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On the marginal laws of one-dimensional stochastic integrals with uniformly elliptic integrand

by

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ABSTRACT. – We show that the law of

$$\int_0^t \sigma_u \, dB_u,$$

where B is a standard Brownian motion, σ a progressive process such that

$$0 < \underline{\sigma} < \sigma_u < \overline{\sigma} < \infty \quad du\, dP\text{-a.s.}$$

for two real numbers $(\underline{\sigma}, \overline{\sigma})$, and $t > 0$, doesn't weight points. © 2000 Éditions scientifiques et médicales Elsevier SAS

Key words: Absolute continuity, marginal law, stochastic integral

RÉSUMÉ. – On montre que la loi de

$$\int_0^t \sigma_u \, dB_u$$

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où B est un mouvement Brownien standard, σ un processus progressif tel que

$$0 < \underline{\sigma} < \sigma_u < \overline{\sigma} < \infty \quad du\, dP\text{-p.s.}$$

pour deux réels $(\underline{\sigma}, \overline{\sigma})$, et $t > 0$, ne charge pas les points.

1. INTRODUCTION

Let us consider on some P-complete filtered probability space $(\Omega, F, (F)_{u \geqslant 0}, P)$ the stochastic integral

$$M_t = \int_0^t \sigma_u \, dB_u,$$

where B is a one-dimensional standard P-$(F)_{u \geqslant 0}$ Brownian motion and σ is a $(F)_{u \geqslant 0}$-progressive process such that

$$0 < \underline{\sigma} < \sigma_u < \overline{\sigma} < \infty \quad du\, dP\text{-a.s.} \tag{1}$$

for two real numbers $(\underline{\sigma}, \overline{\sigma})$, and $t > 0$.

Let A be a Lebesgue null set. Then

$$\begin{aligned} 0 \leqslant \underline{\sigma}^2 \int_0^T 1_A(M_u)\, du &\leqslant \int_0^T 1_A(M_u)\, d\langle M\rangle_u \\ &= \int 1_A(a) L_T^a(M)\, da = 0 \quad P \text{ a.s.}, \end{aligned}$$

where $L_T^a(M)$ denotes the local time of M. This entails

$$\int_0^T P(M_u \in A)\, du = 0,$$

which says that the set of time indexes u at which the law of M_u gives a weight to a given Lebesgue null set A is not a big one, at least of zero measure with respect to the Lebesgue measure on $[0, T]$.

Thus the following question is very natural: for every $t > 0$, is the law of M_t absolutely continuous? This is a much stronger statement: in the reasoning above it could happen that for a fixed u the law of M_u weights A. In fact such a phenomenon does happen for some processes σ: Fabes and Kenig[2] in [2] have designed a uniformly continuous function $\sigma : [0, T] \times \mathbb{R}_+ \to [1, 2]$ such that the law of the solution of the s.d.e.

$$X_0 = 0, \qquad dX_u = \sigma(u, X_u)\, dB_u$$

is singular at time T, such yielding a negative answer to the issue of absolute continuity.

The purpose of this paper is to show the following:

THEOREM 1.1. – *For every $t > 0$, the law of M_t doesn't weight points.*

First of all let us remark that because of assumption (1), σ is a progressive process with respect to the natural filtration of M and B a standard Brownian motion with respect to the same filtration. Therefore we may "normalize" the situation by considering the image law of M on the canonical space $(C(\mathbb{R}_+, \mathbb{R}), \mathcal{B}(C(\mathbb{R}_+, \mathbb{R})), (G_u)_{u\geqslant 0})$ where $(G_u)_{u\geqslant 0}$ is the coordinate filtration. Let still denote by P the image law of M. Then under P the canonical process $(\omega_u)_{u\geqslant 0}$ is a $(G_u)_{u\geqslant 0}$-martingale such that for some $(G_u)_{u\geqslant 0}$-progressive process σ and some $(G_u)_{u\geqslant 0}$-standard Brownian motion B

$$\omega = \int_0^{\cdot} \sigma_u \, dB_u.$$

Let us fix $(\underline{\sigma}, \overline{\sigma})$ and denote by $\mathcal{P}(\underline{\sigma}^2, \overline{\sigma}^2)$ the set of probability laws on

$$\big(C(\mathbb{R}_+, \mathbb{R}), \mathcal{B}\big(C(\mathbb{R}_+, \mathbb{R})\big), (G_u)_{u\geqslant 0}\big)$$

for which the canonical process may be written this way, or equivalently (cf. [3, Chapter 5]) for which the canonical process is a martingale with bracket almost surely equivalent to du satisfying:

$$0 < \underline{\sigma}^2 < \frac{d\langle \omega \rangle_u}{du} < \overline{\sigma}^2 < \infty \quad du\, dP\text{-a.s.}$$

We take a stochastic control-looking route. We shall work with the symmetrized laws of $\mathcal{P}(\underline{\sigma}^2, \overline{\sigma}^2)$: consider for any P in $\mathcal{P}(\underline{\sigma}^2, \overline{\sigma}^2)$ the

[2] I thank Yuyin Hu for this reference.

law $\widehat{P}$ which is the image law of the process

$$u \mapsto \frac{1}{\sqrt{2}}(\omega_u - \omega'_u)$$

defined on the product of the canonical spaces. Notice that thanks to the $\frac{1}{\sqrt{2}}$ factor, $\widehat{P} \in \mathcal{P}(\underline{\sigma}^2, \overline{\sigma}^2)$. Set for a function f bounded and Borel

$$\widehat{C}(f)(t,x) = \sup_{P \in \mathcal{P}(\underline{\sigma}^2, \overline{\sigma}^2)} E^{\widehat{P}}\big[f(x+\omega_t)\big].$$

If a law P gives some weight to a point in $\mathbb{R}$, then $\widehat{P}$ weights $\{0\}$ so that we must prove for any $t > 0$ $\widehat{C}(1_{\{0\}})(t,0) = 0$.

Observe that by Brownian scaling, for every t, x,

$$\widehat{C}(f)(t,x) = \widehat{C}(f)\left(1, \frac{x}{\sqrt{t}}\right),$$

in particular for $x = 0$

$$\widehat{C}(1_{\{0\}})(t,0) = \widehat{C}(1_{\{0\}})(1,0). \tag{2}$$

We proceed as follows: in the next section we show that if f satisfies:

$$\begin{aligned} &\text{for every } x, \quad f(x) \leqslant f(0), \\ &\lim_{|x|\to\infty} f(x) = 0, \end{aligned} \tag{3}$$

then $\sup_{P \in \mathcal{P}(\underline{\sigma}^2, \overline{\sigma}^2)} E^P[f(\omega_t)]$ (with no hat) for big enough t is smaller than $\lambda^* f(0)$ with $\lambda^* < 1$. This relies on a rough upper bound for $\sup_{P \in \mathcal{P}(\underline{\sigma}^2, \overline{\sigma}^2)} E^P[f(\omega_t)]$ obtained from the Dambins–Dubins–Schwarz representation theorem.

The second ingredient is to show that the function $x \mapsto \widehat{C}(1_{\{0\}})(1,x)$ satisfies (3). The third ingredient (Section 3) is the following super-harmonic type property of $\widehat{C}(1_{\{0\}})$:

$$\widehat{C}(1_{\{0\}})(t,0) \leqslant \sup_{P \in \mathcal{P}(\underline{\sigma}^2, \overline{\sigma}^2)} E^P\big[\widehat{C}(1_{\{0\}})(1,\omega_t)\big].$$

The result then readily follows from the scaling property (2).

Throughout the paper f stands for a bounded Borel function.

2. A ROUGH UPPER BOUND

By the Dambins–Dubins–Schwarz theorem

$$M_t = \int_0^t \sigma_u \, dB_u = \beta_{\int_0^t \sigma_u^2 \, du}$$

for some Brownian motion β. Note that $\int_0^t \sigma_u^2 \, du$ is a stopping time T_t of the filtration G with respect to which β is a Brownian motion with

$$\underline{\sigma}^2 t \leqslant T_t = \int_0^t \sigma_u^2 \, du \leqslant \overline{\sigma}^2 t \quad \text{a.s.}$$

One can wonder if this property of range of T_t is enough to grant that the law of β_{T_t} does not weight points. Of course it is not: take for instance the crossing time of a fixed level by β between times $\underline{\sigma}^2 t$ and $\overline{\sigma}^2 t$. Clearly the stopping times T_t are very particular stopping times: they satisfy for instance the property that $\overline{\sigma}^2 t - T_t = \int_0^t (\overline{\sigma}^2 - \sigma_u^2) \, du$ is also a stopping time of the same filtration.

Nevertheless, we derive in this section a control of $E^P[f(\beta_T)]$ where T ranges over all stopping times between times $\underline{\sigma}^2 t$ and $\overline{\sigma}^2 t$, with a few assumptions on f, which gives in particular a contraction property for big times (Corollary 2.3).

PROPOSITION 2.1. – *The following holds*:

$$\sup_{P \in \mathcal{P}(\underline{\sigma}^2, \overline{\sigma}^2)} E^P\big[f(x + \omega_t)\big] \leqslant \sup_G \sup_{\tau \in T(G, \underline{\sigma}^2 t, \overline{\sigma}^2 t)} E\big[f(x + \beta_\tau)\big],$$

where $T(G, \underline{\sigma}^2 t, \overline{\sigma}^2 t)$ is the set of the G-stopping times with values almost surely in $[\underline{\sigma}^2 t, \overline{\sigma}^2 t]$

It is easy to give a more explicit bound for this upper bound:

PROPOSITION 2.2. – *Let $-\infty < a \leqslant b < \infty$, $M \geqslant 0$, $\varepsilon \geqslant 0$. Assume $|f| \leqslant M 1_{[a,b]} + \varepsilon 1_{[a,b]^c}$. Then*

$$\begin{aligned} \sup_G \sup_{\tau \in T(G, \underline{\sigma}^2 t, \overline{\sigma}^2 t)} \big|E\big[f(\beta_\tau)\big]\big| \leqslant{} & \varepsilon + M P\big(\beta_{\underline{\sigma}^2 t} \in [a,b]\big) \\ & + 2M P(\beta_{\underline{\sigma}^2 t} < a < \beta_{\overline{\sigma}^2 t}) \\ & + 2M P(\beta_{\overline{\sigma}^2 t} < b < \beta_{\underline{\sigma}^2 t}). \end{aligned}$$

Proof. – We have

$$\sup_{\tau\in T(G,\underline{\sigma}^2t,\overline{\sigma}^2t)} \left|E\left[f(\beta_\tau)\right]\right| \leqslant M \sup_{\tau\in T(G,\underline{\sigma}^2t,\overline{\sigma}^2t)} E\left[\mathbf{1}_{[a,b]}(\beta_\tau)\right] + \varepsilon.$$

Observe now that

$$\sup_{\tau\in T(G,\underline{\sigma}^2t,\overline{\sigma}^2t)} E\left[\mathbf{1}_{[a,b]}(\beta_\tau)\right] = E\left[\mathbf{1}\left(T_{[a,b]} \leqslant \overline{\sigma}^2 t\right)\right] = P\left(T_{[a,b]} \leqslant \overline{\sigma}^2 t\right),$$

where $T_{[a,b]} = \inf\{u \geqslant \underline{\sigma}^2 t,\ \beta_u \in [a,b]\}$. Now by the reflection principle

$$\begin{aligned} P\left(T_{[a,b]} \leqslant \overline{\sigma}^2 t\right) &\leqslant P\left(\beta_{\underline{\sigma}^2 t} \in [a,b]\right) \\ &\quad + 2P(\beta_{\underline{\sigma}^2 t} < a < \beta_{\overline{\sigma}^2 t}) + 2P(\beta_{\overline{\sigma}^2 t} < b < \beta_{\underline{\sigma}^2 t}) \end{aligned}$$

whence the result. □

As an application:

COROLLARY 2.3. – *Assume* $0 \leqslant f \leqslant f(0)$ *and* $f(x) \underset{|x|\to\infty}{\to} 0$. *Then*

$$\limsup_{t\to\infty} \left(\sup_G \sup_{\tau\in T(G,\underline{\sigma}^2t,\overline{\sigma}^2t)} \left|E\left[f(\beta_\tau)\right]\right|\right) \leqslant \lambda^* f(0),$$

where

$$\lambda^* = 1 - \frac{2}{\pi} \arctan\left(\frac{\underline{\sigma}}{\sqrt{\overline{\sigma}^2 - \underline{\sigma}^2}}\right).$$

In particular $\lambda^* < 1$.

Proof. – By setting $a = b = 0$ in the proposition the result is clear with $2P(\beta_{\underline{\sigma}^2 t} < 0 < \beta_{\overline{\sigma}^2 t}) + 2P(\beta_{\overline{\sigma}^2 t} < 0 < \beta_{\underline{\sigma}^2 t})$ for the value of λ^*. Now

$$\begin{aligned} P\left(\beta_{\underline{\sigma}^2 t} < 0 < \beta_{\overline{\sigma}^2 t}\right) &= P\left(\beta_{\underline{\sigma}^2 t} < 0, -\beta_{\underline{\sigma}^2 t} < \beta_{\overline{\sigma}^2 t} - \beta_{\underline{\sigma}^2 t}\right) \\ &= P\left(X < 0, -\underline{\sigma}\sqrt{t}\, X < \sqrt{\overline{\sigma}^2 t - \underline{\sigma}^2 t}\, Y\right), \end{aligned}$$

whence

$$\begin{aligned} \frac{\lambda^*}{2} &= P\left(X < 0, -\underline{\sigma}\sqrt{t}\, X < \sqrt{\overline{\sigma}^2 t - \underline{\sigma}^2 t}\, Y\right) \\ &\quad + P\left(X > 0, -\underline{\sigma}\sqrt{t}\, X > \sqrt{\overline{\sigma}^2 t - \underline{\sigma}^2 t}\, Y\right) \\ &= P\left(\frac{Y}{X} < -\frac{\underline{\sigma}}{\sqrt{\overline{\sigma}^2 - \underline{\sigma}^2}}\right), \end{aligned}$$

where X and Y are two independant standard gaussian random variables. Then $\frac{X}{Y}$ is Cauchy, whence

$$\frac{\lambda^*}{2} = \int_{-\infty}^{-\underline{\sigma}/\sqrt{\overline{\sigma}^2-\underline{\sigma}^2}} \frac{dx}{\pi(1+x^2)}$$
$$= \frac{1}{\pi}\left[\frac{\pi}{2} - \arctan\left(\frac{\underline{\sigma}}{\sqrt{\overline{\sigma}^2-\underline{\sigma}^2}}\right)\right]. \qquad \square$$

We also get from the proposition, by replacing $f(.)$ by $f(x+.)$:

LEMMA 2.4. – *If* $f(x) \underset{|x|\to\infty}{\to} 0$ *then for every* t

$$\sup_G \sup_{\tau\in T(G,\underline{\sigma}^2 t,\overline{\sigma}^2 t)} |E[f(x+\beta_\tau)]| \to 0 \quad as\ |x| \to \infty.$$

3. SUPERHARMONIC-TYPE PROPERTY OF $\widehat{C}(1_{\{0\}})$

In fact we prove in this section the announced property for any f. Notice first:

LEMMA 3.1. – *For every* $f, t > 0$, *the function* $x \mapsto \widehat{C}(f)(t,x)$ *is Borel.*

Proof. – Take first a (bounded) Lipschitz f with Lipschitz constant k.Then for any P in $\mathcal{P}(\underline{\sigma}^2, \overline{\sigma}^2)$

$$|E^{\widehat{P}}[f(x+\omega_t)] - E^{\widehat{P}}[f(x+\varepsilon+\omega_t)]| \leqslant k\varepsilon,$$

which entails

$$|\widehat{C}(f)(t,x) - \widehat{C}(f)(t,x+\varepsilon)| \leqslant k\varepsilon$$

so that the function $\widehat{C}(f)$ is Lipschitz, therefore Borel. The result follows by a monotone class argument, more precisely by the version given in Theorem 21, Chapter 2 of [1] of the monotone class theorem. $\square$

LEMMA 3.2. – *For any* $f, x, s > 0, t > 0$

$$\widehat{C}(f)(t+s,x) \leqslant \sup_{P\in\mathcal{P}(\underline{\sigma}^2,\overline{\sigma}^2)} E^P[\widehat{C}(f)(t,x+\omega_s)].$$

Proof. – For any P in $\mathcal{P}(\underline{\sigma}^2, \overline{\sigma}^2)$, with transparent notations

$$\begin{aligned} &E^{\widehat{P}}[f(x+\omega_{t+s})] \\ &\quad = E^{P\otimes P}[f(x+\omega'_{t+s}-\omega''_{t+s})] \\ &\quad = E^{P\otimes P}\big[E^{P\otimes P}\big[f\big(x+(\omega'_s-\omega''_s)+(\omega'_{t+s}-\omega''_{t+s}) \\ &\qquad\qquad -(\omega'_s-\omega''_s)\big) \mid F'_s\otimes F''_s\big]\big]. \end{aligned}$$

Observe now that the conditional law with respect to $F'_s \otimes F''_s$ is the product of the conditional laws, therefore it is a symmetrized law of $\mathcal{P}(\underline{\sigma}^2, \overline{\sigma}^2)$ so that

$$\begin{aligned} &E^{P\otimes P}\big[f\big(x+(\omega'_s-\omega''_s)+(\omega'_{t+s}-\omega''_{t+s})-(\omega'_s-\omega''_s)\big) \mid F'_s\otimes F''_s\big] \\ &\quad \leqslant \widehat{C}(f)\big(t, x+(\omega'_s-\omega''_s)\big) P\otimes P \text{ a.s.} \end{aligned}$$

Whence

$$E^{\widehat{P}}[f(x+\omega_{t+s})] \leqslant E^{\widehat{P}}[\widehat{C}(f)(t, x+\omega_s)]$$

so that

$$E^{\widehat{P}}[f(x+\omega_{t+s})] \leqslant \sup_{\mathcal{P}(\underline{\sigma}^2,\overline{\sigma}^2)} E^P[\widehat{C}(f)(t, x+\omega_s)].$$

The result follows.

4. CONCLUSION

In order to apply Corollary 2.3 to the function $\widehat{C}(1_{\{0\}})$ we need the following property:

LEMMA 4.1. – *For every $t > 0$, $\widehat{C}(1_{\{0\}})(t,x) \leqslant \widehat{C}(1_{\{0\}})(t,0)$.*

Proof. – Indeed the result is true if $\widehat{C}(1_{\{0\}})(t,x) = 0$. If not, take P such that P weights some points, let I be the (denumerable) set of such points. Then

$$\widehat{P}(x) = \sum_{y\in I/y-x\in I} P(y-x)P(y)$$

so that

$$\begin{aligned} 2\widehat{P}(x) &= 2\sum_{y\in I/y-x\in I} P(y-x)P(y) \\ &\leqslant \sum_{y\in I/y-x\in I} P^2(y) + \sum_{y-x\in I/y\in I} P^2(y-x) \end{aligned}$$

$$\leqslant \sum_{y\in I} P^2(y) + \sum_{z\in I} P^2(z)$$
$$= 2\widehat{P}(0).$$

The result follows. □

In fact, the above property of the set of symmetrized laws explains why we work with these instead of $\mathcal{P}(\underline{\sigma}^2, \overline{\sigma}^2)$: we were not able to prove the corresponding a priori inequality for $\mathcal{P}(\underline{\sigma}^2, \overline{\sigma}^2)$.

It is easy to conclude now: by Lemmas 2.4 and 4.1 we can apply Corollary 2.3 to the function $x \mapsto C(1_{\{0\}})(1, x)$. Next Lemma 3.2 altogether with the scaling property (2) yield

$$C(1_{\{0\}})(1,0) = \limsup_{t\to\infty} C(1_{\{0\}})(t,0) \leqslant \lambda^* C(1_{\{0\}})(1,0),$$

whence $C(1_{\{0\}})(1,0) = 0$ since $\lambda^* < 1$.

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