor [11]–[14], Révész [16] and Scheffer [18]. Besides theoretical interest, one of the motivations is the close relation between ranked excursion lengths and size-biased random permutation of the Poisson–Dirichlet distribution, the latter being a subject intervening in various branches of probability and number theory. See Pitman and Yor [12] for a full story, and many references. We also mention that ranked excursion lengths recently find applications in financial mathematics, cf. Chesney et al. [4].

Consider the identity:

$$(1.1) \quad \sum_{j=1}^{\infty} V_j(t) = t.$$

Of course, the longest excursion lengths have the most important contributions in (1.1). It is known (cf. Chung and Erdős [5], Révész [15, Chap. 13]) that infinitely often $V_1(t)$ alone may play an overwhelming role. More precisely, $\limsup_{t \rightarrow \infty} V_1(t)/t = 1$ with probability one. However, one may still wonder the contributions in (1.1) from short excursion lengths. Are they totally negligible?

The problem is attacked by Csáki et al. [6], who prove that for any finite constant $r > 0$ and positive function $k(t) \uparrow \infty$,

$$(1.2) \quad \limsup_{t \rightarrow \infty} \frac{1}{t} \sum_{j > r \log_2 t} V_j(t) > 0, \quad \text{a.s.},$$

$$(1.3) \quad \lim_{t \rightarrow \infty} \frac{1}{t} \sum_{j > k(t) \log_2 t} V_j(t) = 0, \quad \text{a.s.},$$

where $\log_2 t \stackrel{\text{def}}{=} \log \log t$. The estimates (1.2) and (1.3) together confirm that, for any monotone positive function $k(t)$, there is asymptotically no contribution in (1.1) from $\{V_j(t); j > k(t) \log_2 t\}$ if and only if $k(t) \uparrow \infty$.

It is certainly interesting to see how negligible the shortest excursion lengths are, in case $k(t) \uparrow \infty$. Let us recall a recent result of Révész [16].

THEOREM A (Révész [16]). – *For $k(t) \uparrow \infty$ such that $t/k(t)$ is non-decreasing,*

$$(1.4) \quad \limsup_{t \rightarrow \infty} \frac{k(t)}{t} \sum_{j > k(t) \log_2 t} V_j(t) \leq (\sqrt{2} + 1) \pi, \quad \text{a.s.}$$

Since $k(t)$ tends to infinity, (1.4) is clearly a quantitative refinement of (1.3). More precisely, it tells us that with probability one, the asymptotic proportion of the contributions in (1.1) from $\{V_j(t); j > k(t) \log_2 t\}$ are no more than (a constant multiplication of) $1/k(t)$. Is this accurate?

The answer is affirmative.

THEOREM 1.1. – *If $k(t) \uparrow \infty$ and if $t/k(t)$ is non-decreasing, then*

$$(1.5) \quad \limsup_{t \rightarrow \infty} \frac{k(t)}{t} \sum_{j > k(t) \log_2 t} V_j(t) = \frac{4}{\pi}, \quad \text{a.s.}$$

Even in case $k(t)$ has a finite limit (to which we refer as the “critical” case), (1.5) still remains true, except for the constant on the right hand side which should be replaced by another one. For more details, cf. Section 4.

Theorem 1.1 gives us the largest contributions possible from $\{V_j(t); j > k(t) \log_2 t\}$ along some exceptional (random) subsequences. It may also be interesting to seek the smallest contributions possible. We consider this problem for all $\{V_j(t); j > \ell(t)\}$.

THEOREM 1.2. – *For any non-decreasing functions $\ell(t) > 1$ and $\varphi(t) > 0$ such that $t/\ell(t)$ is non-decreasing,*

$$\liminf_{t \rightarrow \infty} \frac{\ell(t) \varphi(t)}{t} \sum_{j > \ell(t)} V_j(t) = 0 \text{ or } \infty, \quad \text{a.s.,}$$

according as whether

$$\int^{\infty} \frac{dt}{t \varphi^{1/2}(t)}$$

diverges or converges.

Remark 1.3. – The condition “ $\ell(t) > 1$ ” is to ensure that we are dealing with a non-trivial situation. In case $\ell(t)$ has a finite limit, Theorem 1.2 is a particular case of [8, Theorem 1.1].

Remark 1.4. – There is a discrete-time analogue (i.e. for simple symmetric random walks) for Theorems 1.1 and 1.2, using Skorokhod’s embedding (for details of this kind of argument for excursion lengths, cf. Csáki et al. [6]). We also mention that the corresponding problem in discrete-time setting in dimension 2 is studied by Csáki et al. [7], who prove that the two longest excursion lengths eventually play an overwhelming role. In dimension 2 the problem is posed only for random walks, since any point is polar for Brownian motion.

Remark 1.5. – Theorems A and 1.1 remain true if, instead of the monotonicity of $t/k(t)$, we assume that $k(t)$ is of slow variation.

The rest of the paper is organized as follows. Section 2 is devoted to the study of tail behaviours of the excursion lengths. Theorems 1.1 and 1.2 are proved in Section 3. In Section 4, we treat the critical case, i.e. the limsup behaviour of $\sup_{j>r \log_2 t} V_j(t)$ for each fixed constant $r > 0$. Finally, we extend our results for all recurrent Bessel processes in Section 5.

Throughout the paper, we adopt the usual notation $a(x) \sim b(x)$ ($x \rightarrow x_0$) to denote $\lim_{x \rightarrow x_0} a(x)/b(x) = 1$.

2. DISTRIBUTIONS OF THE EXCURSION LENGTHS

We first recall two important results. The first is a decomposition theorem due to Pitman and Yor [12]. The second is the classical Chernoff's large deviation theorem for iid variables (cf. for example Bahadur [1] for detailed discussions).

THEOREM B (Pitman and Yor [12]). – *For any $n \geq 2$,*

$$(2.1) \quad \frac{1}{\sum_{j=1}^n V_j(1)} - 1 \stackrel{\text{law}}{=} \frac{\sum_{j=1}^n \beta_j}{n + \sum_{j=1}^{n-1} \alpha_j},$$

where " $\stackrel{\text{law}}{=}$ " denotes identity in distribution, $\{\alpha_j\}_{j \geq 1}$ and $\{\beta_j\}_{j \geq 1}$ are independent positive random variables whose laws are determined by: for all $j \geq 1$,

$$(2.2) \quad \mathbb{P}(\alpha_j > x) = (1+x)^{-1/2}, \quad x \geq 0,$$

$$(2.3) \quad \mathbb{E} e^{-\lambda \beta_j} = \left(1 + \int_0^1 \frac{1 - e^{-\lambda y}}{2y^{3/2}} dy\right)^{-1}, \quad \lambda \geq 0.$$

THEOREM C (Chernoff [3]). – *Let $\{X_i\}_{i \geq 1}$ be iid variables with $\psi(t) \stackrel{\text{def}}{=} \mathbb{E} \exp(tX_1) \in (0, \infty]$ for $t \geq 0$. If $\mathbb{P}(X_1 > b) > 0$, then*

$$\log \mathbb{P} \left[\sum_{i=1}^n X_i \geq nb \right] \sim -\rho(b)n, \quad n \rightarrow \infty,$$

where

$$\rho(b) \stackrel{\text{def}}{=} \sup_{t \geq 0: \psi(t) < \infty} (bt - \log \psi(t)).$$

In view of (2.1) and the scaling property, the study of the $V_j(t)$'s, when t is fixed, is reduced to that of the α_j 's and β_j 's. Throughout the section, let $\mathcal{N}, \mathcal{N}_1, \mathcal{N}_2, \dots$ denote independent Gaussian $\mathcal{N}(0, 1)$ random variables, independent of all the other processes and variables. It is well-known that \mathcal{N}^{-2} is a stable variable of index $(1/2)$, in the sense that

$$(2.4) \quad \sum_{j=1}^n \frac{1}{\mathcal{N}_j^2} \stackrel{\text{law}}{=} \frac{n^2}{\mathcal{N}^2},$$

for all $n \geq 1$ (an easy way to see this is to consider the first hitting times for Brownian motion).

LEMMA 2.1. – For each $\varepsilon \in (0, 1)$, there exists a finite constant $M(\varepsilon) > 0$, depending only on ε , such that for all $x > 0$,

$$(2.5) \quad \mathbb{P}\left(\alpha_1 + 1 > x\right) \geq \mathbb{P}\left[\frac{1}{\mathcal{N}^2} > \frac{2x}{\pi}\right],$$

$$(2.6) \quad \mathbb{P}(\alpha_1 > x) \leq \mathbb{P}\left[\frac{1}{\mathcal{N}^2} + M(\varepsilon) > (1 - \varepsilon)\frac{2x}{\pi}\right].$$

Proof. – Elementary. □

LEMMA 2.2. – For $\varepsilon \in (0, 1)$, there exist $M_1(\varepsilon) > 0$ and $M_2(\varepsilon) > 0$ such that for all $n \geq M_1(\varepsilon)$,

$$(2.7) \quad \mathbb{P}\left[\sum_{j=1}^n \beta_j \geq (1 + \varepsilon)n\right] \leq e^{-M_2(\varepsilon)n},$$

$$(2.8) \quad \mathbb{P}\left[\sum_{j=1}^n \beta_j \leq (1 - \varepsilon)n\right] \leq e^{-M_2(\varepsilon)n}.$$

Proof. – From (2.3), it follows that $E\beta_1 = 1$ and that $Ee^{\lambda\beta_1} < \infty$ when $\lambda > 0$ is sufficiently small. Now Lemma 2.2 is a straightforward application of Theorem C. □

THEOREM 2.3. – Fix $\lambda > 0$. For $k(t) \uparrow \infty$,

$$(2.9) \quad \log \mathbb{P}\left[\sum_{j>k(t)\log_2 t} V_j(1) > \frac{\lambda}{k(t)}\right] \sim -\frac{\pi\lambda}{4} \log_2 t, \quad t \rightarrow \infty.$$

Proof. – Let Δ_1 denote the probability term on the left hand side of (2.9). For notational simplicity, we write $k \stackrel{\text{def}}{=} k(t)$ in the proof. By (1.1) and (2.1), for all sufficiently large t ,

$$\Delta_1 = \mathbb{P} \left[\left(\frac{k}{\lambda} - 1 \right) \sum_{j \leq k \log_2 t} \beta_j > [k \log_2 t] + \sum_{j \leq k \log_2 t-1} \alpha_j \right].$$

To get the upper bound for Δ_1 , let us fix $\varepsilon \in (0, 1)$. Observe that by (2.7), for all large t ,

$$\begin{aligned} \Delta_1 &\leq \mathbb{P} \left[\sum_{j \leq k \log_2 t} \beta_j > (1 - \varepsilon) \lambda \log_2 t + \frac{\lambda}{k} \sum_{j \leq k \log_2 t-1} \alpha_j \right] \\ (2.10) \quad &\leq \Delta_2 + \exp(-M_2(\varepsilon) [k \log_2 t]), \end{aligned}$$

where

$$\begin{aligned} \Delta_2 &\stackrel{\text{def}}{=} \mathbb{P} \left[(1 + \varepsilon) [k \log_2 t] > (1 - \varepsilon) \lambda \log_2 t + \frac{\lambda}{k} \sum_{j \leq k \log_2 t-1} \alpha_j \right] \\ &\leq \mathbb{P} \left[\sum_{j \leq k \log_2 t-1} \alpha_j < (1 + 2\varepsilon) \frac{k^2 \log_2 t}{\lambda} \right] \\ &\leq \mathbb{P} \left[\sum_{j \leq k \log_2 t-1} (\alpha_j + 1) < (1 + 3\varepsilon) \frac{k^2 \log_2 t}{\lambda} \right]. \end{aligned}$$

Applying (2.5) and then using (2.4), this leads to

$$\begin{aligned} \Delta_2 &\leq \mathbb{P} \left[\sum_{j \leq k \log_2 t-1} \frac{1}{\mathcal{N}_j^2} < (1 + 3\varepsilon) \frac{2k^2 \log_2 t}{\lambda \pi} \right] \\ &= \mathbb{P} \left[\frac{([k \log_2 t] - 1)^2}{\mathcal{N}^2} < (1 + 3\varepsilon) \frac{2k^2 \log_2 t}{\lambda \pi} \right], \end{aligned}$$

which, by the usual estimate for Gaussian tails, yields

$$\limsup_{t \rightarrow \infty} \frac{\log \Delta_2}{\log_2 t} \leq -\frac{\lambda \pi}{4(1 + 3\varepsilon)}.$$

Going back to (2.10), we have

$$\limsup_{t \rightarrow \infty} \frac{\log \Delta_1}{\log_2 t} \leq -\frac{\lambda \pi}{4(1 + 3\varepsilon)},$$

which implies the desired upper bound for Δ_1 by sending ε to 0.

The lower bound can be proved exactly in the same way, using (2.8) and (2.6) in lieu of (2.7) and (2.5) respectively. \square

LEMMA 2.4. – *There exists a universal constant $c > 0$ such that*

$$(2.11) \quad \mathbb{P} \left[\beta_1 + \beta_2 < x \right] \leq cx, \quad 0 < x \leq 1,$$

$$(2.12) \quad \limsup_{n \rightarrow \infty} n^{1/2} \mathbb{E} \left[\left(\sum_{j=1}^n \beta_j \right)^{-1/2} \right] \leq 1.$$

Proof. – The estimate (2.11) is trivial. Indeed, by Chebyshev’s inequality and (2.3),

$$\mathbb{P} \left[\beta_1 + \beta_2 < x \right] \leq \frac{e}{\left(1 + \int_0^1 (1 - e^{-y/x}) dy / (2y^{3/2}) \right)^2} \leq cx,$$

for all $0 < x \leq 1$. To verify (2.12), observe that for each $\varepsilon \in (0, 1)$,

$$\begin{aligned} \mathbb{E} \left[\left(\sum_{j=1}^n \beta_j \right)^{-1/2} \right] &= \int_0^\infty \mathbb{P} \left[\sum_{j=1}^n \beta_j < \frac{1}{z^2} \right] dz \\ &= \left(\int_0^{1/\sqrt{(1-\varepsilon)n}} + \int_{1/\sqrt{(1-\varepsilon)n}}^n + \int_n^\infty \right) \mathbb{P} \left[\sum_{j=1}^n \beta_j < \frac{1}{z^2} \right] dz \\ &\leq \frac{1}{\sqrt{(1-\varepsilon)n}} + n \mathbb{P} \left[\sum_{j=1}^n \beta_j < (1-\varepsilon)n \right] + \int_n^\infty \mathbb{P} \left[\beta_1 + \beta_2 < \frac{1}{z^2} \right] dz. \end{aligned}$$

Applying (2.8) and (2.11), the above is smaller than (for large n)

$$\frac{1}{\sqrt{(1-\varepsilon)n}} + n \exp(-M_2(\varepsilon)n) + c \int_n^\infty \frac{dz}{z^2},$$

which implies (2.12), as ε can be arbitrarily close to 0. \square

THEOREM 2.5. – *For $\ell(t) \uparrow \infty$ and $\varphi(t) \uparrow \infty$,*

$$(2.13) \quad \mathbb{P} \left[\sum_{j>\ell(t)} V_j(1) < \frac{1}{\ell(t)\varphi(t)} \right] \sim \frac{1}{\varphi^{1/2}(t)}, \quad t \rightarrow \infty.$$

Proof. – The main difference between the proofs of Theorems 2.5 and 2.3 is that we can no longer use Lemma 2.2, since $e^{-M_2(\varepsilon)\ell(t)}$ is not always negligible with respect to $\varphi^{-1/2}(t)$. However, with minor modifications, the lower bound in (2.13) can be obtained as in the proof of Theorem 2.3.

Indeed, let Δ_3 denote the probability term on the left hand side of (2.13). Write $\ell \stackrel{\text{def}}{=} \ell(t)$ and $\varphi \stackrel{\text{def}}{=} \varphi(t)$. By an abuse of notation, we shall write ℓ instead of $[\ell]$, treating ℓ as an integer. By (2.1), for large t ,

$$(2.14) \quad \Delta_3 = \mathbb{P} \left[(\ell\varphi - 1) \sum_{j=1}^{\ell} \beta_j < \ell + \sum_{j=1}^{\ell-1} \alpha_j \right].$$

Fix an $\varepsilon \in (0, 1)$. Since $\mathbb{E}(\beta_j) = 1$, by law of large numbers, for all t sufficiently large, $\mathbb{P}(\sum_{j=1}^{\ell} \beta_j < (1 + \varepsilon)\ell) \geq 1 - \varepsilon$, which, in view of independence, implies

$$\begin{aligned} \Delta_3 &\geq (1 - \varepsilon) \mathbb{P} \left[(\ell\varphi - 1)(1 + \varepsilon)\ell < \ell + \sum_{j=1}^{\ell-1} \alpha_j \right] \\ &\geq (1 - \varepsilon) \mathbb{P} \left[\sum_{j=1}^{\ell-1} \alpha_j > (1 + \varepsilon)\ell^2\varphi \right]. \end{aligned}$$

According to (2.5) and (2.4),

$$\begin{aligned} \Delta_3 &\geq (1 - \varepsilon) \mathbb{P} \left[\sum_{j=1}^{\ell-1} \frac{1}{\mathcal{N}_j^2} > \frac{2((1 + \varepsilon)\ell^2\varphi + \ell - 1)}{\pi} \right] \\ &= (1 - \varepsilon) \mathbb{P} \left[\frac{(\ell - 1)^2}{\mathcal{N}^2} > \frac{2((1 + \varepsilon)\ell^2\varphi + \ell - 1)}{\pi} \right], \end{aligned}$$

which yields

$$\liminf_{t \rightarrow \infty} \varphi^{1/2} \Delta_3 \geq \frac{1 - \varepsilon}{(1 + \varepsilon)^{1/2}}.$$

This leads to the lower bound in (2.13) by sending ε to 0.

To prove the upper bound, let us go back to (2.14), fix an ε and use (2.6) to arrive at:

$$\begin{aligned} \Delta_3 &\leq \mathbb{P} \left[\sum_{j=1}^{\ell-1} \frac{1}{\mathcal{N}_j^2} > (1 - \varepsilon) \frac{2}{\pi} \left((\ell\varphi - 1) \sum_{j=1}^{\ell} \beta_j - \ell \right) - (\ell - 1)M(\varepsilon) \right] \\ &\leq \mathbb{P} \left[\sum_{j=1}^{\ell} \frac{1}{\mathcal{N}_j^2} > (1 - 3\varepsilon) \frac{2}{\pi} \ell\varphi \sum_{j=1}^{\ell} \beta_j \right] + \mathbb{P} \left[\sum_{j=1}^{\ell} \beta_j < \frac{M(\varepsilon) + 1}{\varepsilon\varphi} \right], \end{aligned}$$

for all large t . By means of (2.4) and the boundedness of the Gaussian density, the above is

$$\leq \sqrt{\frac{\ell}{(1 - 3\varepsilon)\varphi}} \mathbb{E} \left[\left(\sum_{j=1}^{\ell} \beta_j \right)^{-1/2} \right] + \mathbb{P} \left[\beta_1 + \beta_2 < \frac{M(\varepsilon) + 1}{\varepsilon\varphi} \right].$$

This yields the upper bound in Theorem 2.5 by means of Lemma 2.4. \square

3. PROOFS OF THEOREMS 1.1 AND 1.2

Recall the statement of Theorem 1.1: for $k(t) \uparrow \infty$ such that $t/k(t)$ is non-decreasing,

$$(3.1) \quad \limsup_{t \rightarrow \infty} \frac{k(t)}{t} \sum_{j > k(t) \log_2 t} V_j(t) \leq \frac{4}{\pi}, \quad \text{a.s.}$$

$$(3.2) \quad \limsup_{t \rightarrow \infty} \frac{k(t)}{t} \sum_{j > k(t) \log_2 t} V_j(t) \geq \frac{4}{\pi}, \quad \text{a.s.}$$

We first observe that the excursion lengths inherit the scaling property in the following sense: for any fixed $t > 0$,

$$(3.3) \quad \{V_j(t); j \geq 1\} \stackrel{\text{law}}{=} \{t V_j(1); j \geq 1\}.$$

Proof of Theorem 1.1. – The upper bound (3.1) is a straightforward consequence of Theorem 2.3. Indeed, fix $\theta > 1$, $\varepsilon \in (0, 1)$ and define $t_n \stackrel{\text{def}}{=} \theta^n$. By scaling (cf. (3.3)) and Theorem 2.3, for all sufficiently large n ,

$$\mathbb{P} \left[\sum_{j > k(t_n) \log_2 t_n} V_j(t_{n+1}) > \frac{(1 + 2\varepsilon)4}{\pi} \frac{t_{n+1}}{k(t_n)} \right] \leq \exp \left(-(1 + \varepsilon) \log_2 t_n \right),$$

which is summable for n . By the Borel–Cantelli lemma, almost surely for all large n ,

$$\sum_{j > k(t_n) \log_2 t_n} V_j(t_{n+1}) \leq \frac{(1 + 2\varepsilon)4\theta}{\pi} \frac{t_n}{k(t_n)}.$$

By monotonicity, this yields

$$\limsup_{t \rightarrow \infty} \frac{k(t)}{t} \sum_{j > k(t) \log_2 t} V_j(t) \leq \frac{(1 + 2\varepsilon)4\theta}{\pi}, \quad \text{a.s.},$$

which leads to (3.1) by letting θ and ε tend to 1 and 0 respectively.

To verify (3.2), we have to overcome the dependence difficulty. Let \mathcal{F} denote the natural filtration of the Brownian motion W . For each $t > 0$, define

$$(3.4) \quad G(t) \stackrel{\text{def}}{=} \sup \left\{ s \leq t : W(s) = 0 \right\},$$

$$(3.5) \quad D(t) \stackrel{\text{def}}{=} \inf \left\{ s \geq t : W(s) = 0 \right\}.$$

In words, $G(t)$ and $D(t)$ are respectively the left and right extremities of the excursion interval straddling t . Note that for each t , $D(t)$ is a stopping time. Fix an $\varepsilon \in (0, 1)$. Let $t_n \stackrel{\text{def}}{=} n^{4n}$. Consider the event

$$E_n \stackrel{\text{def}}{=} \left\{ \sum_{j > k(t_n) \log_2 t_n} V_j(t_n) > \frac{(1 - 2\varepsilon)4}{\pi} \frac{t_n}{k(t_n)} \right\}.$$

Of course, for each n , E_n is $\mathcal{F}_{D(t_n)}$ -measurable. If we could verify

$$(3.6) \quad \sum_n \mathbb{P}\left(E_n \mid \mathcal{F}_{D(t_{n-1})}\right) = \infty, \quad \text{a.s.},$$

then according to Paul Lévy’s Borel–Cantelli lemma (cf. for example, Shiryaev [19, p. 518]), we would have proved $\mathbb{P}(\limsup_n E_n) = 1$ which would in turn yield (3.2).

It remains to check (3.6). Consider the process $\{\widetilde{W}(s) \stackrel{\text{def}}{=} W(s + D(t_{n-1})); s \geq 0\}$, which by the strong Markov property is again one-dimensional Brownian motion, independent of $\mathcal{F}_{D(t_{n-1})}$. For each $t > 0$, we can define

$$\widetilde{V}_1(t) \geq \widetilde{V}_2(t) \geq \dots,$$

the ordered excursion lengths of \widetilde{W} before time t . Clearly,

$$\left\{ \sum_{j > m} V_j(s + D(t_{n-1})) > x \right\} \supset \left\{ \sum_{j > m} \widetilde{V}_j(s) > x \right\},$$

for any positive m , s and x . Consequently,

$$\begin{aligned} & \mathbb{P}\left(E_n \mid \mathcal{F}_{D(t_{n-1})}\right) \\ & \geq \mathbf{1}_{\{D(t_{n-1}) < \varepsilon t_n\}} \mathbb{P}\left[\sum_{j > k(t_n) \log_2 t_n} V_j((1 - \varepsilon)t_n) > \frac{(1 - 2\varepsilon)4}{\pi} \frac{t_n}{k(t_n)} \right], \end{aligned}$$

where $\mathbf{1}$ denotes the indicator function. By means of the scaling property (cf. (3.3)) and Theorem 2.3, it is easily seen that the probability term on the right hand side is greater than $\exp(-\log_2 t_n) = 1/(4n \log n)$ for all sufficiently large n , which yields

$$(3.7) \quad \mathbb{P}\left(E_n \mid \mathcal{F}_{D(t_{n-1})}\right) \geq \frac{\mathbf{1}_{\{D(t_{n-1}) < \varepsilon t_n\}}}{4n \log n}.$$

To complete the proof of (3.6), we recall a classical result of Chung and Erdős [5] for the lower functions of $G(t)$: for any non-decreasing function

$f > 0$, $\liminf_{t \rightarrow \infty} f(G(t))/t$ equals 0 or ∞ almost surely, according as whether $\int^\infty dt/\sqrt{t f(t)}$ diverges or converges. (This test is extended to general Markov processes by Bertoin [2]). In particular, the test confirms that with probability one, $G(t)(\log G(t))^3 > t$ for all large t . This yields $D(t) < t(\log t)^3$ for large t . Since $t_{n-1}(\log t_{n-1})^3 < \varepsilon t_n$ when n is sufficiently large (how large depending on ε), we have $D(t_{n-1}) < \varepsilon t_n$ almost surely. In view of (3.7), we have proved (3.6), hence Theorem 1.1. \square

We restate Theorem 1.2 here: if $\ell(t) > 1$ and $\varphi(t) > 0$ are non-decreasing such that $t/\ell(t)$ is non-decreasing,

$$(3.8) \quad \int^\infty \frac{dt}{t\varphi^{1/2}(t)} < \infty \implies \liminf_{t \rightarrow \infty} \frac{\ell(t)\varphi(t)}{t} \sum_{j>\ell(t)} V_j(t) = \infty, \quad \text{a.s.}$$

$$(3.9) \quad \int^\infty \frac{dt}{t\varphi^{1/2}(t)} = \infty \implies \liminf_{t \rightarrow \infty} \frac{\ell(t)\varphi(t)}{t} \sum_{j>\ell(t)} V_j(t) < \infty, \quad \text{a.s.}$$

Proof of Theorem 1.2. – When $\ell(t)$ has a finite limit, Theorem 1.2 is proved in [8] (cf. Remark 1.3). Therefore we assume $\ell(t) \uparrow \infty$. Fix $\theta > 1$, and let $t_n \stackrel{\text{def}}{=} \theta^n$.

The proof of (3.8) is routine, using Theorem 2.5, Borel–Cantelli, monotonicity, and the fact that $\int^\infty dt/t\varphi^{1/2}(t) < \infty$ if and only if $\sum_n \varphi^{-1/2}(t_n) < \infty$.

To verify (3.9), assume $\int^\infty dt/t\varphi^{1/2}(t) = \infty$. In the rest of the proof, by an abuse of notation, we shall write $\ell(t_n)$ instead of $[\ell(t_n)]$, treating $\ell(t_n)$ as an integer. Recall $G(t)$ and $D(t)$ from (3.4)–(3.5). Fix constants $b > 0$ and $0 < \varepsilon < 1$. Consider the event

$$F_n \stackrel{\text{def}}{=} \left\{ \sum_{j>\ell(t_n)} V_j(t_n) < \frac{b t_n}{\ell(t_n)\varphi(t_n)} \right\}.$$

By scaling (cf. (3.3)) and Theorem 2.5,

$$(3.10) \quad \mathbb{P}(F_n) \sim \frac{b^{1/2}}{\varphi^{1/2}(t_n)}, \quad n \rightarrow \infty,$$

which implies

$$(3.11) \quad \sum_n \mathbb{P}(F_n) = \infty.$$

We now apply a second moment argument. Let $m > n + 1$. Let \mathcal{F} denote as before the natural filtration of W . Define $\{\widehat{W}(s) \stackrel{\text{def}}{=} W(s + D(t_n)); s \geq 0\}$ which is again Brownian motion independent of \mathcal{F}_{t_n} , and denote by $\widehat{V}_1(t) \geq \widehat{V}_2(t) \geq \dots$ the ranked excursion lengths of \widehat{W} over the interval $(0, t)$ for each $t > 0$. Since for all positive i and λ ,

$$\left\{ D(t_n) < t_{m-1}; \sum_{j>i} V_j(t_m) < \lambda \right\} \subset \left\{ \sum_{j>i} \widehat{V}_j(t_m - t_{m-1}) < \lambda \right\},$$

we obtain:

$$\begin{aligned} & \mathbb{P}\left(F_n; F_m; D(t_n) < t_{m-1}\right) \\ & \leq \mathbb{P}(F_n) \mathbb{P}\left[\sum_{j>\ell(t_m)} V_j(t_m - t_{m-1}) < \frac{b t_m}{\ell(t_m) \varphi(t_m)}\right] \\ & \leq (1 + \varepsilon) \mathbb{P}(F_n) \sqrt{\frac{b}{(1 - \theta^{-1}) \varphi(t_m)}} \\ (3.12) \quad & \leq \frac{1 + 2\varepsilon}{\sqrt{1 - \theta^{-1}}} \mathbb{P}(F_n) \mathbb{P}(F_m). \end{aligned}$$

(We have used respectively Theorem 2.5 and (3.10) in the last two inequalities). On the other hand, by the Markov property, one can write

$$D(t_n) = t_n + \frac{W^2(t_n)}{\mathcal{N}^2},$$

where \mathcal{N} is a Gaussian $\mathcal{N}(0, 1)$ variable, independent of \mathcal{F}_{t_n} . According to Brownian path decomposition (cf. for example Revuz and Yor [17, Exercise XII.3.8]), the process

$$\left\{ \Lambda(s) \stackrel{\text{def}}{=} \frac{|W(s(t_n - G(t_n)) + G(t_n))|}{(t_n - G(t_n))^{1/2}}; 0 \leq s \leq 1 \right\},$$

often referred to as the ‘‘Brownian meander’’, is independent of $\mathcal{F}_{G(t_n)}$. Moreover, $\mathbb{P}(\Lambda(1) \geq x) = \exp(-x^2/2)$ for any $x > 0$. Of course, $\{\Lambda(s); 0 \leq s \leq 1\}$ and \mathcal{N} are independent. Accordingly,

$$D(t_n) = t_n + (t_n - G(t_n)) \frac{\Lambda^2(1)}{\mathcal{N}^2} \leq t_n + \frac{t_n \Lambda^2(1)}{\mathcal{N}^2},$$

which implies

$$\begin{aligned} \mathbb{P}\left(D(t_n) \geq t_{m-1} \mid \mathcal{F}_{G(t_n)}\right) & \leq \mathbb{P}\left(\Lambda^2(1) \geq \frac{\mathcal{N}^2(t_{m-1} - t_n)}{t_n}\right) \\ & = \mathbb{E}\left[\exp\left(-\frac{\mathcal{N}^2}{2}(\theta^{m-n-1} - 1)\right)\right] \\ & = \theta^{-(m-n-1)/2}. \end{aligned}$$

Now, a key observation is that, for each n , the event F_n is $\mathcal{F}_{G(t_n)}$ -measurable. Therefore

$$(3.13) \quad \mathbb{P}\left(F_n; F_m; D(t_n) \geq t_{m-1}\right) \leq \mathbb{P}\left(F_n; D(t_n) \geq t_{m-1}\right) \leq \theta^{-(m-n)/2} \mathbb{P}(F_n).$$

Combining (3.12)–(3.13) with (3.11) gives

$$\limsup_{N \rightarrow \infty} \sum_{n=1}^N \sum_{m=1}^N \mathbb{P}(F_n \cap F_m) \Big/ \left(\sum_{n=1}^N \mathbb{P}(F_n)\right)^2 \leq \frac{1}{\sqrt{1 - \theta^{-1}}}.$$

By the Borel–Cantelli lemma of Kochen and Stone [9], $\mathbb{P}(\limsup_n F_n) \geq \sqrt{1 - \theta^{-1}}$. Therefore,

$$\mathbb{P}\left[\liminf_{t \rightarrow \infty} \frac{\ell(t) \varphi(t)}{t} \sum_{j > \ell(t)} V_j(t) \leq b\right] \geq \sqrt{1 - \theta^{-1}}.$$

This yields (3.9) by sending θ and b respectively to ∞ and 0. □

4. CRITICAL CASE

Theorem 1.1 concerns the limsup behaviour of $\sum_{j \geq k(t) \log_2 t} V_j(t)$ when $k(t)$ tends to infinity. What happens if it has instead a constant limit? In other words, we ask whether it is possible to get more information about (1.2). Recall the following theorem.

THEOREM D (Csáki *et al.* [6]). – *Let $\varepsilon > 0$,*

$$\limsup_{t \rightarrow \infty} \frac{1}{t} \sum_{j > (\sqrt{3\pi}/\varepsilon) \log_2 t} V_j(t) \leq \varepsilon, \quad \text{a.s.}$$

The main result of this section is to determine the exact value of the “limsup” term.

THEOREM 4.1. – *For any $r > 0$,*

$$\limsup_{t \rightarrow \infty} \frac{1}{t} \sum_{j > r \log_2 t} V_j(t) = \frac{\lambda(r)}{1 + \lambda(r)}, \quad \text{a.s.}$$

Here, $\lambda(r) \in (0, \infty)$ denotes the unique solution to the equation (for $\lambda > 0$)

$$\Xi(\lambda) = \exp\left(\frac{1}{r}\right),$$

where

$$(4.1) \quad \Xi(\lambda) \stackrel{\text{def}}{=} \sup_{a \geq 0} \frac{g(a)}{f(\lambda a)},$$

$$(4.2) \quad f(b) \stackrel{\text{def}}{=} \int_1^\infty \frac{e^{-bx}}{2x^{3/2}} dx,$$

$$(4.3) \quad g(a) \stackrel{\text{def}}{=} 1 - \int_0^1 \frac{e^{ay} - 1}{2y^{3/2}} dy = 1 - \sum_{n=1}^\infty \frac{a^n}{n!(2n-1)}.$$

Remark 4.2. – Let γ be the unique positive number satisfying

$$(4.4) \quad \sum_{n=1}^\infty \frac{\gamma^n}{n!(2n-1)} = 1.$$

Then $g(a) \leq 0$ for $a \geq \gamma$, which means that the supremum in (4.1) can be taken over $a \in (0, \gamma)$. A few lines of elementary computations confirm that $\lambda \mapsto \Xi(\lambda)$ is a (strictly) increasing and continuous function, with $\Xi(0+) = 1$ and $\Xi(\infty) = \infty$, which guarantees that the constant $\lambda(r)$ is well-defined.

Remark 4.3. – We have the following asymptotic estimates for $\lambda(r)$:

$$\lambda(r) \sim \begin{cases} 4/(\pi r) & r \rightarrow \infty, \\ 1/(\gamma r) & r \rightarrow 0, \end{cases}$$

γ being the constant in (4.4). That $\lim_{r \rightarrow \infty} r\lambda(r) = 4/\pi$ is in “formal agreement” with Theorem 1.1.

The proof of Theorem 4.1 relies on the following analogue estimate of Theorem 2.3 for the critical case:

THEOREM 4.4. – Fix $\lambda > 0$,

$$(4.5) \quad \log \mathbb{P} \left[\sum_{j=n+1}^\infty V_j(1) > \frac{\lambda}{1+\lambda} \right] \sim -(\log \Xi(\lambda)) n, \quad n \rightarrow \infty,$$

where $\Xi(\lambda)$ is as in (4.1).

From (4.5), Theorem 4.1 can be proved exactly along the same lines of the proof of Theorem 1.1 in Section 3. We feel free to omit the details. The rest of the section is devoted to the proof of (4.5).

Proof of Theorem 4.4. – The main ingredient in the proof is again Theorems B and C (cf. Section 2). Let Δ_4 denote the probability expression on the left hand side of (4.5).

To prove the upper bound, we fix a constant $a > 0$. By (2.1) and Chebyshev’s inequality,

$$(4.6) \quad \Delta_4 = \mathbb{P} \left[\lambda n + \lambda \sum_{j=1}^{n-1} \alpha_j < \sum_{j=1}^n \beta_j \right]$$

$$(4.7) \quad \leq e^{-na\lambda} p^{n-1}(a\lambda) q^n(a),$$

where $p(z) \stackrel{\text{def}}{=} \mathbb{E}e^{-z\alpha_1}$ and $q(z) \stackrel{\text{def}}{=} \mathbb{E}e^{z\beta_1}$ (for $z \geq 0$) are, respectively, the Laplace transform of α_1 and the moment generating function of β_1 . From (2.2), it is easily checked that $p(z) = e^z f(z)$ (f being defined in (4.2)), whereas using (2.3) and analytic continuation, we have $q(z) = 1/g(z)$ (for $0 \leq z < \gamma$), where g is as in (4.3). Minimising the expression on the right hand side of (4.7), we obtain, for some unimportant constant $C > 0$,

$$\Delta_4 \leq C (\Xi(\lambda))^{-n},$$

($\Xi(\lambda)$ being as in (4.1)) which yields the upper bound in Theorem 4.4.

It remains to verify the lower bound. Going back to (4.6), we have, for any $a > 0$,

$$\Delta_4 \geq \mathbb{P} \left[\sum_{j=1}^n (-\alpha_j) > -na \right] \mathbb{P} \left[\sum_{j=1}^n \beta_j > (a+1)\lambda n \right].$$

According to Theorem C (cf. Section 2), for any fixed $\varepsilon > 0$, when n is sufficiently large,

$$(4.8) \quad \Delta_4 \geq \exp \left[-(1+\varepsilon)(\Delta_5(a) + \Delta_6(a)) n \right],$$

where

$$\Delta_5(a) \stackrel{\text{def}}{=} \sup_{s \geq 0} \left(-(a+1)s - \log f(s) \right),$$

$$\Delta_6(a) \stackrel{\text{def}}{=} \sup_{0 \leq t < \gamma} \left((a+1)\lambda t + \log g(t) \right),$$

(f, g and γ being as in (4.2)–(4.4)). In view of the exact form of f , it is easily seen that

$$\Delta_5(a) = -(a+1)s(a) - \log f(s(a)),$$

where $s(a) > 0$ is the unique solution to $f'(s)/f(s) = -(a + 1)$. Similarly, writing $a_0 \stackrel{\text{def}}{=} (\lambda^{-1} - 1)^+ = \max(\lambda^{-1} - 1, 0)$, for each $a > a_0$,

$$\Delta_6(a) = (a + 1)\lambda t(a) + \log g(t(a)),$$

where $t(a) \in (0, \gamma)$ is the unique solution to $g'(t)/g(t) = -(a + 1)\lambda$. By some tedious but elementary computations, it is seen that $a \mapsto s(a)$ (over $(0, \infty)$) and $a \mapsto t(a)$ (over (a_0, ∞)) are both monotone continuous functions, with $s(0+) = \infty$, $s(\infty) = 0$, $t(a_0+) = 0$ and $t(\infty) = \gamma$ (recalling γ from (4.4)). Therefore, there exists $a^* > a_0$ such that $s(a^*)/\lambda = t(a^*) \in (0, \gamma)$, which yields

$$\Delta_5(a^*) + \Delta_6(a^*) = -\log f(\lambda t(a^*)) + \log g(t(a^*)) \leq \log \Xi(\lambda),$$

by definition of $\Xi(\lambda)$. Since (4.8) holds for all $a > 0$, taking $a \stackrel{\text{def}}{=} a^*$ gives the lower bound in Theorem 4.4. \square

5. BESSEL EXCURSIONS

Looking at Sections 1–4, the Brownian properties we have exploited essentially consist of scaling, strong Markov, path decomposition and Theorem B, all of which are also enjoyed by recurrent Bessel processes. Therefore, the main results of the paper can be extended to the latter processes, which we briefly describe as follows. Let $\{R_d(t); 0 \leq t \leq 1\}$ be a Bessel process of dimension d , with $R_d(0) = 0$. We assume $0 < d < 2$ to ensure the recurrence (of course, in the special case $d = 1$, R_d can be realized as reflecting Brownian motion). As for Brownian motion, we define

$$V_{1,d}(t) \geq V_{2,d}(t) \geq \dots$$

the ordered excursion lengths of R_d over the interval $[0, t]$. Recall the following

THEOREM E (Pitman and Yor [12]). – For any $n \geq 2$,

$$\frac{1}{\sum_{j=1}^n V_{j,d}(1)} - 1 \stackrel{\text{law}}{=} \frac{\sum_{j=1}^n \beta_j^{(d)}}{n + \sum_{j=1}^{n-1} \alpha_j^{(d)}},$$

where “ $\stackrel{\text{law}}{=}$ ” denotes identity in distribution, $\{\alpha_j^{(d)}\}_{j \geq 1}$ and $\{\beta_j^{(d)}\}_{j \geq 1}$ are independent positive random variables whose laws are determined by: for

all $j \geq 1$,

$$\begin{aligned} \mathbb{P}\left(\alpha_j^{(d)} > x\right) &= (1+x)^{-(1-d/2)}, \quad x \geq 0, \\ \mathbb{E} e^{-\lambda \beta_j^{(d)}} &= \left(1 + \frac{2-d}{2} \int_0^1 \frac{1-e^{-\lambda y}}{y^{2-d/2}} dy\right)^{-1}, \quad \lambda \geq 0. \end{aligned}$$

THEOREM 5.1. – *Let $k(t) \uparrow \infty$ such that $t/k(t)$ is non-decreasing,*

$$\limsup_{t \rightarrow \infty} \frac{(k(t))^{d/(2-d)}}{t} \sum_{j > k(t) \log_2 t} V_{j,d}(t) = \left(\Gamma\left(\frac{d+2}{2}\right)\right)^{-2/(2-d)}, \quad \text{a.s.}$$

THEOREM 5.2. – *For non-decreasing functions $\ell(t) > 1$ and $\varphi(t) > 0$ such that $t/\ell(t)$ is non-decreasing, with probability one,*

$$\begin{aligned} \liminf_{t \rightarrow \infty} \frac{(\ell(t))^{d/(2-d)} \varphi(t)}{t} \sum_{j > \ell(t)} V_{j,d}(t) \\ = \left\{ \begin{array}{l} 0 \\ \infty \end{array} \right\} \iff \int^\infty \frac{dt}{t \varphi^{(2-d)/2}(t)} \left\{ \begin{array}{l} = \\ < \end{array} \right\} \infty. \end{aligned}$$

THEOREM 5.2. – *For any $r > 0$,*

$$\limsup_{t \rightarrow \infty} \frac{1}{t} \sum_{j > r \log_2 t} V_{j,d}(t) = \frac{\lambda_d(r)}{1 + \lambda_d(r)}, \quad \text{a.s.},$$

where $\lambda_d(r) \in (0, \infty)$ is the unique solution to the equation (for $\lambda > 0$)

$$\Xi_d(\lambda) = \exp\left(\frac{1}{r}\right),$$

with

$$\begin{aligned} \Xi_d(\lambda) &\stackrel{\text{def}}{=} \sup_{a \geq 0} \frac{g_d(a)}{f_d(\lambda a)}, \\ f_d(b) &\stackrel{\text{def}}{=} \left(1 - \frac{d}{2}\right) \int_1^\infty \frac{e^{-bx}}{x^{2-d/2}} dx, \\ g_d(a) &\stackrel{\text{def}}{=} 1 - \left(1 - \frac{d}{2}\right) \int_0^1 \frac{e^{ay} - 1}{y^{2-d/2}} dy. \end{aligned}$$

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