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## Limit velocity for a driven particle in a random medium with mass aggregation

by

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**ABSTRACT.** – We study a one-dimensional infinite system of particles driven by a constant positive force  $F$  which acts only on the leftmost particle which is regarded as the tracer particle (t.p.). All other particles are field neutral, do not interact among themselves, and independently of each other with probability  $0 < p \leq 1$  are either perfectly inelastic and “stick” to the t.p. after the first collision, or with probability  $1 - p$  are perfectly elastic, mechanically identical and have the same mass  $m$ . At initial time all particles are at rest, and the initial measure is such that the interparticle distances  $\xi_i$ ’s are i.i.d. r.v.’s. with absolutely continuous density. We show that for any value of the field  $F > 0$ , the velocity of

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the t.p. converges to a limit value, which we compute. © 2000 Éditions scientifiques et médicales Elsevier SAS

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**RÉSUMÉ.** – On étudie un système de particules infini unidimensionnel soumis à une force constante positive  $F$  qui agit seulement sur la particule la plus à gauche, vue comme la particule traceuse (p.t.).

Toutes les autres particules sont en champ neutre, et n'interagissent pas entre elles. Indépendamment des autres, elles sont soit, avec probabilité  $0 < p < 1$ , parfaitement inélastiques et “se collent” à la p.t. après première collision, soit, avec probabilité  $1 - p$ , parfaitement élastiques, mécaniquement identiques avec même masse  $m$ . A l'origine toutes les particules sont au repos, et la mesure initiale est telle que les distances interparticules  $\xi_i$  sont des variables aléatoires i.i.d. de densité absolument continue. On montre que pour toute valeur du champ  $F > 0$ , la vitesse de la p.t. converge vers une valeur limite que nous calculons. © 2000 Éditions scientifiques et médicales Elsevier SAS

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## 1. INTRODUCTION

In this work we study a “sticky particle model”, namely we investigate the long time behaviour of the tracer particle (the t.p.), which is subject to a positive constant force  $F$ , and which interacts with the field-neutral random media made of initially standing particles of two possible types. Each neutral particle, with probability  $0 < p \leq 1$  and independently of all other particles, is declared to be perfectly inelastic (further referred as an  $s$ -type particle). After the first interaction with the t.p. an  $s$ -type particle is “incorporated” into (sticks to) the t.p. according to the usual Newtonian mechanics laws. With probability  $1 - p$  each particle is declared to be perfectly elastic (further referred to as an  $e$ -type particle). It interacts elastically with the t.p. during the evolution. “Sticky particle models” as a subject of research have quite a long history and are related mostly to problems of gelation and to formation of large scale structures in the universe. Nevertheless, in spite of a big amount of physics literature (see [17] for a recent survey), starting from the paper

of Zeldovich ([18]), mathematical understanding of this subject remains rather unsatisfactory. Most of the effort has been concentrated on the study of the qualitative behaviour of the solutions of the corresponding hydrodynamic equations, small density fluctuations and their influence on the formation of shocks and mass concentration (see [5]). On the “particle level”, one-dimensional models with mass aggregation and self-gravitational force for finite systems of particles have been studied in [1,8,9]. Ergodic properties of one-dimensional semi-infinite systems of similar type but with only elastic interactions ( $p = 0$ ) were studied intensively in the middle of the last decade (see for instance [2,3,11]), and more recently ([10,13]), where the limiting behaviour of the t.p. was determined either by a relation between the pressure of neutral particles and the force, or by the distribution of the initial positions of standing particles. In our situation, the motion of the t.p. can be interpreted as the motion of a single, mass-aggregating point through a “dust” of light point particles. The interaction with this dust of elastic and inelastic particles creates a net force opposite to the direction of the flow which therefore competes with the external force  $F$ . We prove here that for any value of the field  $F$ , and any  $0 < p \leq 1$  the velocity of the t.p. converges to a limiting value, which we compute exactly as a function of  $F$ ,  $p$  and the initial density of particles.

The article is organized as follows. In Section 2 we give a precise description of the model, establishing some necessary notation as we move along, and state our main results. Section 3 is dedicated to the study of an auxiliary Markovian process, sometimes also called “Markov approximation dynamics” associated with the original one, and we show that for this process, the velocity of the t.p. converges to a limiting value. In Section 4 we go back to the original problem and show that its dynamics converges to the Markov approximation dynamics (in a suitable sense) as times goes to infinity.

## 2. THE MODEL AND MAIN RESULTS

We consider a semi-infinite system of particles of two types (0 and 1) on the half line  $\mathbb{R}_+ = [0, +\infty)$ , with nonnegative velocities. The state of the system at a given time is specified by a choice of a type, a position and a velocity for each particle and we take  $X = (\mathbb{R}_+ \times \mathbb{R}_+ \times \{0, 1\})^{\mathbb{N}}$  as the phase space of the system.  $X$ , endowed with the natural topology, is a Polish space (see [7]). Sometimes it is convenient to represent a

configuration  $x \in X$  as a sequence  $\{q_n, v_n, \eta_n\}_{n \geq 0}$  where  $q_n$ ,  $v_n$ , and  $\eta_n$  will denote the position, velocity and the type of the  $n$ th particle. The particle that corresponds to the label 0 (0-th particle) will be called the tracer particle (t.p.). The particles are all assumed to be pointlike, and initially have the same mass  $m = 1$ . The tracer particle is subjected to a constant force  $F$ . All other particles are “force neutral”, and do not interact among themselves. Particles of type-1 are perfectly inelastic with respect to collisions with the tracer particle. By this we mean that after the collision the energy of the two particle system (t.p. plus the particle type 1) is the smallest possible subjected to classical mechanics collision laws which correspond to the type-1 particle being incorporated into the t.p., making the mass of the t.p. increase by 1, and the velocity of the t.p. being immediately modified by the following rule:

$$V' = \frac{M_t}{M_t + 1} V; \quad (2.1)$$

where  $V$  and  $V'$  are respectively the incoming and the outgoing velocities of the t.p., and  $M_t$  is the mass of the t.p. at the moment of the collision. Due to this behavior we sometimes refer to type-1 particles as *sticky particles* or *s-particles*. The force acting on the t.p. is always the same ( $F$ ) and therefore its acceleration decreases after each collision with a type-1 particle as a result of the mass increase. Type-0 particles are perfectly elastic with respect to collisions with the tracer particle. We will refer to them as *e-particles*. More precisely, all *e-particles* keep their velocity until they collide with the t.p., when the velocities are changed according to the usual collision rules satisfying momentum and energy conservation. If  $v$  and  $V$  are the incoming velocities of the neutral particle and the t.p., respectively, and  $v'$  and  $V'$  are the corresponding outgoing velocities, then

$$\begin{aligned} V' &= \alpha_t V + (1 - \alpha_t)v; \\ v' &= (1 + \alpha_t)V - \alpha_t v; \end{aligned} \quad (2.2)$$

where  $\alpha_t \stackrel{\text{def}}{=} (M_t - 1)/(M_t + 1) \geq 0$ , and  $M_t$  is the mass of the t.p. at time  $t$ . Since the *e-particles* are indistinguishable among themselves and do not interact with *s-particles*, it is convenient to think of them as pulses, which may cross each other. These rules define the dynamics on the phase space  $\{q_n, v_n, \eta_n\}_{n \geq 0}$  in a natural way. For each  $t \geq 0$  all the sticky particles that collided with the tracer particle before  $t$  will have the same position

$(q_0(t))$  and the same velocity  $(v_0(t))$ . Also the mass of the tracer particle at time  $t$  is given by  $M_t := 1 + \sum_{i=1}^{\infty} \eta_i \mathbf{1}_{q_i(0) < q_0(t)}$ . At the initial time ( $t = 0$ ) all particles are at rest, i.e.  $v_i = 0, \forall i \geq 0$ , and the t.p. is located at the origin ( $q_0 = 0$ ). The initial measure  $\mu_0$  is then described by:

- (A) the interparticle distances  $\xi_n = q_n - q_{n-1}, n \geq 1$ , are i.i.d. positive random variables with an absolutely continuous distribution, such that  $\mathbb{E}_{\mu_0} \xi_1 < +\infty$ ;
- (B) the types  $(\eta_n)_{n \geq 1}$  are i.i.d. random variables independent of  $(\xi_n)_{n \geq 1}$  with  $\mu_0(\eta_n = 1) = p, 0 < p \leq 1$ .

Our main result is

THEOREM 2.1. –

$$\lim_{t \rightarrow \infty} v_0(t) = \sqrt{\frac{F \mathbb{E}_{\mu_0} \xi_1}{2 - p}} \quad \mu_0\text{-a.s.} \quad (2.3)$$

Even though the proof of this result is somewhat involved, the heuristics are quite simple. Acting on the t.p. there are two competing forces. One is the forward constant force  $F$ . The other is an effective (*mean-field*) friction force arising from the t.p. loss of momenta after each collision. This effective force increases as the velocity of the t.p. increases and therefore we expect the t.p. velocity to remain bounded. Both forces act on a particle that gets heavier as the time evolves and the resulting acceleration should decrease. The limit velocity would be the one for which both forces have equal intensities. It is not hard to guess the value of this velocity. After a time interval  $\Delta t$  the t.p. traveling with velocity  $v$  collides with  $v \Delta t / \mathbb{E}_{\mu_0} \xi_1$  particles, with a proportion  $1 - p$  being elastic. The momentum transfer to a particle is  $v$  if it is a sticky particle and approximately  $2v$  if it is an elastic one when the t.p. is very heavy (see (2.2)). Thus the total momentum transfer per unit time is  $(pv + (1 - p)2v)v / \mathbb{E}_{\mu_0} \xi_1$ . As time goes to infinity, equilibrium should be reached with an identity between the latter expression and  $F$ . This identity yields (2.3). As is often the case for this sort of problem, even though the ideas are simple, the corresponding rigorous analysis is nontrivial, and one has to deal with several technical difficulties. The central one is to provide a good control on the “influence of the past”, i.e., one has to show that, on large time scales, recollisions with the moving  $e$ -particles do not affect the motion of the t.p. much. We choose the following strategy: first consider an auxiliary dynamics (called the  $A$ -dynamics), in which all  $e$ -particles are annihilated immediately after the first collision with the t.p. By doing so we get rid of possible recollisions

of the t.p. with  $e$ -particles and thus the “influence of the past” may arise only through the t.p.’s present velocity, which is manageable enough so that we are able to prove time convergence for the  $A$ -dynamics. The next and final step is to show that the original dynamics converges in time to the auxiliary  $A$ -dynamics. At this point it is more convenient to use a “pathwise” approach, i.e., we will show that for any  $x \in X$  for which the original dynamics is well defined and the velocity of the t.p. in the  $A$ -dynamics converges, the velocity of the t.p. in the original dynamics also converges to the same limit.

Before ending this section we briefly stop on the question of the  $\mu_0$ -a.s. existence of the above described dynamics. The following three situations (see [15], for instance) lead to problems for defining the dynamics: (i) infinitely many neutral particles appear in a finite neighbourhood of the t.p.; (ii) occurrence of *multiple collisions*, i.e., the t.p. collides simultaneously with several neutral particles, or with the neutral particle which has velocity equal to the velocity of the t.p.; (iii) occurrence of infinitely many collisions in a finite interval of time. Let us denote by  $\mathcal{X}$  the set of all initial configurations for which all velocities are equal to zero and define

$$X_\infty = \left\{ \begin{array}{l} x \in \mathcal{X}; \text{ such that infinitely many particles occur in} \\ \text{some bounded neighbourhood of the t.p. in a finite time;} \end{array} \right.$$

$$X_T = \{x \in \mathcal{X}; \text{ such that a multiple collision occurs}\};$$

$$X_{R_\infty} = \left\{ \begin{array}{l} x \in \mathcal{X}; \text{ such that infinitely many} \\ \text{recollisions occur in a finite time;} \end{array} \right.$$

where  $x = \{\xi_1, \xi_2, \dots, \xi_n, \dots\}$ . From the choice of the initial velocities, for our system  $X_\infty = \{x: \sum_1^{+\infty} \xi_i < +\infty\}$ , and thus  $\mu_0(X_\infty) = 0$ . On the other hand Lemma 2.1 of [12] implies that  $\mu_0(X_T) = 0$ . We observe that the presence of inelastic particles does not affect the general scheme of the proof in [12], and all arguments go through. Finally, keeping in mind the fact that  $\mu_0(X_\infty \cup X_T) = 0$ , for any time interval between two consecutive inelastic collisions (since the mass of the t.p. is not changing), we can apply the *barycenter* argument of [4] (see page 374 there), which implies that  $\mu_0(X_{R_\infty}) = 0$ . Thus we get that  $\mu_0(X_\infty \cup X_T \cup X_{R_\infty}) = 0$ . We denote by  $X_D$  the set of all initial configurations for which the dynamics is well defined.

### 3. AN AUXILIARY ANNIHILATING PROCESS AND ITS PROPERTIES

We introduce an auxiliary dynamics (we refer to it further as *A*-dynamics), which is a deterministic evolution of the t.p. governed by the two following rules:

- (a') If the t.p. collides with an *e*-particle then the velocity of the t.p. is changed according to the formula (2.2), after which the *e*-particle is immediately annihilated;
  - (b') If the t.p. collides with an *s*-particle then the velocity of the t.p. is changed according to the formula (2.1), after which its mass increases by 1.
- (3.1)

*Remark.* – Further on all quantities related to the *A*-dynamics (like the velocity of the t.p., hitting times, flight times, etc.) will be equipped with a ‘bar’, e.g.,  $\bar{v}_0(t)$ ,  $\bar{t}_z$ , etc. Moreover, through the article we will use the following notation: for any  $z \in \mathbb{R}_+$ ,  $\bar{t}_z$  and  $t_z$  denote respectively the time when  $\bar{q}_0(\bar{t}_z) = z$ , and the time when  $q_0(t_z) = z$  for the *A*- and the original dynamics; if at the point  $z \in \mathbb{R}_+$  a collision occurs, by  $v_0(t_z^-)$ ,  $v_0(t_z^+)$  resp. we will denote the velocity of the t.p. just before and after the collision.

**PROPOSITION 3.1.** – *In the A-dynamics the velocity process of the t.p. converges  $\mu_0$ -a.s. and*

$$\lim_{t \rightarrow \infty} \bar{v}_0^p(t) = \bar{v}_0^p \stackrel{\text{def}}{=} \sqrt{\frac{F \mathbb{E}_{\mu_0} \xi_1}{2 - p}} \quad \mu_0\text{-a.s.} \tag{3.2}$$

*Proof.* – First we will prove (3.2) for the discrete *A*-dynamics that arises by looking at the continuous time *A*-dynamics only at collision times. Collision rules (3.1) give us the following relation between the velocities of the t.p. immediately before and immediately after collisions:

$$\begin{aligned} \text{(a)} \quad & \bar{v}_0(t_{q_i}^+) = \alpha_i \bar{v}_0(t_{q_i}^-); \\ \text{(b)} \quad & \bar{v}_0^2(t_{q_i}^-) = \bar{v}_0^2(t_{q_{i-1}}^+) + \frac{2F \xi_i}{M_{i-1}}; \end{aligned} \tag{3.3}$$

where as before  $\alpha_i = \frac{M_{i-1} + \eta_{i-1}}{M_{i-1} + 1}$ ,  $M_i = 1 + \sum_{j=1}^i \eta_j$ , and we set  $\bar{v}_0(0) = 0$ .

Equations (3.3 a,b) yield

$$\bar{v}_0^2(t_{q_i}^-) = \alpha_{i-1}^2 \bar{v}_0^2(t_{q_{i-1}}^-) + \beta_i; \quad (3.4)$$

where  $\beta_i = \frac{2F\xi_i}{M_{i-1}}$ , and iterating (3.4) we have

$$\bar{v}_0^2(t_{q_i}^-) = \sum_{j=1}^{i-1} \left( \prod_{k=j}^{i-1} \alpha_k \right)^2 \beta_j + \beta_i. \quad (3.5)$$

From the stationarity of the sequence  $\{\xi_i\}_{i \geq 1}$  and the law of large number for the variables  $\sum_{j=1}^i \eta_j$  we get that  $\beta_i \rightarrow 0$   $\mu_0$ -a.s., so we only need to study the convergence of

$$\sum_{j=1}^{i-1} \left( \prod_{k=j}^{i-1} \alpha_k \right)^2 \beta_j. \quad (3.6)$$

We rewrite  $\prod_{j=i}^n \alpha_j$  as  $e^{-\sum_{j=i}^n \log \alpha_j^{-1}}$ , and  $\alpha_j^{-1}$  as  $1 + \gamma_j$ , where  $\gamma_j = (2 - \eta_j) / \sum_{i=1}^j \eta_i$ , with the usual conventions for division by 0 and exponentiation of  $\infty$ . We rewrite (3.6)

$$\sum_{i=1}^n e^{-2 \sum_{j=i}^n \log(1+\gamma_j)} \beta_i. \quad (3.7)$$

For the sake of clarity, let us make the underlying set of realizations of the randomness  $\Omega$  explicit. A single realization will be denoted by  $\omega$ . For arbitrary  $0 < \varepsilon < p$ , and  $\omega$  in a set of full measure  $A \subset \Omega$ , let us choose  $m = m(\varepsilon, \omega) > 0$  such that  $(p - \varepsilon)j < \sum_{i=1}^j \eta_i < (p + \varepsilon)j$  for all  $j > m$ . Breaking the sum in (3.7) into the two pieces:  $i \leq m$  and  $i > m$ , the first sum can be estimated

$$\sum_{i=1}^m e^{-2 \sum_{j=i}^n \log(1+\gamma_j)} \beta_i \leq e^{-2 \sum_{j=m}^n \log(1+\gamma_j)} \sum_{i=1}^m \beta_i. \quad (3.8)$$

There is a positive number  $c$  depending only on  $m$ ,  $p$  and  $\varepsilon$  such that  $\log(1 + \gamma_j) \geq c/j$ , for  $j \geq m$  so that

$$e^{-c' \sum_{j=m}^n 1/j} \sum_{i=1}^m \beta_i, \quad (3.9)$$

is an upper bound for (3.8), where  $c'$  is a positive number depending only on  $m, p$  and  $\varepsilon$ . The first sum is thus seen to converge to 0 as  $n \rightarrow \infty$  for  $\omega \in A$ . We then only have to consider

$$\sum_{i=m}^n e^{-2 \sum_{j=i}^n \log(1+\gamma_j)} \beta_i. \tag{3.10}$$

By taking a larger  $m$  if necessary, we have

$$\frac{1 - \varepsilon \lambda + \psi_j}{p j} \leq \log(1 + \gamma_j) \leq \frac{1 + \varepsilon \lambda + \psi_j}{p j}$$

for all  $j > m$  and  $\omega \in A$ , where  $\lambda = 2 - p$  and  $\psi_j = p - \eta_j$ . We will then have an upper and a lower bound to (3.10) of the form

$$\sum_{i=m}^n e^{-\frac{2(1\pm\varepsilon)}{p} (\lambda \sum_{j=i}^n 1/j + \sum_{j=i}^n \psi_j/j)} \beta_i.$$

By the three series theorem  $\sum_{j=1}^n \psi_j/j$  converges almost surely, so

$$\sum_{j=i}^n \psi_j/j \rightarrow 0$$

as  $i, n \rightarrow \infty$  almost surely. Therefore taking  $m$  larger if necessary, we will have the upper and lower bounds

$$e^{\pm\varepsilon} \sum_{i=m}^n e^{-\frac{2\lambda(1\pm\varepsilon)}{p} \sum_{j=i+1}^n 1/j} \beta_i$$

for (3.10) for  $\omega$  in a subset of full measure of  $\Omega$ . Notice that the sum in the exponent can be written as  $\sum_{j=1}^n 1/j - \sum_{j=1}^i 1/j$  and since  $\sum_{j=1}^i 1/j - \log i$  converges to a constant (Euler’s constant!) as  $i \rightarrow \infty$ , we can substitute it, increasing  $m$  if necessary, by  $\log(n/i)$  incorporating the error in the factor  $e^{\pm\varepsilon}$ . We thus have almost sure upper and lower bounds for (3.9) of the form

$$\frac{e^{\pm\varepsilon}}{n^\zeta} \sum_{i=m}^n i^\zeta \beta_i,$$

where  $\zeta = 2\lambda(1 \pm \varepsilon)/p$ . Incorporating the error in the factor  $e^{\pm\varepsilon}$ , we can substitute  $\beta_i$  in the above sum by  $2F\xi_i/(pi)$ , thus getting

$$e^{\pm\varepsilon} \frac{2F}{pn^\zeta} \sum_{i=m}^n i^{\zeta-1} \xi_i.$$

Since  $\frac{1}{n^\zeta} \sum_{i=1}^m i^{\zeta-1} \xi_i \rightarrow 0$  as  $n \rightarrow \infty$ , and since  $\varepsilon$  is arbitrary, the proposed limit will equal that of

$$\frac{2F}{pn^{\zeta'}} \sum_{i=1}^n i^{\zeta'-1} \xi_i,$$

where  $\zeta' = 2\lambda/p$ , provided the latter exists and is continuous in  $\zeta' > 0$ . The above expression equals

$$\frac{2F\mathbb{E}_{\mu_0}\xi_1}{pn^{\zeta'}} \sum_{i=1}^n i^{\zeta'-1} + \frac{2F}{pn^{\zeta'}} \sum_{i=1}^n i^{\zeta'-1} (\xi_i - \mathbb{E}_{\mu_0}\xi_1). \tag{3.11}$$

The first term in (3.11) is then seen to converge to  $\frac{2F\mathbb{E}_{\mu_0}\xi_1}{p^{\zeta'}} = F\mathbb{E}_{\mu_0}\xi_1/(2-p)$  as  $n \rightarrow \infty$ . By Kolmogorov’s inequality, in the case of having second moments, or through a more involved argument starting with it, in the case of having only the first moment of  $\xi_i$ , the second term in (3.11) is seen to converge almost surely to 0 as  $n \rightarrow \infty$ . This finishes the argument for the discrete dynamics. For the full dynamics, the result follows from the fact that for  $t_{q_i} < t < t_{q_{i+1}}$ , we have  $\bar{v}_0(t_{q_i}^+) < \bar{v}_0^p(t) < \bar{v}_0(t_{q_{i+1}}^-)$ , and from (3.3) and the fact that  $\alpha_i \rightarrow 1$  as  $i \rightarrow \infty$ .  $\square$

We denote by  $X_A$  the set of all initial configurations for which (3.2) holds.

### 4. PROOF OF THEOREM 2.1

Before going into details we describe the strategy of the proof. As mentioned in Section 2, we will prove that the velocity process in the original dynamics converges to the one of the associated  $A$ -dynamics. Speaking informally we will show that after some time the original dynamics behaves in some sense as an “ $A$ -dynamics with a small perturbation”, with this perturbation getting negligible as time evolves. The first thing we show is that, from some point of the evolution on,

all standing  $e$ -particles collide only once with the t.p. The keystone for this is Proposition 4.1, or more directly its corollary (Corollary 4.1), which shows that for almost all initial configurations the dynamics has the following property: at infinitely many times during the evolution *all moving particles* have velocity larger than  $\bar{v}_0^p/2$ . On the other hand, due to mass aggregation, the t.p. becomes more and more “insensitive” to collisions with standing particles. Using comparisons from Lemmas 4.1 and 4.2 we get that there exists,  $\mu_0$ -a.s., a finite time after which the velocity of the t.p. never drops below  $\bar{v}_0^p/2$ . Both previous facts imply that starting from some random, but  $\mu_0$ -a.s. finite time  $\tau$ , the velocities of  $e$ -neutral particles which collide for the first time with the t.p. after this time will be larger than  $\bar{v}_0^p$ , and again, using comparisons from Lemma 4.2, we will get that the t.p. will interact with them only once. Thus, starting from this time, the dynamics behaves “almost” as an  $A$ -dynamics: perturbations might come only via recollisions with finitely many  $e$ -particles which collided with the t.p. before time  $\tau$ . In the last step of the argument we show that fluctuations which come through these recollisions go to zero with time and thus the velocity process converges. The argument is organized in several steps.

LEMMA 4.1. – *Consider two  $A$ -dynamics evolving on the same configuration  $x \in X_D$ , the set of all initial configurations for which the dynamics is well defined, of initially standing particles, with initial velocities  $\bar{v}_0$  and  $\bar{v}'_0$  respectively, then:*

$$\lim_{t \rightarrow +\infty} \bar{v}_0(t) - \bar{v}'_0(t) = 0, \tag{4.1}$$

and moreover, if  $0 \leq \bar{v}_0 - \bar{v}'_0 \leq \delta$ , then

$$0 \leq \bar{v}_0(t_z) - \bar{v}'_0(t_z) \leq \delta \tag{4.2}$$

for any  $z \in \mathbb{R}_+$ .

*Proof.* – (4.2) follows from the following observation: if  $\bar{v}_0(t_{q_k}) - \bar{v}'_0(t_{q_k}) \leq \delta$ , then  $\bar{v}_0(t_z) - \bar{v}'_0(t_z) \leq \delta$  for any  $z \in (q_k, q_{k+1})$ . Thus  $\bar{v}_0(t_{q_{k+1}}^-) - \bar{v}'_0(t_{q_{k+1}}^-) \leq \delta$ , and, depending on the type of collision with the  $(k + 1)$ -th particle ( $e$ - or  $s$ -type), we have

$$\bar{v}_0(t_{q_{k+1}}) - \bar{v}'_0(t_{q_{k+1}}) = a_k (\bar{v}_0(t_{q_{k+1}}^-) - \bar{v}'_0(t_{q_{k+1}}^-)) < \delta, \tag{4.3}$$

where  $a_k = (M_{t_{q_{k+1}}} - 1)/(M_{t_{q_{k+1}}} + 1)$ , if  $\eta_{k+1} = 0$ , or  $a_k = (M_{t_{q_{k+1}}})/(M_{t_{q_{k+1}}} + 1)$ , if  $\eta_{k+1} = 1$ . Iterating the argument we get (4.2). (4.1)

follows immediately from (4.3), since  $\prod_{k=1}^{\infty} a_k = 0$  on  $X_A$ , the set of all initial configurations for which (3.2) holds.  $\square$

LEMMA 4.2. – Consider two dynamics, the original one and the A-dynamics, evolving from the same configuration  $x \in X_D$  of initially standing particles, starting with velocities  $v_0$  and  $\bar{v}_0$ , respectively. If  $v_0 \leq \bar{v}_0$ , then

$$v_0(t_z) \leq \bar{v}_0(\bar{t}_z) \tag{4.4}$$

for all  $z \in \mathbb{R}_+$ .

*Proof.* – Let  $x \in X_D$ . For any  $z \in \mathbb{R}_+$  we then have that  $t_z < +\infty$  and the number of collisions up to  $t_z$  is finite. We shall prove (4.4) by induction on the number of collisions  $K$  for the t.p. in the original dynamics, before reaching point  $z$ .

The statement is obviously true for  $K = 0$  and  $K = 1$ . Let us assume it is true for  $K \leq n$  and we will prove it for  $K = n + 1$ . Let us denote by  $z_n < z_{n+1} < z$  the points where the  $n$ -th and  $(n + 1)$ -th collision of the t.p. took place. By the induction assumption, if  $\bar{z} < z_{n+1}$ , then  $v_0(t_{\bar{z}}) \leq \bar{v}_0(\bar{t}_{\bar{z}})$ . From this it follows at once that

$$v_0(t_{z_{n+1}}^-) \leq \bar{v}_0(\bar{t}_{z_{n+1}}^-) \tag{4.5}$$

and, independently of the fact that at the point  $z_{n+1}$  we have a recollision or a collision with a standing particle, inequality (4.5) implies that  $v_0(t_{z_{n+1}}^+) \leq \bar{v}_0(\bar{t}_{z_{n+1}}^+)$ , as well. Now, by assumption, in between  $z_{n+1}(x)$  and  $z$ , in both dynamics, the t.p. moves without interaction, and with constant acceleration  $F$ . Thus we get (4.4).  $\square$

The next proposition provides a lower bound for the mean velocity of the tracer particle.

PROPOSITION 4.1. –

$$\liminf_{t>0} \frac{q_0(t)}{t} \geq \sqrt{\frac{F \mathbb{E}_{\mu_0} \xi_1}{2}} \stackrel{\text{def}}{=} \bar{v}_0^0 \quad \mu_0\text{-a.s.} \tag{4.6}$$

*Proof.* – For any initial configuration  $\xi = \{(q_1, \eta_1), \dots, (q_n, \eta_n), \dots\}$  such that the dynamics is well defined, it follows from energy dissipation that

$$q_n \geq \sum_{r=1}^{n-M_{n-1}} \frac{v_{i_r}^2(t_{q_n}^-)}{2F} + M_{n-1} \frac{v_0^2(t_{q_n}^-)}{2F}$$

$$\geq \frac{(\sum_{r=1}^{n-M_{n-1}} v_{i_r}(t_{q_n}^-) + M_{n-1}v_0)^2}{(n+1)2F}, \quad n \geq 1,$$

where we assume that the particles initially located at positions  $q_{i_r}, r = 1, \dots, n - M_{n-1}$  are  $e$ -particles, and  $v_{i_r}(t_{q_n}^-)$  represents the velocity of  $i_r$ -th  $e$ -particle at time  $t_{q_n}^-$ , and  $M_{n-1} = 1 + \sum_{i=1}^{n-1} \eta_i$  is the mass of the t.p. upon collision with the  $n$ -th initially standing particle. If  $0 < q_1 < q_2 < \dots$ , then equality is obtained if and only if  $M_{n-1} = 1$ . By momenta conservation,  $\sum_{r=1}^{n-M_{n-1}} v_{i_r}(t_{q_n}^-) + M_{n-1}v_0(t_{q_n}^-) = Ft_{q_n}$ , and thus we get the following inequality.

$$t_{q_n} \leq \sqrt{\frac{2(n+1)q_n}{F}}, \quad n \geq 1.$$

Thus, for any  $t \in (t_{q_n}, t_{q_{n+1}}]$  we have

$$\frac{q_0(t)}{t} \geq \frac{q_n}{t_{q_{n+1}}} \geq \frac{q_{n+1} - \xi_{n+1}}{\sqrt{\frac{2(n+2)q_{n+1}}{F}}} = \sqrt{\frac{Fq_{n+1}}{2(n+2)}} - \sqrt{\frac{F\xi_{n+1}^2}{2(n+2)q_{n+1}}}.$$

Since  $q_{n+1}/(n+2) \rightarrow \mathbb{E}_{\mu_0} \xi_1$   $\mu_0$ -a.s. and  $\xi_{n+1}^2/(n+2)q_{n+1} \rightarrow 0$   $\mu_0$ -a.s., we get (4.6).  $\square$

Now, since  $\bar{v}_0^0 > \bar{v}_0^p/2$  for all values of the parameter  $p \in [0, 1]$ , defining

$$c_1 \stackrel{\text{def}}{=} \frac{1}{2} \left( \bar{v}_0^0 - \frac{1}{2} \sqrt{F \mathbb{E}_{\mu_0} \xi_1} \right),$$

we have that

$$0 < c_1 < \frac{1}{2} \left( \bar{v}_0^0 - \frac{1}{2} \bar{v}_0^p \right). \tag{4.7}$$

**COROLLARY 4.1.** –  $\mu_0$ -a.s. there exists an infinite increasing sequence of finite random indices  $\{\phi_n\}_{n=1}^\infty$ , growing to  $+\infty$ , such that at time  $t_{q_{\phi_n}}, n = 1, 2, \dots$ , all moving particles have velocity at least  $\bar{v}_0^0 - c_1/2$ .

*Proof.* – Assume the opposite, i.e. that with positive  $\mu_0$  probability there exists a finite random time, say  $\theta$ , such that from this time on there always exists in the system at least one moving particle with velocity smaller than  $\bar{v}_0^0 - c_1/2$ . Consider the motion of the rightmost such particle from time  $\theta$  on. It is well defined since there is always at least one

such particle by assumption and their number is finite  $\mu_0$ -a.s. It need not always be the same particle and it could be the t.p. (in case there is no slow moving  $e$ -particle). Since it is always in front of the t.p. and at time  $\theta$  it is at a finite distance from the t.p., this immediately implies that

$$\lim_{t \rightarrow +\infty} \frac{q_0(t) - q_0(\theta)}{t - \theta} \leq \bar{v}_0^0 - c_1/2,$$

which contradicts (4.6).  $\square$

*Remark 4.1.* – From now on we assume that  $x \in X_0 \stackrel{\text{def}}{=} X_A \cap X_D$ , which is subset of  $\mathcal{X}$  of full measure.

Next we let

$$c_2 \stackrel{\text{def}}{=} \left[ \frac{8\bar{v}_0^0 + c_1}{c_1} \right] + 1, \quad (4.8)$$

where  $[\cdot]$  stands for the integer part and we define a mass index:

$$m_c \stackrel{\text{def}}{=} \min \left\{ j: \sum_{i=1}^j \eta_i \geq c_2 \right\}. \quad (4.9)$$

It is obvious that  $m_c$  is finite  $\mu_0$ -a.s.

*Remark 4.2.* – The choice of the constant  $c_2$  in the definition of the mass index  $m_c$  is motivated by the two following properties: first, if the t.p. at the moment of collision with a standing  $e$ -particle has velocity at least  $\bar{v}_0^0 - c_1$  and its mass is at least  $c_2$ , then the velocity transfer to the standing  $e$ -particle will be larger than  $\bar{v}_0^0 + c_1/8$ ; and second, if at the moment of collision of the t.p. with a standing  $n$ -particle the t.p. has velocity at least  $\bar{v}_0^0 - c_1/2$  and its mass is at least  $c_2$ , then the velocity of the t.p. after collision is at least  $\bar{v}_0^0 - 3c_1/4$ .

**COROLLARY 4.2.** – *Given  $c_1 > 0$  there exists  $\mu_0$ -a.s. a finite index  $I$  such that*

$$\bar{v}_0^0 - c_1 \leq v_0(t) \leq \bar{v}_0^0 + c_1/8; \quad (4.10)$$

for all  $t \geq t_{q_I}$ .

*Proof.* – Proposition 3.1 implies that  $\mu_0$ -a.s. there exists an index  $r \equiv r(p, c_1) < +\infty$  such that

$$\bar{v}_0(t) \in [\bar{v}_0^0 - c_1/8, \bar{v}_0^0 + c_1/8] \quad (4.11)$$

for all  $t \geq t_{q_r}$ . The r.h.s. inequality of (4.10) follows then immediately from Proposition 3.1 and Lemma 4.2. The first inequality in (4.11) will require some work. Let us define the random index:

$$\kappa \stackrel{\text{def}}{=} \max\{m_c, r\}, \tag{4.12}$$

and a random subindex:

$$\varkappa \stackrel{\text{def}}{=} \min\{j: \phi_j \geq \kappa\}. \tag{4.13}$$

The choice of  $\varkappa$  gives us the following property of the system at time  $t_{q_{\phi_\varkappa}}^+$ : (a) all moving particles have velocity at least  $\bar{v}_0^0 - c_1/2$ ; (b)  $M_{t_{q_{\phi_\varkappa}}^+} \geq m_c$ ; (c) for the associated  $A$ -dynamics  $\bar{v}_0(t) \in [\bar{v}_0^p - c_1/8, \bar{v}_0^p + c_1/8]$  for all  $t \geq t_{q_{\phi_\varkappa}}$ .

Let us denote by  $\tau = \{\tau_0 = t_{q_{\phi_\varkappa}}, \tau_1, \dots\}$  an increasing sequence of times (which is random), at which the t.p. recollides with the  $e$ -particles which have indices not larger than  $\phi_\varkappa$ . We claim that during the time interval  $(\tau_{i-1}, \tau_i]$ ,  $i = 1, 2, \dots$ , the velocity process in the original dynamics never drops below the value  $\bar{v}_0^0 - c_1$ , and, as a consequence of this and of our choice of  $m_c$  we get that the velocities of the  $e$ -particles with which the t.p. interacts (for the first time) during  $(\tau_{i-1}, \tau_i]$  are bigger than  $\bar{v}_0^p + c_1/8$ . Thus according to the r.h.s. inequality of (4.10) the t.p. will not be able to interact with them anymore.

First we consider an additional  $A$ -dynamics starting at time  $t'_{q_{\phi_\varkappa}} := t_{q_{\phi_\varkappa}}$ , point  $q_{\phi_\varkappa}$ , with the same realization of standing particles in front of it as the original dynamics, and with initial velocity of t.p. being  $\bar{v}'_0 - \frac{3}{4}c_1 \leq \bar{v}'_0 \leq \bar{v}_0(t)$ . We claim that

$$\bar{v}'_0(\bar{t}'_{q_{\phi_\varkappa+n}}) \geq \bar{v}_0^0 - c_1, \quad \text{for all } n = 1, 2, \dots \tag{4.14}$$

Indeed, by Lemma 4.1 we get that

$$0 \leq \bar{v}_0(\bar{t}_z) - \bar{v}'_0(\bar{t}'_z) \leq \bar{v}_0^p - \bar{v}_0^0 + 7c_1/8$$

for all  $z \geq q_{\phi_\varkappa}$ , and since  $\bar{v}_0(\bar{t}_z) \geq \bar{v}_0^p - c_1/8$  for all  $z \geq q_{\phi_\varkappa}$ , we get (4.14).

Now take  $k_j = \max\{n: t_{q_n} \leq \tau_j\}$ , if  $\tau_j < +\infty$ , otherwise  $k_j = +\infty$ , i.e.  $k_j$  is the index of the standing particle such that the  $j$ -th recollision of the t.p. with some particle of the past will occur in between points  $q_{k_j}$  and  $q_{k_j+1}$ .

First we prove the claim above for  $(\tau_0, \tau_1]$ . If  $k_1 = \phi_x$  the case becomes somewhat trivial: since the t.p. will recollide with some particle of the past, this implies that its velocity at the moment of collision with  $q_{\phi_x+1}$  is larger than  $\bar{v}_0^0 - c_1/2$  and  $v_0(\tau_1^+) \geq \bar{v}_0^0 - 3c_1/4$  follows from the choice of  $m_c$ .

Suppose now  $k_1 > \phi_x$ . Then by our assumptions, in between the points  $q_{\phi_x}$  and  $q_{k_1}$  no recollisions will occur and thus, by (4.14),  $v_0(t_z^+) \geq \bar{v}_0^0 - c_1$ , for all  $q_{\phi_x} \leq z \leq q_{k_1}$ . Moreover if some of the particles among  $p_{\phi_x+1}, \dots, p_{k_1}$  are  $e$ -particles (here by  $p_j$  we denote the particle which is initially located at  $q_j$ ), their velocity will be bigger than  $\bar{v}_0^p + c_1/8$ , and thus the t.p. will not interact with them anymore. Now, since between points  $q_{k_1}$  and  $q_{k_1+1}$  the t.p. will recollide with some particle of the past, this implies that its velocity at the moment of collision with  $p_{k_1+1}$  is already larger than  $\bar{v}_0^0 - c_1/2$  and thus the fact that  $v_0(\tau_1^+) \geq \bar{v}_0^0 - 3c_1/4$  again follows from the choice of  $m_c$ .

The argument can be repeated for the time interval  $(\tau_1, \tau_2]$  above. Proceeding iteratively we get the claim. So setting  $I = \phi_x$  we complete the proof of the corollary.  $\square$

*Remark 4.3.* – At this point, after all preparatory work has been done, we are ready to prove that the velocity of the t.p. in the original dynamics converges to the same limit (2.3) as in the  $A$ -dynamics. Out of the definition of  $I$  and from the claim in the proof of Corollary 4.2 it follows that after a  $\mu_0$ -a.s. finite time  $t_{q_I}$  the t.p. might recollide only with finitely many  $e$ -particles, namely those with indices smaller or equal than  $I$ . It is generally believed that in this situation the number of collisions of each  $e$ -particle with the t.p. would be finite. This would basically complete the proof of the Theorem 2.1, since we would have only a finite number (at most  $I$ ) of  $e$ -particles with which the t.p. would recollide only finitely many times. Thus after the time of the last recollision, say  $t_L$ , the original dynamics and the  $A$ -dynamics which starts at the same point  $q_0(t_L)$  with the same velocity  $v_0(t_L)$  would not differ and convergence would follow. This situation would also lead with little effort to other relevant results as, e.g., the existence of an invariant measure as seen from the t.p. and a central limit theorem (see discussion in the conclusion below). *We do not yet have a proof* of this. In general, only a few rigorous results of this type are known (see, for instance, [6,14,19]). Nevertheless we present below an argument which shows that even taking into account the possibility of infinitely many recollisions of the t.p. with finitely many  $e$ -particles, we can get convergence of the original velocity process to  $\bar{v}_0^p$ .

Proceeding with the proof, suppose the  $e$ -particles  $p_{i_1}, \dots, p_{i_k}$  with  $i_j \leq \phi_l$  interact with the t.p. infinitely many times. We fix some notations. By  $\delta v_l^i$  and  $\theta_l^i$ , respectively, denote the velocity transferred to  $p_{i_l}$  by the t.p. at the moment of its  $i$ -th collision with  $p_{i_l}$  and the time of this collision. It is obvious that

$$\sum_{i=1}^{+\infty} \delta v_l^i < \bar{v}_0^p + c_1/2,$$

for all  $l = 1, \dots, k$ , otherwise  $p_{i_l}$  would attain a velocity which is larger than the upper bound for the velocity of the t.p. after finitely many collisions for all later times (see case 1 of (4.10)), and thus they would not interact as they should under our hypothesis. Thus, for any  $\varepsilon > 0$  we can find  $n \equiv n(\xi, \varepsilon)$  such that

$$\sum_{l=1}^k \sum_{i=n}^{+\infty} \delta v_l^i < \varepsilon. \tag{4.15}$$

We denote  $\tilde{\theta}_n = \max\{\theta_1^n, \dots, \theta_k^n\}$ . Let us compare the two velocity processes: the first is the original one – and we start to observe it from time  $\tilde{\theta}_n$ , so that the t.p. is at the position  $q_0(\tilde{\theta}_n)$  with velocity  $v_0(\tilde{\theta}_n)$ , and the other one is a velocity process in a new  $A$ -dynamics which is realized in the same configuration of standing particles and begins at the space point  $q_0(\tilde{\theta}_n)$  with initial velocity equal to  $v_0(\tilde{\theta}_n)$ . It is a consequence of (4.15) and our choice of  $\tilde{\theta}_n$  that for all  $z \geq q_0(\tilde{\theta}_n)$  the following holds

$$\bar{v}_0(\bar{t}_z) - \varepsilon \leq v_0(t_z). \tag{4.16}$$

Indeed, if by  $\Delta v_l^i$  we denote the transfer of (negative) velocity to the t.p. at the moment of its  $i$ -th collision with  $p_{i_l}$ , then the fact that  $M_{\tilde{\theta}_n} > 1$  will imply that  $|\Delta v_l^i| < \delta v_l^i$ , and together with (4.15) we get

$$\sum_{l=1}^k \sum_{i=n}^{+\infty} \Delta v_l^i < \varepsilon. \tag{4.17}$$

Now applying (4.2) iteratively for both dynamics in between each two consecutive recollisions we obtain (4.16). Since  $x \in X_A \cap X_0$ , by Proposition 3.1 the velocity process  $\bar{v}_0(\cdot)$  in the new  $A$ -dynamics converges to  $\bar{v}_0^p$ . Since  $\varepsilon > 0$  can be chosen arbitrary, the proof of Theorem 2.1 follows.

## 5. FINAL REMARKS

We briefly discuss other relevant questions in this model which the present analysis comes short of reaching. One of them is about invariant measures as seen from the tracer particle. A natural (and almost obvious) conjecture is that, after taking the limit as time goes to infinity, the particles in front of the tracer particle could be described by the superposition of two independent renewal point processes, one for the resting particles, one for the moving ones. The first process has independent interval distributions, identically distributed as  $\xi_1$ , except for the first interval, which is distributed as the residual lifetime associated to  $\xi_1$ . The second also has independent interval distributions, identically distributed as the interval between successive elastic particles in  $x$  (call the associated random variable  $\hat{\xi}_1$  and it is easy to see that it equals  $2(1-p)\xi_1$ , in distribution), except for the first interval, which is distributed as the residual lifetime associated to  $\hat{\xi}_1$  and the moving particles having all the same constant velocity  $2\bar{v}_0^p$ . Another question is about the central limit theorem, for either the velocity of the tracer particle or its position. Both questions require a finer analysis of the motion of the tracer particle and its surroundings, especially particles that might be colliding with it infinitely often. A result which would clear the way for these and other results was mentioned in Remark 4.3 above.

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