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A deviation inequality for non-reversible Markov processes

by

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ABSTRACT. – Using the dissipative criterion of Lumer–Philips for the contraction semigroup, we get in this Note a new deviation inequality for $\int_0^t V(X_s)\,ds$ by means of the symmetrized Dirichlet form. A more explicit version is obtained in the case where the logarithmic Sobolev inequality holds. © 2000 Éditions scientifiques et médicales Elsevier SAS

Key words: Dirichlet forms, Deviation inequality, Logarithmic Sobolev inequality

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RÉSUMÉ. – Par le critère de dissipativité de Lumer–Philips pour la contractivité de semigroupes, on obtient une inégalité nouvelle de déviation pour $\int_0^t V(X_s)\,ds$ via la forme de Dirichlet symmetrisée. Une expression plus explicite est obtenue dans le cas où l'inégalité de Sobolev logarithmique est vraie. © 2000 Éditions scientifiques et médicales Elsevier SAS

Mots Clés: Forme de Dirichlet, Inégalité de déviation, Inégalité de Sobolev logarithmique

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1. Let $(\Omega, (\mathcal{F}_t)_{t\in\mathbf{R}^+}, (X_t)_{t\in\mathbf{R}^+}, (\mathbf{P}_x)_{x\in E})$ be a conservative *càdlàg* Markov process with values in a Polish space E, with semigroup of transition probability $(P_t(x,dy))$. We assume that μ is a probability measure on E (equipped with the Borel σ-field $\mathcal{B}$), which is invariant and ergodic with respect to (P_t). For any initial measure ν on E, write $\mathbf{P}_\nu := \int_E \mathbf{P}_x \nu(dx)$.

We denote by $(\mathcal{L}, \mathbf{D}_p(\mathcal{L}))$ the generator of (P_t) acting on $L^p(E,\mu)$ ($\mathbf{D}_p(\mathcal{L})$ being its domain in L^p), where $1 \leqslant p < +\infty$. The *symmetrized Dirichlet form* is given by

$$\mathcal{E}^\sigma(f,g) := \frac{1}{2}\big[\langle -\mathcal{L}f, g\rangle_\mu + \langle -\mathcal{L}g, f\rangle_\mu\big], \quad \forall f,g \in \mathbf{D}_2(\mathcal{L}), \tag{1}$$

where $\langle\cdot,\cdot\rangle_\mu$ is the usual inner product in $L^2(E,\mu)$.

Under the assumption below

$$(\mathcal{E}^\sigma, \mathbf{D}_2(\mathcal{L})) \text{ is } \textit{closable}, \tag{H1}$$

its closure $(\mathcal{E}^\sigma, \mathbf{D}(\mathcal{E}^\sigma))$ corresponds to a symmetric Markov semigroup $(P_t^\sigma)_{t\geqslant 0}$ on $L^2(E,\mu)$.

Given a measurable function $V : E \to \mathbf{R}$, μ-integrable. In this note we are interested to the probability of deviation of the empirical mean $\frac{1}{t}\int_0^t V(X_s)\,ds$ from its *real* (or asymptotic) mean $m := \int_E V\,d\mu := \langle V\rangle_\mu$, i.e.,

$$\mathbf{P}_\nu\left(\left|\frac{1}{t}\int_0^t V(X_s)\,ds - m\right| > r\right).$$

Introduce

$$J_V(r) := \inf\left\{\mathcal{E}^\sigma(f,f) \,\middle|\, f \in \mathbf{D}(\mathcal{E}^\sigma) \cap L^2(|V|\,d\mu), \int f^2\,d\mu = 1; \right.$$
$$\left. \text{and} \int V f^2\,d\mu = r\right\} \tag{2}$$

for every $r \in \mathbf{R}$ (*Convention*: $\inf \emptyset := +\infty$). As is easily seen, J_V is a convex function on $\mathbf{R}$. Then $[J_V < +\infty]^0$ (interior) is some interval (a,b) where $-\infty \leqslant a \leqslant b \leqslant +\infty$.

Define now I_V as the lower semi-continuous (l.s.c. in short) regularization of J_V. Obviously $I_V(m) = J_V(m) = 0$ and $I_V : \mathbf{R} \to [0, +\infty]$ is convex. Then I_V is non-decreasing on $[m, +\infty)$ and non-increasing on

$(-\infty, m]$. Notice that when $a < b$, then for any $r \in \mathbf{R}$,

$$I_V(r) = \begin{cases} J_V(r), & \text{if } r \in (a, b); \\ J_V(a+), & \text{if } r = a; \\ J_V(b-), & \text{if } r = b; \\ +\infty, & \text{otherwise.} \end{cases} \tag{3}$$

When our Markov process (X_t) is μ-reversible (or (P_t) is μ-symmetric), Deuschel and Stroock [4, Theorem 5.3.10, p. 210] (1989) proved essentially the following large deviation estimation (where a general level-2 large deviation lower bound is given)

$$\lim_{t \to \infty} \frac{1}{t} \log \mathbf{P}_\nu \left(\frac{1}{t} \int_0^t V(X_s)\, ds - m > r \right) = -I_V(m + r), \quad \forall r \geqslant 0, \tag{4}$$

for V bounded. For general unbounded V, (4) is shown in [7] (1993).

In this little note we propose to extend and strengthen (4). Our main observation is

THEOREM 1. – *Assume* (H1). *For any initial measure* ν *such that* $\nu \ll \mu$ *and* $\frac{d\nu}{d\mu} \in L^2(\mu)$, *we have for all* $t > 0$, *all* $r > 0$,

$$\mathbf{P}_\nu \left(\frac{1}{t} \int_0^t V(X_s)\, ds - m > r \right) \leqslant \left\| \frac{d\nu}{d\mu} \right\|_{L^2(\mu)} \cdot \exp\bigl[-t \cdot I_V(m + r)\bigr], \tag{5}$$

$$\mathbf{P}_\nu \left(\frac{1}{t} \int_0^t V(X_s)\, ds - m < -r \right) \leqslant \left\| \frac{d\nu}{d\mu} \right\|_{L^2(\mu)} \cdot \exp\bigl[-t \cdot I_V(m - r)\bigr]. \tag{6}$$

Remark 2. – In the symmetric case, the deviation inequality (5) is sharp in its exponent for large time t, by (4). The main differences between (4) and (5) are:

(i) The symmetry assumption required in (4) is removed for (5);

(ii) In (5), t and r, being arbitrary, are fixed unlike in (4) which is only an asymptotic relation ($t \to +\infty$). Hence (5) is much more stronger and practical.

However in the non-symmetric case, inequality (5) is no longer asymptotically exact. In fact, when the level-2 large deviation principle of Donsker–Varadhan holds and V is bounded, the limit (4) is given by a contraction form of the Donsker–Varadhan entropy functional, which is different from the expression in terms of Dirichlet form. See Deuschel and Stroock [4, Chapter VI] and Ben Arous and Deuschel [1] (1994).

Nevertheless that last large deviation result requires quite restrictive conditions in the non-symmetric case: indeed there exist geometrically ergodic irreducible Markov processes so that the level-1 large deviation principle fails (see Bryc and Smolenski [2] (1993)). While the deviation inequality (5) requires only (H1), which is satisfied in the most part of interesting cases. Moreover (H1) can be removed in case that V is bounded, see Remarks 3(a) below.

2. Proof of Theorem 1. Consider the Feynman–Kac semigroup

$$P_t^V f(x) := \mathbf{E}^x f(X_t) \cdot \exp\left(\int_0^t V(X_s)\,ds\right) \tag{7}$$

where $f \geqslant 0$ is $\mathcal{B}$-measurable. We shall establish for any μ-integrable function $V : E \to \mathbf{R}$,

$$0 < \|P_t^V\|_2 \leqslant \mathrm{e}^{t\Lambda(V)}, \quad \forall t \geqslant 0, \tag{8}$$

where

$$\begin{aligned}\|P_t^V\|_2 &:= \sup\{\|P_t^V f\|_{L^2(\mu)};\ f \geqslant 0 \text{ and } \langle f^2\rangle_\mu \leqslant 1\} \\ &= \sup\{\langle P_t^V f, g\rangle_\mu;\ f, g \geqslant 0 \text{ and } \langle f^2\rangle_\mu \leqslant 1, \langle g^2\rangle_\mu \leqslant 1\},\end{aligned}$$

and

$$\Lambda(V) := \sup\left\{-\mathcal{E}_V^\sigma(f, f) \,|\, f \in \mathbf{D}(\mathcal{E}_V^\sigma), \int f^2 d\mu = 1\right\}. \tag{9}$$

Here

$$\mathbf{D}(\mathcal{E}_V^\sigma) := \mathbf{D}(\mathcal{E}^\sigma) \cap L^2(|V|\,d\mu), \qquad \mathcal{E}_V^\sigma(f, f) = \mathcal{E}^\sigma(f, f) - \int V f^2 d\mu.$$

Let us see quickly why (8) implies (5), by a very classical argument borrowed from the Cramèr theorem [4]. In fact set $P(\lambda) := \Lambda(\lambda V)$, $\forall \lambda \in \mathbf{R}$. By Chebychev's inequality, for all $r, t > 0$ fixed,

$$\begin{aligned}&\mathbf{P}_\nu\left(\frac{1}{t}\int_0^t V(X_s)\,ds - m > r\right) \\ &\quad\leqslant \inf_{\lambda > 0} \exp[-\lambda t(m + r)] \cdot \mathbf{E}^\nu \exp\left[\lambda \int_0^t V(X_s)\,ds\right]\end{aligned}$$

$$
\begin{aligned}
&\leqslant \inf_{\lambda>0} \exp\left[-\lambda t(m+r)\right] \cdot \left\|\frac{d\nu}{d\mu}\right\|_{L^2(\mu)} \cdot \|P_t^{\lambda V}\|_2 && (10)\\
&\leqslant \left\|\frac{d\nu}{d\mu}\right\|_{L^2(\mu)} \cdot \inf_{\lambda>0}\left\{\exp\left[-\lambda t(m+r)\right] \cdot \mathrm{e}^{t\Lambda(\lambda V)}\right\} \quad \text{(by (8))}\\
&= \left\|\frac{d\nu}{d\mu}\right\|_{L^2(\mu)} \exp\left\{-t \cdot \sup_{\lambda>0}\left[\lambda(m+r) - P(\lambda)\right]\right\}.
\end{aligned}
$$

It remains to identify the exponent in the last term of (10).

Since $\Lambda(\lambda V) \geqslant \lambda m$ by the definition (9), m is a sub-differential of $P(\lambda)$ at $\lambda = 0$. Thus for $r > 0$,

$$\sup_{\lambda>0}\left[\lambda(m+r) - P(\lambda)\right] = \sup_{\lambda\in\mathbf{R}}\left[\lambda(m+r) - P(\lambda)\right],$$

which is the Legendre transformation $P^\star(m+r)$ of $P(\lambda)$.

On the other hand, we have by (9)

$$P(\lambda) = \Lambda(\lambda V) = \sup\left\{\lambda z - J_V(z); z \in \mathbf{R}\right\} = \sup\left\{\lambda z - I_V(z); z \in \mathbf{R}\right\}$$

for all $\lambda \in \mathbf{R}$. Hence the famous Fenchel–Legendre theorem gives us

$$P^\star(m+r) = I_V(m+r).$$

Substituting those into (10), we get (5).

Applying (5) to $-V$, we get (6).

Consequently to conclude this theorem, it remains to show (8). We divide its proof into three cases.

Case 1. – *V bounded.* In this bounded case (P_t^V) is a strongly continuous semigroup of bounded operators on $L^2(\mu)$, whose generator is exactly $(\mathcal{L} + V; \mathbf{D}_2(\mathcal{L} + V) = \mathbf{D}_2(\mathcal{L}))$ by the well known Feynman–Kac formula. By the definition (9) of $\Lambda(V)$,

$$\left\langle\left(\mathcal{L} + V - \Lambda(V)\right)f, f\right\rangle_\mu \leqslant 0, \quad \forall f \in \mathbf{D}_2(\mathcal{L}). \tag{11}$$

That means exactly that the generator $\mathcal{L} + V - \Lambda(V)$ with domain $\mathbf{D}_2(\mathcal{L})$ is a dissipative operator on $L^2(E, \mu)$ in the sense of Lumer and Philips [9, Chapter IX, p. 250]. By the Lumer–Philips Theorem [9, Chapter IX, p. 250], the semigroup $(\mathrm{e}^{-t\Lambda(V)}P_t^V)$ generated by $\mathcal{L} + V - \Lambda(V)$ is contractive on $L^2(E,\mu)$. In other words,

$$\|e^{-t\Lambda(V)}P_t^V\|_2 \leqslant 1, \quad \forall t \geqslant 0,$$

which is exactly (8).

Case 2. – *V upper bounded* ($V \leqslant a$). Considering $V-a$ if necessary, we can assume $V \leqslant 0$. Take $V_n = \max\{V, -n\}$ for $n \in \mathbf{N}$. We have by the Case 1,

$$\|P_t^V\|_2 \leqslant \lim_{n\to\infty} \|P_t^{V_n}\|_2 \leqslant \lim_{n\to\infty} e^{t\Lambda(V_n)} = \exp\bigl(t \cdot \inf_{n\geqslant 1} \Lambda(V_n)\bigr). \tag{12}$$

Recall that

$$\begin{aligned}-\Lambda(V_n) &= \inf\left\{\mathcal{E}^\sigma(f,f) - \int V_n f^2\,d\mu \,\middle|\, f \in \mathbf{D}(\mathcal{E}^\sigma) \text{ and } \int f^2 d\mu \leqslant 1\right\}\\ &= \inf\left\{F_n(f) \,\middle|\, \int f^2 d\mu \leqslant 1\right\},\end{aligned}$$

where $F_n : L^2(E,\mu) \to [0,+\infty]$ is given by

$$F_n(f) := \mathcal{E}^\sigma(f,f) - \int V_n f^2\,d\mu, \quad \text{if } f \in \mathbf{D}(\mathcal{E}^\sigma), \text{ and } +\infty \text{ else}.$$

By Kato [5, p. 461, Lemma 3.14a] and our assumption (H1), F_n is lower semicontinuous on $L^2(E,\mu)$ with respect to the strong topology, then with respect to the weak topology $\sigma(L^2, L^2)$ (since F_n, being the sum of two nonnegative quadratic forms, is convex on $L^2(E,\mu)$). Moreover, since the unit ball $\{f \in L^2(\mu); \int f^2 d\mu \leqslant 1\}$ is compact with respect to $\sigma(L^2, L^2)$, by an elementary analytical lemma (see e.g. [8, Proposition 1.2]),

$$\begin{aligned}-\inf_{n\geqslant 1} \Lambda(V_n) &= \sup_{n\geqslant 1} \inf\left\{F_n(f) \,\middle|\, \int f^2\,d\mu \leqslant 1\right\}\\ &= \inf\left\{\sup_n F_n(f) \,\middle|\, \int f^2\,d\mu \leqslant 1\right\} = -\Lambda(V).\end{aligned}$$

Substituting it into (12), we get (8) again.

Case 3. – *General case.* Take $V^N = \min\{V, N\}$ for $N \in \mathbf{N}$. By the monotone convergence theorem,

$$\|P_t^V\|_2 = \sup\bigl\{\langle P_t^V f, g\rangle_\mu \,|\, f, g \geqslant 0 \text{ and } \langle f^2\rangle_\mu \leqslant 1, \langle g^2\rangle_\mu \leqslant 1\bigr\}$$

$$= \sup_{N\geqslant 1} \sup\left\{\left\langle P_t^{V^N} f, g\right\rangle_\mu \middle| f, g \geqslant 0 \text{ and } \langle f^2\rangle_\mu \leqslant 1, \langle g^2\rangle_\mu \leqslant 1\right\}$$
$$\leqslant \sup_{N\geqslant 1} \mathrm{e}^{t\Lambda(V^N)} = \mathrm{e}^{t\Lambda(V)},$$

where the third inequality follows from the Case 2, and the last equality follows from the fact that $\mathbf{D}(\mathcal{E}^\sigma) \cap L^\infty(\mu)$ is a form core for all $\mathcal{E}^\sigma_{V^N}$, $N \geqslant 1$, and for the not necessarily closable quadratic form $\mathcal{E}^\sigma_V$.

The proof of (8) and then that of Theorem 1 are so finished. □

Remark 3. –

(a) When V is bounded, it holds that

$$\|P_t^V\|_2 \leqslant \exp\left[t \cdot \Lambda^0(V)\right]$$

where

$$\Lambda^0(V) := \sup\left\{\int V f^2\, d\mu + \langle \mathcal{L} f, f\rangle_\mu \middle| f \in \mathbf{D}_2(\mathcal{L}) \text{ and } \langle f^2\rangle_\mu \leqslant 1\right\} \tag{13}$$

without the assumption (H1) about the closability of $(\mathcal{E}^\sigma, \mathbf{D}_2(\mathcal{L}))$, by the proof in the Case 1 above. As in the proof of (8) ⇒ (5) above, one can deduce from (13) the deviation inequalities (5) and (6) without (H1), but with I_V substituted by the l.s.c. regularization I_V^0 of

$$J_V^0(r) := \inf\left\{\mathcal{E}^\sigma(f, f) \middle| f \in \mathbf{D}_2(\mathcal{L}), \int f^2 d\mu = 1; \int V f^2\, d\mu = r\right\}.$$

When (H1) is satisfied and V is bounded, $\Lambda^0(\lambda V) = \Lambda(\lambda V)$, $\forall \lambda \in \mathbf{R}$ (by the fact that $\mathbf{D}_2(\mathcal{L})$, being a form core of $\mathcal{E}^\sigma$, is so for $\mathcal{E}^\sigma_{\lambda V}$ because of the boundedness of V), and then $I_V^0 = I_V$.

(b) Note also the following (indicated by the referee): the inequality (8) implies not only (5) and (6), but also (with the same argument)

$$\mathbf{E}^\mu f(X_0) g(X_t) 1_{\left[\frac{1}{t}\int_0^t V(X_s)\, ds - m > r\right]}$$
$$\leqslant \|f\|_{L^2(\mu)} \|g\|_{L^2(\mu)} \cdot \exp\left[-t \cdot I_V(m + r)\right], \quad \forall r, t > 0.$$

(c) Applying the Lumer–Philips theorem to $\mathcal{L} - V$ in $L^p(\mu)$ with $1 \leqslant p < +\infty$, we get, instead of (8), that for any V bounded,

$$\|P_t^V\|_p \leqslant \exp\left(t \Lambda_p(V)\right)$$

where

$$\Lambda_p(V) := \sup\left\{ \int V|f|^p \, d\mu + \left\langle \operatorname{sgn}(f)|f|^{p-1}, \mathcal{L}f \right\rangle_\mu \Big| f \in \mathbf{D}_p(\mathcal{L}), \right.$$
$$\left. \langle |f|^p \rangle_\mu = 1 \right\}.$$

3. In this paragraph we do not require (H1) but we assume the log-Sobolev inequality below: there exists $C > 0$ such that for all $f \in \mathbf{D}_2(\mathcal{L})$,

$$\int_E f^2 \log f^2 - \langle f^2 \rangle_\mu \log \langle f^2 \rangle_\mu \leqslant C \langle -\mathcal{L}f, f \rangle_\mu. \tag{14}$$

Consider the log-Laplace transformation of $V - m$:

$$H(\lambda) = \log \int_E \mathrm{e}^{\lambda V} \, d\mu - \lambda m \tag{15a}$$

and its Legendre transformation

$$H^\star(r) = \sup\{\lambda r - H(\lambda); \lambda \in \mathbf{R}\}. \tag{15b}$$

By the classical Cramèr's theorem [4], $H^\star$ governs the large deviation principle of the i.i.d. sequence of common law $\mu(V - m \in \cdot)$.

The following result says that the log-Sobolev inequality (14) implies a same type of estimation as in the i.i.d. case.

COROLLARY 4. – *Assume* (14) (*not* (H1)). *Then for any* $V \in L^1(\mu)$,

$$\frac{1}{t} \log \|P_t^V\|_2 \leqslant \frac{1}{C} \log \int_E \mathrm{e}^{CV} d\mu. \tag{16}$$

In particular for each initial measure $\nu \ll \mu$ *with* $\frac{d\nu}{d\mu} \in L^2(\mu)$ *and for all* $r > 0, t > 0$

$$\mathbf{P}_\nu\left(\frac{1}{t} \int_0^t V(X_s) \, ds - m > r \right) \leqslant \left\| \frac{d\nu}{d\mu} \right\|_{L^2(\mu)} \cdot \exp\left(-\frac{t}{C} H^\star(r) \right). \tag{17}$$

Proof. – The deviation inequality (17) follows from (16) by Chebychev's inequality as in Theorem 1. To show the key (16), assume at first that V is bounded.

By (13) in Remark 3, we have

$$\begin{aligned}
&\frac{1}{t}\log\|P_t^V\|_2\\
&\quad\leqslant \sup\left\{\int Vf^2\,d\mu+\langle\mathcal{L}f,f\rangle_\mu \,\middle|\, f\in\mathbf{D}_2(\mathcal{L})\quad\text{and } \langle f^2\rangle_\mu=1\right\}\\
&\quad\leqslant \sup\left\{\int Vf^2\,d\mu-\frac{1}{C}\int f^2\log f^2\,d\mu \,\middle|\, f\in\mathbf{D}_2(\mathcal{L})\right.\\
&\qquad\qquad\qquad\qquad\qquad\left.\text{and } \langle f^2\rangle_\mu=1\right\}\quad\text{(by (14))}\\
&\quad=\frac{1}{C}\log\int_E \mathrm{e}^{CV}\,d\mu,
\end{aligned}$$

where the last equality follows from Donsker–Varadhan's variational formula (see e.g. [8]).

Now for V unbounded, set $V_n=\min\{\max\{V,-n\},n\}$. We have

$$\|P_t^V\|_2\leqslant \liminf_{n\to+\infty}\|P_t^{V_n}\|_2\leqslant \lim_{n\to\infty}\left(\int \mathrm{e}^{CV_n}\,d\mu\right)^{t/C}=\left(\int \mathrm{e}^{CV}d\mu\right)^{t/C}$$

by the bounded case shown above and the dominated convergence (and Fatou's lemma if the last integral is infinite). (16) is hence established. □

Remark 5. – Ledoux [6] (1999) develops systematically the so called Herbst method which consists to derive deviation inequalities from a log-Sobolev inequality. The strategy consists to apply a log-Sobolev inequality to $\mathrm{e}^{\lambda F}$ to obtain a differential inequation, from which a control on $\mathbf{E}\mathrm{e}^{\lambda F}$ is deduced by comparison lemma. Nevertheless for that strategy works here for $F=\int_0^t V(X_s)\,ds$, we should assume that a log-Sobolev inequality on the path space $(\mathbf{D}([0,t],E),\mathbf{P}_\nu)$ holds, which is in general not the case here.

Even in case that such a path level log-Sobolev inequality holds, it seems that the Herbst method does not give directly better estimation than (17). For instance, let (B_t) be the Brownian motion on a Riemannian manifold E, with generator $\Delta/2$, where Δ is the Laplace–Beltrami operator. Assume that the Ricci curvature satisfies $|Ric_u|\leqslant K$ for all $u\in O(E)$ (the bundle of orthonormal frames on E). By Capitaine–Hsu–Ledoux [3, (6)], the path level log-Sobolev inequality below holds:

$$\mathbf{E}^x\left(F^2\log F^2\right)-\mathbf{E}^xF^2\log\mathbf{E}^xF^2\leqslant 2\mathrm{e}^{Kt}\mathbf{E}^x|DF|_H^2 \tag{18}$$

for any $x \in E$ and $F : C([0,t]; E) \to \mathbf{R}$ provided that the right side term above is finite, where $|DF|_H$ is the norm in the Cameron–Martin subspace of the Malliavin derivative DF on the path space. Now the Herbst method developed in [6, §2.3] yields: if $|DF|_H^2 \leqslant \sigma^2$, $\mathbf{P}_x$-a.s., then

$$\mathbf{P}_x(F - \mathbf{E}^x F > r) \leqslant \exp\left(-\frac{r^2}{2\mathrm{e}^{Kt}\sigma^2}\right). \tag{19}$$

Using the notations of [3], we can easily prove that for $F = \int_0^t V(B_s)\,ds$ with $\|\nabla V\|_\infty := \sup_{x\in E} |\nabla V(x)| < +\infty$ (where $|\nabla V(x)|$ is the Riemannian norm of the gradient of V at x),

$$|DF|_H^2 \leqslant \int_0^t \left(\int_s^t |\nabla V|(B_u)\,du \right)^2 ds \leqslant \|\nabla V\|_\infty^2 \cdot \frac{t^3}{3}, \quad \mathbf{P}_x\text{-a.s.}$$

We then obtain by (19),

$$\mathbf{P}_x\left(\int_0^t V(B_s)\,ds - \mathbf{E}^x \int_0^t V(B_s)\,ds > rt \right) \leqslant \exp\left(-\frac{3r^2}{2t\mathrm{e}^{Kt}\|\nabla V\|_\infty^2}\right). \tag{20}$$

That estimation is quite interesting and sharp for small t, but not so for large t. On the other hand, when E is compact, the log-Sobolev inequality (14) holds (a well known fact), then (17) is valid and it gives a much better estimation than (20) for large t.

Our approach in Corollary 4 consists to apply log-Sobolev inequality after obtaining the control of $\|P_t^V\|_2$ (in Theorem 1), not before, unlike in the Herbst method. One can regard it as another application of log-Sobolev inequality, complementing those amply developed by Ledoux [6].

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REFERENCES

[1] Ben Arous G., Deuschel J.D., The rate function of hypoelliptic diffusions, Comm. Pure Appl. Math. XLVII (6) (1994) 843–860.

[2] Bryc W., Smolenski W., On the convergence of averages of mixing sequences, J. Theoret. Probab. 6 (3) (1993) 473–484.

[3] Capitaine M., Hsu E.P., Ledoux M., Martingale representation and a simple proof of logarithmic Sobolev inequality on path spaces, Elect. Comm. Probab. 2 (7) (1997).

[4] Deuschel J.D., Stroock D.W., Large deviations, Pure and Appl. Math. 137 (1989).

[5] Kato T., Perturbation Theory for Linear Operators, 2nd ed., Springer, Berlin, 1984; (2nd corrected printing).

[6] Ledoux M., Concentration of measure and logarithmic Sobolev inequalities, in: Séminaire de Probab. XXXIII, Lecture Notes in Math., Vol. 1709, Springer, 1999, pp. 120–216.

[7] Wu L.M., Feynman–Kac semigroups, ground state diffusions and large deviations, J. Funct. Anal. 123 (1) (1994) 202–231.

[8] Wu L.M., An introduction to large deviations, in: Yan J.A., Peng S., Fang S., Wu L. (Eds.), Several Topics in Stochastic Analysis, Academic Press of China, Beijing, 1997, pp. 225–336; (in chinese).

[9] Yosida K., Functional Analysis, 3rd edn., Springer, 1971.