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INFERENCE ON THE VARIANCE AND SMOOTHING OF THE PATHS OF DIFFUSIONS ☆

STATISTIQUE SUR LA VARIANCE ET RÉGULARISATION DES TRAJECTOIRES DE DIFFUSIONS

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ABSTRACT. – We give a hypothesis testing method to fit the diffusion coefficient σ of a d-dimensional stochastic differential equation on the basis of the observation of certain functionals of regularizations of the solution.

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RÉSUMÉ. – Dans cet article, nous donnons une méthode permettant de faire des tests d'hypothèse sur le coefficient de diffusion d'une equation differentielle stochastique en dimension d, sur la base de l'observation de certaines fonctionnelles de régularisées des solutions, obtenues par convolution avec un noyau deterministe. Les méthodes exposées s'appliquent aussi a une classe plus générale de semimartingales continues.

1. Introduction

Let

$$X_t = x_0 + \int_0^t a_s\, dW_s + V_t \tag{1}$$

be an Itô semimartingale with values in $\mathcal{R}^d$, d a positive integer.

We use the following notation:

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$\mathcal{W} = \{W_s\colon s \geqslant 0\} = \{(W_s^1, \dots, W_s^d)^{\mathrm{T}}\colon s \geqslant 0\}$ is Brownian motion in $\mathcal{R}^d$, $(.)^{\mathrm{T}}$ denotes transposition, $x_0 \in \mathcal{R}^d$, $\mathcal{F} = \{\mathcal{F}_s\colon s \geqslant 0\}$ is the filtration generated by $\mathcal{W}$.

$$a_s = \left(a_s^{j,k}\right)_{j,k=1,\dots,d}, \quad V_s = \left(V_s^j\right)_{j=1,\dots,d}, \qquad s \geqslant 0,$$

are stochastic processes with continuous paths adapted to $\mathcal{F}$, the first one taking values in the real matrices of $d \times d$ elements and the second one in $\mathcal{R}^d$. We assume throughout this paper that for $j = 1, \dots, d$, the function $s \rightsquigarrow V_s^j$ has bounded variation in each bounded interval $[0, T]$ and denote $|V|^j(T)$ its total variation on this interval. We denote $a = \{a_s\colon s \geqslant 0\}$, $V = \{V_s\colon s \geqslant 0\}$, $\mathcal{X} = \{X_t\colon t \geqslant 0\}$, $\|.\|$ is Euclidean norm and $sg(y) = \frac{y}{\|y\|}(y \in \mathcal{R}^d,\ y \neq 0)$.

Our purpose is to study inference methods on the noise part in (1) from the observation of a functional of a regularization of the actual path X_t during a time interval $0 \leqslant t \leqslant \tau$. This is done in Examples B and C below for diffusions that verify some additional requirements. A specially interesting case is to test the hypothesis that the noise is purely Brownian, in which explicit formulae are obtained. A well-known difficulty for this problem is that if one considers different values of a the induced measures on the space of trajectories become mutually singular, so that there is no straightforward method based on likelihood.

2. Related results

Several estimation methods for the diffusion coefficient and related problems have been studied in the literature after the pionner work by Dacunha-Castelle and Florens-Zmirou [4]. An example of this approach is the following [5]. Consider a solution of the one-dimensional SDE

$$dX_s = \sigma(X_s)\,dW_s + b(X_s)\,ds$$

(b and σ satisfy certain regularity conditions and $0 < k \leqslant \sigma(x) \leqslant K$ for some constants k, K). Let us denote by T_x the hitting time of x, i.e. $T_x = \inf\{s \in [0, 1]\colon\ X_s = x\}$ and $L_t(x)$ the local time $L_t(x) = \lim_{\delta \to 0^+} \frac{1}{2\delta} \int_0^t \mathbb{1}_{\{|X_s - x| < \delta\}}\,ds$, and pick a sequence h_n such that $nh_n \to \infty$ and $nh_n^3 \to 0$. Based on the observation of $(X_{\frac{i}{n}})_{i \leqslant n}$ (what we will call in the sequel *discrete sampling*), when the trajectory of the diffusion visits x (i.e. $T_x < 1$), $\sigma(x)$ is estimated by

$$S_n(x) = \frac{\sum_{i=1}^{n-1} \mathbb{1}_{\{|X_{\frac{i}{n}} - x| < h_n\}} n(X_{\frac{i+1}{n}} - X_{\frac{i}{n}})^2}{\sum_{i=1}^{n-1} \mathbb{1}_{\{|X_{\frac{i}{n}} - x| < h_n\}}}.$$

More precisely, Florens-Zmirou has shown that, conditionally on the event $\{T_x < 1\}$, $\sqrt{nh_n}((S_n(x)/\sigma^2(x)) - 1)$ converges in distribution to $(L_1(x))^{-1/2} Z$, where Z is a standard normal random variable independent of $L_1(x)$.

Other related results and refinements can be found in the literature. See for instance [6, 7,9]. These papers deal with discrete sampling and are based on various approximations of the local time. Azaïs [1] and Jacod [8] have obtained approximation methods for local times, including, in the second reference, the speed of convergence. More precisely, if X is a Brownian motion and $h(x, y)$ satisfies certain boundedness conditions and $\theta(h)$ is a suitable centering constant, Jacod [8] has shown that

$$n^{1/4}\left(n^{-1/2}\sum_{i=1}^{[nt]} h\left(\sqrt{n}\left(X_{\frac{(i-1)}{n}} - x\right), \sqrt{n}\left(X_{\frac{i}{n}} - X_{\frac{(i-1)}{n}}\right)\right) - \theta(h)L_t(x)\right)$$

converges to a conditionally Gaussian martingale. This result extends to diffusion processes and it is used in [9] for non-parametric kernel estimation of the diffusion coefficient.

Despite the considerable amount of results on the estimation of the diffussion coefficient, two statistical problems remained, to the best of our knowledge, without satisfactory solutions: (a) Testing hypothesis on the diffusion coefficient function σ. (b) Instead of discrete sampling, using functionals defined on regularization of the diffusion.

In this paper we present a method for hypothesis testing on the diffusion coefficient, that is based upon the observation of functionals defined on regularizations of the path of the underlying process $\mathcal{X}$ instead of the discrete sampling framework. The same results can be applied to make inference on the variance in a continuous time regression model but we will not pursue the subject here. Our approach is based on certain integral functionals related to the occupation measure. Theorem 3.1 below is the key result in order to obtain the asymptotic distribution of our estimates. In [11] a similar statement to that of Theorem 3.1 has been proved in dimension 1. A first result in the spirit of Theorem 3.1, valid when $\mathcal{X}$ is a one-dimensional Brownian Motion, was given in [3].

On the other hand, and more important from the standpoint of applications, the statement of Theorem 3.2 is more adequate for statistical purposes, since Theorem 3.1 explicitly involves the values of the (unknown) process X.

3. Main results

We assume the following additional hypotheses on the process $\{a_s\colon\ s \geqslant 0\}$:

(i) (Strong ellipticity) For each $s_0 > 0$, there exists a positive constant C_{s_0} such that for every $v \in \mathcal{R}^d$ one has $v^{\mathrm{T}} a_s a_s^{\mathrm{T}} v \geqslant C_{s_0}\|v\|^2$ for all $s \in [0, s_0]$.

(ii) For each $j, k = 1, \ldots, d,\ s \geqslant 0,\ \varepsilon > 0$:

$$\varepsilon^{-1/2}\left(a_{s+\varepsilon}^{j,k} - a_s^{j,k}\right) = \left(\vec{a}_s^{\,j,k}\right)^{\mathrm{T}} Z_{s,\varepsilon}^{j,k} + r_{s,\varepsilon}^{j,k} \tag{2}$$

where $\vec{a}_s^{\,j,k},\ Z_{s,\varepsilon}^{j,k}$ are random vectors with values in $\mathcal{R}^d$. We also put for $j, k = 1, \ldots, d$, $Z_{s,\varepsilon}^{i,j,k}$ (respectively $\vec{a}_s^{\,i,j,k}$) for the ith coordinate of $Z_{s,\varepsilon}^{j,k}$ (respectively $\vec{a}_s^{\,j,k}$) $i = 1, \ldots, d$, and

$$Z_{s,\varepsilon} = \left(Z_{s,\varepsilon}^{i,j,k}\right)_{i,j,k=1,\ldots,d}, \qquad \vec{a}_s = \left(\vec{a}_s^{\,i,j,k}\right)_{i,j,k=1,\ldots,d}.$$

$r_{s,\varepsilon}^{j,k}$ is a random variable with values in $\mathcal{R}^1$, $\vec{a}_s^{j,k}$ is $\mathcal{F}_s$-measurable and $Z_{s,\varepsilon}^{j,k}$, $r_{s,\varepsilon}^{j,k}$ are $\mathcal{F}_{s+\varepsilon}$-measurable, and verify for almost every pair $s, t,\ s \neq t$:

$$\big(Z_{s,\varepsilon}, Z_{t,\varepsilon}, W_{\cdot}^{\varepsilon,s}, W_{\cdot}^{\varepsilon,t}\big) \Rightarrow \big(Z_s, Z_t, \widetilde{W}_{\cdot}^{s}, \widetilde{W}_{\cdot}^{t}\big) = \zeta(s,t) \quad \text{as } \varepsilon \to 0 \tag{3}$$

where $\Rightarrow$ denotes weak convergence of probability measures in the space

$$\mathcal{R}^{d^3} \times \mathcal{R}^{d^3} \times \big[C\big([0,+\infty), \mathcal{R}^d\big)\big] \times \big[C\big([0,+\infty), \mathcal{R}^d\big)\big]$$

and, for each $\varepsilon > 0,\ t \geqslant 0$, $W_{\cdot}^{\varepsilon,t}$ is a new Brownian motion with values in $\mathcal{R}^d$ defined as

$$W_u^{\varepsilon,t} = \varepsilon^{-1/2}(W_{t+\varepsilon u} - W_t), \quad u \geqslant 0;$$

$\{\widetilde{W}_{\cdot}^{t}\colon\ t \geqslant 0\}$ is a collection of independent Brownian motions with values in $\mathcal{R}^d$; the distribution of $\zeta(s,t)$ is symmetric, that is, $\zeta(s,t)$ and $-\zeta(s,t)$ have the same law, $\zeta(s,t)$ is independent of $\mathcal{F}_\infty$ and for $\{s,t\} \cap \{s',t'\} = \phi$ one has $\zeta(s,t)$ and $\zeta(s',t')$ are independent.

(iii) Finally, we assume the following boundedness properties.

First:

$$\sup_{s\in[0,T]} \sup_{j,k=1,\dots,d} E\big\{\big|r_{s,\varepsilon}^{j,k}\big|^p\big\} \to 0 \quad \text{as } \varepsilon \to 0 \text{ for every } p > 0$$

and second, for every T, $\varepsilon_0 > 0$ and every $p > 0$ the $L^p(\Omega)$-norms of the coordinates of:

$$a_s, \quad \vec{a}_s, \quad Z_{s,\varepsilon}$$

are uniformly bounded as $0 \leqslant s \leqslant T,\ 0 < \varepsilon \leqslant \varepsilon_0$.

Let us check that solutions of SDE with sufficiently regular coefficients satisfy the above hypotheses. There exist also some other general examples that we will not consider here, as semimartingales of the form (1) such that a.s. $s \to a_s$ is Holder-continuous with exponent greater than $1/2$, and non-Markovian Itô integrals described in [10, p. 106].

With the above notations, let $a_s = \sigma(s, X_s)$ and $V_{(.)}$ absolutely continuous,

$$V_t = \int_0^t b_s\,ds \quad \text{with } b_s = b(s, X_s),$$

where

$$\sigma(s,x) = \big(\sigma^{j,k}(s,x)\big)_{j,k=1,\dots,d}; \quad b(s,x) = \big(b^j(s,x)\big)_{j=1,\dots,d}, \qquad s \geqslant 0,\ x \in \mathcal{R}^d$$

satisfy the usual hypotheses to ensure the existence and uniqueness of strong solution of the system of stochastic differential equations

$$dX_t = \sigma(t, X_t)\,dW_t + b(t, X_t)\,dt, \quad X_0 = x_0$$

such as Lipshitz local behaviour and degree one polynomial bound at ∞.

Furthermore we assume that σ is of class C_b^2 (that is C^2 with bounded derivatives) and satisfies the strong ellipticity assumption $v^{\mathrm{T}}\sigma(s,x)\sigma^{\mathrm{T}}(s,x)v \geqslant C_{s_0}\|v\|^2$ for some $C_{s_0} > 0$ and all $s \in [0, s_0]$, $x, v \in \mathcal{R}^d$.

It is easy to check that (2) and the subsequent conditions hold true and more precisely, that

$$\varepsilon^{-1/2}\big(a_{s+\varepsilon}^{j,k} - a_s^{j,k}\big) = \big(D_x\sigma^{j,k}\big)(s, X_s).\sigma(s, X_s).\varepsilon^{-1/2}(W_{s+\varepsilon} - W_s) + \mathrm{o}_{L^p}(1)$$

which means that we have (2) with:

$$\big(\vec{a}_s^{\,j,k}\big)^{\mathrm{T}} = \big(D_x\sigma^{j,k}\big)(s, X_s).\sigma(s, X_s), \qquad Z_{s,\varepsilon}^{j,k} = \varepsilon^{-1/2}(W_{s+\varepsilon} - W_s) = W_1^{\varepsilon,s}.$$

The notation $A^{j,k}(s,\varepsilon) = \mathrm{o}_{L^p}(1)$ means that

$$\sup_{j,k=1,\ldots,d} E\Big\{\sup_{0\leqslant s\leqslant T}\big|A^{j,k}(s,\varepsilon)\big|^p\Big\} \to 0 \quad \text{as } \varepsilon \downarrow 0.$$

for each $T > 0$.

(3) is easily verified.

Let us now turn to the description of the general procedure that we will follow, that is, smoothing of the paths and CLT results. Instead of observing the path of the underlying stochastic process $\{X_t\colon\ 0 \leqslant t \leqslant \tau\}$ during a time interval, which generally speaking is not feasible from physical point of view, we will observe a regularization

$$X_\varepsilon(t) = \int_{-\infty}^{+\infty} \psi_\varepsilon(t-s)X_s\,ds,$$

where $\mathcal{X}$ has been extended by $X_s = x_0$ for $s < 0$, $\varepsilon > 0$, and for each $x \in \mathcal{R}$, $\psi(x) = (\psi^{j,k}(x))_{j,k=1,\ldots,d}$ is a deterministic matrix kernel, each function $\psi^{j,k}(x)$ being C^∞ real-valued, support contained in the interval $[-1, 1]$,

$$\int_{-\infty}^{+\infty} \psi(x)\,dx = \left(\int_{-\infty}^{+\infty} \psi^{j,k}(x)\,dx\right)_{j,k=1,\ldots,d} = I$$

(I denotes the identity matrix $d \times d$) and $\psi_\varepsilon(t) = \varepsilon^{-1}\psi(\varepsilon^{-1}t)$.

We also add the following technical condition. Denote $\underline{\lambda}(x)$ (respectively $\bar{\lambda}(x)$) the minimal (respectively maximal) eigenvalue of $\psi(x)\psi^{\mathrm{T}}(x)$. We will assume that there exists a positive constant L_ψ such that $\bar{\lambda}(x) \leqslant L_\psi\underline{\lambda}(x)$ for all $x \in \mathcal{R}$. This condition plays some role only if $d > 1$ and limits the anisotropy that is allowed for the regularization.

In fact we do not observe the complete smoothed path but only a functional defined on it having the general form:

$$\theta_{\varepsilon,\tau} = \theta_{\varepsilon,\tau}(f,g) = \int_0^\tau f\big(X_\varepsilon(t)\big)g\big(\big\|\sqrt{\varepsilon}X_\varepsilon'(t)\big\|\big)\,dt \tag{4}$$

$'$ denotes differentiation with respect to t, $f:\mathcal{R}^d \to \mathcal{R}$ is of class C_b^2 and $g:\mathcal{R}^+ \to \mathcal{R}$ is of class C^2, $|g'(r)| \leqslant C_g(1+r^m)$ for some $m \geqslant 1$, some constant C_g and all $r \in \mathcal{R}^+$.

Our aim is to give a Central Limit Theorem for $\theta_{\varepsilon,\tau}$ as a first step to statistical inference on a.

Two interesting functionals are: 1) (Normalized curve length) Let $g(r) = r$ and $f(.)$ a C_b^2- approximation of $\mathbb{1}_B$ the indicator function of a subset in $\mathcal{R}^d$ with a sectionally smooth boundary. In this case, the functional $\theta_{\varepsilon,\tau}$ is an approximation of $\sqrt{\varepsilon}.l_\varepsilon(\tau;B)$, $l_\varepsilon(\tau;B)$ denoting the length of the part of the curve $t \rightsquigarrow X_\varepsilon(t)$ $(0 \leqslant t \leqslant \tau)$ that is contained in the "observation window" B, a subset of the state space. In the relevant situations $l_\varepsilon(\tau;B) \to +\infty$ as $\varepsilon \downarrow 0$ and $\varepsilon^{1/2}$ is the appropriate renormalization of the length to have a non-trivial limiting behaviour. 2) (Normalized kinetic energy) Let $g(r) = r^2$ and $f(.)$ is as in the previous example, in which case $\theta_{\varepsilon,\tau}$ is an approximation of $\varepsilon.E_\varepsilon(\tau;B)$, $E_\varepsilon(\tau;B)$ denoting the kinetic energy of the same part of the smoothed path.

THEOREM 3.1. – *With the hypotheses above*

$$\left(W_\tau, \varepsilon^{-1/2}\left[\theta_{\varepsilon,\tau} - \int_0^\tau f(X_t)E\{g(\|\Sigma_t^{1/2}\xi\|)/\mathcal{F}_\infty\}\,dt\right]\right) \Rightarrow (W_\tau, W^*_{\bar\sigma^2(\tau)}) \tag{5}$$

as $\varepsilon \downarrow 0$ *where*

- $\Rightarrow$ *denotes weak convergence of probability measures in the space* $C([0,+\infty),\mathcal{R}^d) \times C([0,+\infty),\mathcal{R})$,
- W^* *denotes a new one-dimensional Wiener process independent of* $\mathcal{F}_\infty$,
- *for* $u \in \mathcal{R}$, $\Sigma_u = \int_{-1}^1 \psi(v)a_u a_u^{\mathrm{T}}\psi^{\mathrm{T}}(v)\,dv$,
- ξ *is a Gaussian standard random variable with values in* $\mathcal{R}^d$, *independent of* $\mathcal{F}_\infty$,
- $\bar\sigma^2(\tau) = \int_0^\tau du \iint_{-1}^1 s(u,v,v')\,dv\,dv'$ *where*

$$s(u,v,v') = E\{f^2(X_u)g'(\|\eta_{u,v}\|)g'(\|\eta'_{u,v'}\|).(sg(\eta_{u,v}))^{\mathrm{T}} \times \psi(-v)a_u a_u^{\mathrm{T}}\psi^{\mathrm{T}}(-v')sg(\eta'_{u,v'})/\mathcal{F}_\infty\}$$

and the conditional distribution of the pair of $\mathcal{R}^d$*-valued random variables* $(\eta_{u,v}, \eta'_{u,v'})$ *given the* σ*-algebra* $\mathcal{F}_\infty$ *is centered Gaussian and*

$$E\{\eta_{u,v}\eta_{u,v}^{\mathrm{T}}/\mathcal{F}_\infty\} = E\{\eta'_{u,v'}\eta'^{\mathrm{T}}_{u,v'}/\mathcal{F}_\infty\} = \Sigma_u,$$

$$E\{\eta_{u,v}\eta'^{\mathrm{T}}_{u,v'}/\mathcal{F}_\infty\} = \int_{-1}^{v\wedge v'} \psi(-w)a_u a_u^{\mathrm{T}}\psi^{\mathrm{T}}(-w+|v'-v|)\,dw.$$

Remark 1. – It is not possible to use for statistical purposes the above theorem as it has been stated, since its application requires the knowledge of the path $\{X_t\ (0 \leqslant t \leqslant \tau)\}$ which can not be observed. The next theorem points to solve this problem.

Remark 2. – Note that the drift V does not appear in the statement of Theorem 3.1, either in the centering term or in the asymptotic probability distribution. The same happens in the statistical version below. This is of course useful to make inference on a.

For the next theorem we will add a certain number of restrictions to the preceeding framework. We only consider the case of diffusions with coefficients that do not depend on time, that is $a_s = \sigma(X_s),\ b_s = b(X_s)$ where σ and b satisfy the hypotheses stated above. We will also assume that $\psi(x) = \psi^*(x).I$ where ψ^* is real-valued (isotropic regularization). Then, we may replace X_t by $X_\varepsilon(t)$ in the centering term which becomes:

$$m_\varepsilon(\tau) = \int_0^\tau f\big(X_\varepsilon(t)\big) E\big\{g\big[\|\psi^*\|_2\|\sigma^{\mathrm{T}}\big(X_\varepsilon(t)\big).\xi\|\big]/\mathcal{F}_\infty\big\}\,dt$$

($\|\psi^*\|_2$ denotes the L^2-norm ot the function ψ^*) and the asymptotic law of $W^*_{\bar\sigma^2_{\varepsilon'}(\tau)}$ is the same as the one of $W^*_{\bar\sigma^2(\tau)}$, where

$$\bar\sigma^2_\varepsilon(\tau) = \int_0^\tau f^2\big(X_\varepsilon(u)\big)\,du \iint_{-1}^{1} \psi^*(-v)\psi^*(-v')\,dv\,dv'$$
$$\times E\big\{g'(\|\eta_{u,v}\|)g'(\|\eta'_{u,v'}\|).\big(sg(\eta_{u,v})\big)^{\mathrm{T}}.\sigma\big(X_\varepsilon(u)\big).\sigma^{\mathrm{T}}\big(X_\varepsilon(u)\big).\big(sg(\eta'_{u,v'})\big)/\mathcal{F}_\infty\big\},$$

Σ_u is replaced by $\|\psi^*\|^2\sigma(X_\varepsilon(u)).\sigma^{\mathrm{T}}(X_\varepsilon(u))$ and $E\{\eta_{u,v}(\eta'_{u,v'})^{\mathrm{T}}/\mathcal{F}_\infty\}$ by $K(v,v')$. $\sigma(X_\varepsilon(u)).\sigma^{\mathrm{T}}(X_\varepsilon(u))$ with

$$K(v,v') = \int_{-1}^{v\wedge v'} \psi^*(-w)\psi^*(-w+|v-v'|)\,dw. \tag{6}$$

THEOREM 3.2. – *Under the above conditions,*

$$\big(W_\tau, \varepsilon^{-1/2}[\theta_{\varepsilon,\tau} - m_\varepsilon(\tau)]\big) \Rightarrow \big(W_\tau, W^*_{\bar\sigma^2(\tau)}\big) \quad \text{as } \varepsilon \downarrow 0$$

and

$$\bar\sigma^2_\varepsilon(\tau) \approx \bar\sigma^2(\tau),$$

where weak convergence, ξ, W^* and $\bar\sigma^2(\tau)$ are as in Theorem 3.1.

4. Examples

Example A. – In Theorem 3.1 let us put $d = 1$ and $g(r) = r$. We have: $\Sigma_u = |a_u|^2\|\psi\|_2^2$, $\theta_{\varepsilon,\tau} = \int_0^\tau f(X_\varepsilon(t))\sqrt{\varepsilon}|X'_\varepsilon(t)|\,dt = \sqrt{\varepsilon}\int_{-\infty}^{+\infty} f(u)N_u^{X_\varepsilon}([0,\tau])\,du$ where for $F: I \to \mathcal{R}$, $N_u^F(I)$ denotes the number of roots of the equation $F(t) = u$ in the interval I. In this case, the centering term becomes

$$\int_0^\tau f(X_t)|a_t|\|\psi\|_2\sqrt{\frac{2}{\pi}}\,dt = \sqrt{\frac{2}{\pi}}\|\psi\|_2\int_{-\infty}^{+\infty} f(u)\widetilde{L}_u^X([0,\tau])\,du.$$

Here $\widetilde{L}_u^X([0,\tau])$ stands for the local time of the process $\mathcal{X}$ for the mesure having density $|a_t|$ with respect to Lebesgue measure λ. That is, if $\mu_\tau(B) = \int_0^\tau \mathbb{1}_{\{X_t \in B\}} |a_t| \, dt$, then $\widetilde{L}_u^X([0,\tau]) = \frac{d\mu_\tau}{d\lambda}(u)$.

The asymptotic variance is given by

$$\bar{\sigma}^2(\tau) = \int_0^\tau f^2(X_u)|a_u|^2 \, du \iint_{-1}^{1} \psi(-v)\psi(-v')\big[2.P(\eta_v \eta_{v'} > 0) - 1\big] \, dv \, dv',$$

where the distribution of the random variable $(\eta_v, \eta_{v'})^{\mathrm{T}}$ in $\mathcal{R}^2$ is centered Gaussian with covariance matrix

$$\begin{pmatrix} \|\psi\|_2^2 & \int_{-1}^{v \wedge v'} \psi(-w)\psi(-w + |v - v'|) \, dw \\ \int_{-1}^{v \wedge v'} \psi(-w)\psi(-w + |v - v'|) \, dw & \|\psi\|_2^2 \end{pmatrix}.$$

Summing up, the statement of Theorem 3.1 takes the form:

$$\varepsilon^{-1/2}\left[\sqrt{\varepsilon} \int_{-\infty}^{+\infty} f(u) N_u^{X_\varepsilon}([0,\tau]) \, du - \sqrt{\frac{2}{\pi}} \|\psi\|_2 \int_{-\infty}^{+\infty} f(u) \widetilde{L}_u^X([0,\tau]) \, du\right] \Rightarrow W^*_{\bar{\sigma}^2(\tau)}$$

when $\varepsilon \downarrow 0$, the convergence taking place in the space $C([0,+\infty), \mathcal{R})$.

It is known that if $\mathcal{X}$ is a continuous semimartingale, as $\varepsilon \downarrow 0$ the expression in brackets tends to zero almost surely (this is essentially the result in [2]). The convergence above is a result on the fluctuations.

Example B. – Suppose that we are in the conditions of Theorem 3.2, with $d = 1$ and $g(r) = r$. We also assume that $\inf_{x \in \mathcal{R}} \sigma(x) > 0$. Suppose that we want to test the null hypothesis

$$H_0\colon\ \sigma(x) = \sigma_0(x) \quad \text{for all } x \in \mathcal{R}$$

against the alternative

$$H_\varepsilon\colon\ \sigma(x) = \sigma_\varepsilon(x) = \sigma_0(x) + \sqrt{\varepsilon}\sigma_1(x) + \gamma(x,\varepsilon) \quad \text{for all } x \in \mathcal{R}$$

where $\gamma(x,\varepsilon) = \mathrm{o}(\sqrt{\varepsilon})$ and $D_x\gamma(x,\varepsilon) = \mathrm{o}(\sqrt{\varepsilon})$ as $\varepsilon \downarrow 0$, uniformly on $x \in \mathcal{R}$. Here $\sigma_0(.)$, $\sigma_1(.)$ and $\gamma(.,\varepsilon)$ are given functions of class C_b^2 with at most degree one polynomial growth at ∞.

The application of Theorems 3.1 and 3.2 is not straightforward under the present conditions, since the process X_t depends now on ε. However, one can check that the same proofs remain valid, replacing X_t by the process X_t^ε, which is the solution of

$$dX_t^\varepsilon = \sigma_\varepsilon\big(X_t^\varepsilon\big) \, dW_t + b\big(X_t^\varepsilon\big) \, dt, \qquad X_0^\varepsilon = x_0$$

and setting $X_\varepsilon(t) = (\psi_\varepsilon * X^\varepsilon)(t)$, under the hypothesis H_ε, one has:

$$\varepsilon^{-1/2}\left[\sqrt{\varepsilon}\int_{-\infty}^{+\infty} f(u)N_u^{X_\varepsilon}([0,\tau])\,du - \sqrt{\frac{2}{\pi}}\|\psi\|_2\int_0^\tau f\big(X_\varepsilon(t)\big)\sigma_0\big(X_\varepsilon(t)\big)\,dt\right]$$
$$\approx \sqrt{\frac{2}{\pi}}\|\psi\|_2\int_0^\tau f\big(X_\varepsilon(t)\big)\sigma_1\big(X_\varepsilon(t)\big)\,dt + W^*_{\bar\sigma^2_\varepsilon(\tau)}. \tag{7}$$

One should interpret (7) in the following way:

As $\varepsilon \downarrow 0$ the law of the left-hand member converges in $C([0,+\infty),\mathcal{R})$ to the law of the random process

$$\sqrt{\frac{2}{\pi}}\|\psi\|_2\int_0^\tau f\big(X(t)\big)\sigma_1\big(X(t)\big)\,dt + W^*_{\bar\sigma^2_0(\tau)}$$

and *furthermore* the right-hand member in (7) converges to this process as $\varepsilon \downarrow 0$. Note that this is well adapted to statistical purposes since both the centering term and the asymptotic distribution in (7) can be computed from the hypotheses and from functionals defined on the smooth path $\{X_\varepsilon(t)\colon\ 0 \leqslant t \leqslant \tau\}$.

Example C. – Suppose again that we are in the conditions of Theorem 3.2, $g(r) = r$ and $d > 1$.

Suppose that we want to test the null hypothesis

$$H_0\colon\ \Gamma(x) = \sigma(x).\sigma^{\mathrm T}(x) = \Gamma_0(x) \quad \text{for all } x \in \mathcal{R}^d$$

against the alternative

$$H_\varepsilon\colon\ \Gamma(x) = \Gamma_0(x) + \sqrt{\varepsilon}\Gamma_1(x) + \Gamma_2(x,\varepsilon) \quad \text{for all } x \in \mathcal{R}^d,$$

where $\|\Gamma_2(x,\varepsilon)\|_{d\times d} = \mathrm{o}(\sqrt{\varepsilon})$ and $\|D_x\Gamma_2(x,\varepsilon)\| = \mathrm{o}(\sqrt{\varepsilon})$ as $\varepsilon \to 0$, uniformly on $x \in \mathcal{R}^d$. $\Gamma_0(.)$ and $\Gamma_1(.)$ are positive semidefinite $d\times d$ matrices having elements that are functions of class C_b^2 and at most degree two polynomial growth at ∞ and $\|.\|_{d\times d}$ is any norm on $d \times d$ matrices. Furthermore, assume that $\Gamma_0(.)$ satisfies a strong ellipticity condition of the type $y^{\mathrm T}\Gamma_0(x)y \geqslant \delta\|y\|^2$ for some $\delta > 0$ and any $x, y \in \mathcal{R}^d$.

The result takes the form that under H_ε:

$$\varepsilon^{-1/2}\left[\int_0^\tau f\big(X_\varepsilon(t)\big)\big[\|\sqrt{\varepsilon}X'_\varepsilon(t)\| - \|\psi^*\|_2.J_0\big(\Gamma_0(X_\varepsilon(t))\big)\big]\,dt\right]$$
$$\approx \frac{1}{2}\|\psi^*\|_2\int_0^\tau f\big(X_\varepsilon(t)\big).J_1\big(\Gamma_0\big(X_\varepsilon(t)\big),\Gamma_1\big(X_\varepsilon(t)\big)\big)\,dt + W^*_{\bar\sigma^2_\varepsilon(\tau)},$$

$$\bar\sigma^2_\varepsilon(\tau) \approx \int_0^\tau f^2\big(X_\varepsilon(u)\big)\,du \iint_{-1}^{1}\psi^*(-v)\psi^*(-v')J_2\big(\Gamma_0\big(X_\varepsilon(u)\big),\widetilde K(v,v')\big)\,dv\,dv',$$

where for A positive definite and B, C positive semidefinite $d \times d$ matrices, ξ a standard normal random vector in $\mathcal{R}^d$ and k a real number $|k| \leqslant 1$, we put:

$$J_0(A) = E\left\{\left(\xi^{\mathrm{T}} A\xi\right)^{1/2}\right\}, \qquad J_1(A,B) = E\left\{\frac{\xi^{\mathrm{T}} B\xi}{(\xi^{\mathrm{T}} A\xi)^{\frac{1}{2}}}\right\},$$

$$J_2(C,k) = E\left\{\frac{\eta^{\mathrm{T}} C\eta'}{\|\eta\|\|\eta'\|}\right\},$$

where η, η' are normal centered random vectors in $\mathcal{R}^d$, $E\{\eta.\eta'\} = k.C$, $E\{\eta.\eta^{\mathrm{T}}\} = E\{\eta'.\eta'^{\mathrm{T}}\} = C$ and $\widetilde{K}(v,v') = \frac{K(v,v')}{\|\psi^*\|_2^2}$ (see (6)).

An important case in which J_0, J_1, J_2 can be computed by means of closed formulae is $\Gamma_0 = I$, i.e. pure Brownian noise under the null hypothesis. One obtains:

$$J_0(I) = \begin{cases} \dfrac{(2p)!}{(2^p p!)^2}\sqrt{8\pi}\, p & \text{if } d = 2p, \\[2ex] \dfrac{(2^p p!)^2}{(2p)!}\dfrac{1}{\sqrt{2\pi}} & \text{if } d = 2p+1, \end{cases}$$

$$J_1(I,B) = \frac{J_0(I)}{d} tr(B),$$

$$J_2(I,k) = \frac{1}{\sqrt{\pi}\, 2^{\frac{d}{2}-1}\Gamma(\frac{d-1}{2})} \int_{-\infty}^{+\infty} x\mathcal{I}_d(x) \mathrm{e}^{-\frac{1}{2}(x - \frac{k}{\sqrt{1-k^2}})^2}\, dx,$$

$$\mathcal{I}_d(x) = \int_0^{+\infty} \frac{\rho^{d-2}}{\sqrt{x^2+\rho^2}} \mathrm{e}^{-\frac{1}{2}\rho^2}\, d\rho.$$

5. Proofs

Proof of Theorem 3.1. – We do not perform here the detailed computations of the proof.

Notice first that it suffices to consider convolution kernels such that $\operatorname{supp}(\psi)$ lies in $\{t \leqslant 0\}$. In fact,

$$X_\varepsilon(t) = \int_{-1}^{+1} \psi(u) X_{t-\varepsilon u}\, du = \int_{-2}^{0} \psi(u+1) X_{t-\varepsilon-\varepsilon u}\, du = \widetilde{X}_\varepsilon(t-\varepsilon)$$

where $\widetilde{X}_\varepsilon$ is the regularization that corresponds to the kernel $\widetilde{\psi}(u) = \psi(u+1)$ wich has support contained in $[-2,0]$. It follows that

$$\int_\varepsilon^{\tau+\varepsilon} f\left(X_\varepsilon(t)\right) g\left(\left\|\sqrt{\varepsilon} X'_\varepsilon(t)\right\|\right) dt = \int_0^\tau f\left(\widetilde{X}_\varepsilon(t)\right) g\left(\left\|\sqrt{\varepsilon}\widetilde{X}'_\varepsilon(t)\right\|\right) dt$$

and the study of the asymptotic behaviour of $\theta_{\varepsilon,\tau}$ can be reduced to the case when the support of ψ lies in $\{t \leqslant 0\}$ if one knows that the integrals over $[0,\varepsilon]$ and $[\tau, \tau+\varepsilon]$ can be neglected. This is easy to prove. With no loss of generality, we can assume that $\operatorname{supp}(\psi) \subset [-1,0]$.

A localization argument shows that it suffices to prove the theorem when the components of $a, \vec{a}$ and V are bounded by a non-random constant.

We introduce the following additional notation: For $\gamma \in [0,1]$, $x, y \in \mathcal{R}^d$ we put

$$G_t(y,\gamma) = \int_{\mathcal{R}^d} g(\|w\|) n_{A_t(\gamma)}(y-w)\,dw = E\{g(\|y + [A_t(\gamma)]^{1/2}\xi\|)/\mathcal{F}_\infty\},$$

where $A_t(\gamma) = \int_\gamma^1 \psi(-u) a_t a_t^{\mathrm{T}} \psi^{\mathrm{T}}(-u)\,du$; $n_\Sigma(.)$ is the normal density in $\mathcal{R}^d$ with mean 0 and variance matrix Σ and ξ is a random vector in $\mathcal{R}^d$ with standard normal distribution, ξ independent of $\mathcal{F}_\infty$. One easily checks that

$$\frac{\partial G_t}{\partial \gamma} = \frac{1}{2} tr[\acute{A}_t(\gamma)(D_{yy}G_t)]; \qquad \lim_{\gamma \to 1^-} G_t(y,\gamma) = g(\|y\|),$$

where $\acute{A}_t(\gamma)$ denotes the derivative with respect to γ. We put $X'_\varepsilon(t) = \varepsilon^{-1/2} \times \int_0^1 \psi(-u)\,d_u X_u^{\varepsilon,t}$ where

$$X_u^{\varepsilon,t} = \varepsilon^{-1/2}(X_{t+\varepsilon u} - X_t) = \int_0^u a_{t+\varepsilon v}\,d_v W_v^{\varepsilon,t} + \varepsilon^{-1/2}(V_{t+\varepsilon u} - V_t), \quad u \geqslant 0$$

and $Y_\gamma^{\varepsilon,t} = \int_0^\gamma \psi(-u)\,d_u X_u^{\varepsilon,t}$. Apply Itô's formula to the random process $\eta_\gamma^{\varepsilon,t} = G_t(Y_\gamma^{\varepsilon,t},\gamma)$, $0 \leqslant \gamma \leqslant 1$:

$$\begin{aligned}\eta_1^{\varepsilon,t} - \eta_0^{\varepsilon,t} &= g(\|\sqrt{\varepsilon}X'_\varepsilon(t)\|) - E\{g(\|[A_t(0)]^{1/2}\xi\|)/\mathcal{F}_\infty\} \\ &= \int_0^1 (D_y G_t)(Y_\gamma^{\varepsilon,t},\gamma)\,d_\gamma Y_\gamma^{\varepsilon,t} + \int_0^1 (D_\gamma G_t)(Y_\gamma^{\varepsilon,t},\gamma)\,d\gamma \\ &\quad + \frac{1}{2}\int_0^1 (d_\gamma Y_\gamma^{\varepsilon,t})^{\mathrm{T}}(D_{yy}G_t)(Y_\gamma^{\varepsilon,t},\gamma)\,d_\gamma Y_\gamma^{\varepsilon,t}. \end{aligned} \tag{8}$$

We write the left-hand side of (5) as

$$\begin{aligned}&\varepsilon^{-1/2}\left[\theta_{\varepsilon,\tau} - \int_0^\tau f(X_t)E\{g(\|\Sigma_t^{1/2}\xi\|)/\mathcal{F}_\infty\}\,dt\right] \\ &\quad = \mathrm{o}_\varepsilon(1) + \varepsilon^{-1/2}\int_0^\tau dt \int_0^1 f(X_t)(D_y G_t)(Y_\gamma^{\varepsilon,t},\gamma)\psi(-\gamma)a_t\,d_\gamma W_\gamma^{\varepsilon,t}, \end{aligned} \tag{9}$$

where the notation $\mathrm{o}_\varepsilon(1)$ means weak convergence to 0 in $C([0,+\infty),\mathcal{R})$ as $\varepsilon \to 0$.

To prove (9) we proceed in two steps:

First, we show that

$$\varepsilon^{-1/2}\int_0^\tau [f(X_\varepsilon(t)) - f(X_t)]\,g(\|\sqrt{\varepsilon}X'_\varepsilon(t)\|)\,dt = \mathrm{o}_\varepsilon(1). \tag{10}$$

This can be done on the basis of computations that are similar to those in [11], Step 1 and Lemma 3, c), which correspond to $d = 1$, $g(y) = y$ and V absolutely continuous, and can be adapted to our present context with some further work.

Second, in $\varepsilon^{-1/2}\int_0^\tau f(X_t)g(\|\sqrt{\varepsilon}X'_\varepsilon(t)\|)\,dt$ substitute $g(\|\sqrt{\varepsilon}X'_\varepsilon(t)\|)$ using (8).

Then, at the cost of adding a new error term $R_\varepsilon(\tau)$ we can replace $d_\gamma Y_\gamma^{\varepsilon,t}$ by $\psi(-\gamma)a_t\,d_\gamma W_\gamma^{\varepsilon,t}$ and

$$\left(d_\gamma Y_\gamma^{\varepsilon,t}\right)^{\mathrm{T}}(D_{yy}G_t)\left(Y_\gamma^{\varepsilon,t},\gamma\right)d_\gamma Y_\gamma^{\varepsilon,t}$$

by

$$-tr\left[\acute{A}_t(\gamma)\left(D_{yy}G_t(Y_\gamma^{\varepsilon,t},\gamma)\right)\right]dt.$$

One has: $R_\varepsilon(\tau) = A_\varepsilon^1(\tau) + A_\varepsilon^2(\tau) + A_\varepsilon^3(\tau)$ with

$$A_\varepsilon^1(\tau) = \int_0^\tau f(X_t)\,dt\int_0^1 (D_yG_t)\left(Y_\gamma^{\varepsilon,t},\gamma\right)\psi(-\gamma)\varepsilon^{-1/2}(a_{t+\varepsilon\gamma}-a_t)\,d_\gamma W_\gamma^{\varepsilon,t},$$

$$A_\varepsilon^2(\tau) = \frac{1}{2}\int_0^\tau f(X_t)\,dt\int_0^1 tr\left[\varepsilon^{-1/2}\left(a_{t+\varepsilon\gamma}a_{t+\varepsilon\gamma}^{\mathrm{T}} - a_ta_t^{\mathrm{T}}\right)\right.$$
$$\left.\times\,\psi^{\mathrm{T}}(-\gamma)(D_{yy}G_t)\left(Y_\gamma^{\varepsilon,t},\gamma\right)\psi(-\gamma)\right]d\gamma,$$

$$A_\varepsilon^3(\tau) = \varepsilon^{-1}\int_0^\tau f(X_t)\,dt\int_0^1 (D_yG_t)\left(Y_\gamma^{\varepsilon,t},\gamma\right)\psi(-\gamma)\,d_\gamma V_{t+\varepsilon\gamma}.$$

Tightness of $A_\varepsilon^1(\tau)$ in $C([0,+\infty),\mathcal{R})$ follows from the hypotheses and the fact that $(D_yG_t)(Y_\gamma^{\varepsilon,t},\gamma)$ is bounded in L^p for all $p > 0$.

For $A_\varepsilon^2(\tau)$ proceed as follows: let $H(y,\Sigma) = \int_{\mathcal{R}^d} g(\|w\|)n_\Sigma(y-w)\,dw$ so that $G_t(y,\gamma) = H(y, A_t(\gamma))$. Check that under the hypothesis we have done on g, the matrix $\Sigma^{1/2}(D_{yy}H)$ has elements that are polynomially bounded functions of y. Then, using the strong ellipticity hypothesis on $a_sa_s^{\mathrm{T}}$, plus the control on the relation between the maximal and minimal eigenvalues of $\psi(x)\psi^{\mathrm{T}}(x)$ and the conditions on a, one can prove that the trace in the integrand that appears in $A_\varepsilon^2(\tau)$ is bounded by a random variable with bounded L^p-norm, times the function $[\int_\gamma^1 \bar{\lambda}(u)\,du]^{-1/2}\bar{\lambda}(\gamma)$. Tightness follows now using standard bounds on the moments of the increments of $A_\varepsilon^2(\tau)$.

For $A_\varepsilon^3(\tau)$ assume that $\tau \in [0,s_0]$, $0 < \varepsilon < 1$ and $|V|_t = \sum_{j=1}^d |V^j|_t \leqslant \bar{v}$ for $t \in [0, s_0+1]$, $\bar{v}$ a non-random constant. The equicontinuity in probability of $A_\varepsilon^3(\tau)$ follows from the equicontinuity of

$$\widetilde{A}_\varepsilon^3(\tau) = \varepsilon^{-1}\int_0^\tau \left[|V|_{t+\varepsilon} - |V|_t\right]dt.$$

Let η, $0 < \eta < 1$, be given and $0 < \tau < \tau' < s_0$. If $\varepsilon > \frac{\tau'-\tau}{\eta}$, then $\widetilde{A}_\varepsilon^3(\tau') - \widetilde{A}_\varepsilon^3(\tau) \leqslant \bar{v}.\eta$. If $\tau' - \tau < \varepsilon \leqslant \frac{\tau'-\tau}{\eta}$, then $\widetilde{A}_\varepsilon^3(\tau') - \widetilde{A}_\varepsilon^3(\tau) \leqslant \omega_{|V|}(\frac{\tau'-\tau}{\eta})$ (ω_f denotes the continuity modulus of the function f). If $0 < \varepsilon < \tau' - \tau$ we have

$$\widetilde{A}^3_\varepsilon(\tau') - \widetilde{A}^3_\varepsilon(\tau) \leqslant \int_\tau^{\tau'} d|V|_s \varepsilon^{-1} \int_{(s-\varepsilon)\vee 0}^{s} dt + \int_{\tau'}^{\tau'+\varepsilon} d|V|_s \varepsilon^{-1} \int_{(s-\varepsilon)\vee 0}^{\tau'} dt$$
$$\leqslant 2\omega_{|V|}(\tau' - \tau)$$

and equicontinuity of $\widetilde{A}^3_\varepsilon$ follows.

Weak convergence to zero of $A^i_\varepsilon(\tau)$ $(i = 1, 2, 3)$ can be proved now in a similar way as in [11] for the case $d = 1$. This finishes the proof of (9).

At this point, in the last integral in (9) make the change of variables $u = t + \varepsilon\gamma$, obtaining the first equality in the following chain:

$$\varepsilon^{-1/2} \int_0^\tau dt \int_0^1 f(X_t)(D_y G_t)\big(Y^{\varepsilon,t}_\gamma, \gamma\big)\psi(-\gamma) a_t \, d_\gamma W^{\varepsilon,t}_\gamma$$
$$= \varepsilon^{-1} \int_0^\tau f(X_t)\, dt \int_t^{t+\varepsilon} (D_y G_t)\left(Y^{\varepsilon,t}_{\frac{u-t}{\varepsilon}}, \frac{u-t}{\varepsilon}\right)\psi\left(-\frac{u-t}{\varepsilon}\right) a_t \, dW_u$$
$$= \mathrm{o}_\varepsilon(1) + \varepsilon^{-1} \int_\varepsilon^\tau dW_u \int_{u-\varepsilon}^{u} f(X_t)(D_y G_t)\left(Y^{\varepsilon,t}_{\frac{u-t}{\varepsilon}}, \frac{u-t}{\varepsilon}\right)\psi\left(-\frac{u-t}{\varepsilon}\right) a_t \, dt$$
$$= \mathrm{o}_\varepsilon(1) + \int_\varepsilon^\tau dW_u \int_0^1 f(X_{u-\varepsilon\gamma})(D_y G_{u-\varepsilon\gamma})\big(Y^{\varepsilon,u-\varepsilon\gamma}_\gamma, \gamma\big)\psi(-\gamma) a_{u-\varepsilon\gamma} \, d\gamma$$
$$= \mathrm{o}_\varepsilon(1) + \int_\varepsilon^\tau f(X_u)\, dW_u \int_0^1 (D_y G_{u-\varepsilon\gamma})$$
$$\times \left(\int_0^\gamma \psi(-v) a_{u-\varepsilon\gamma} \, d_v W^{\varepsilon,u-\varepsilon\gamma}_v, \gamma \right)\psi(-\gamma) a_u \, d\gamma$$
$$= \mathrm{o}_\varepsilon(1) + \int_0^\tau f(X_u) K_\varepsilon(u)\, dW_u.$$

The second equality follows from a Fubini-type theorem applied to the double Itô × Lebesgue integral; for the third undo the change of variables. The other equalities are plain.

The remaining of the proof of Theorem 3.1 consists in proving the weak convergence of $(W_\tau, \int_0^\tau K_\varepsilon(u)\, dW_u)$ as $\varepsilon \to 0$ to the limit law in the statement. This is done by standard weak convergence arguments and the calculations are variants to those in [11] for $d = 1$. Theorem 3.2 follows from Theorem 3.1 once one proves that it is possible to replace X_t by $X_\varepsilon(t)$ in the centering term. This is similar – easier – to (10).

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