*Ann. I. H. Poincaré – PR* 38, 6 (2002) 973–990

S0246-0203(02)01127-5/FLA

# ON SHARP BURKHOLDER–ROSENTHAL-TYPE INEQUALITIES FOR INFINITE-DEGREE $U$-STATISTICS

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Received 15 March 2001, revised 11 March 2002

ABSTRACT. – In this paper, we present a method that allows one to obtain a number of sharp inequalities for expectations of functions of infinite-degree $U$-statistics. Using the approach, we prove, in particular, the following result: Let $D$ be the class of functions $f:\mathbf{R}_+ \to \mathbf{R}_+$ such that the function $f(x+z) - f(x)$ is concave in $x \in \mathbf{R}_+$ for all $z \in \mathbf{R}_+$. Then the following estimate holds:

$$Ef\left(\sum_{l=0}^{m} \sum_{1\leqslant i_1<\cdots<i_l\leqslant n} Y_{i_1,\ldots,i_l}(X_{i_1},\ldots,X_{i_l})\right)$$
$$\leqslant \sum_{q=0}^{m} \sum_{1\leqslant j_1<\cdots<j_q\leqslant n} Ef\left(\sum_{l=q}^{m} \sum_{i_1<\cdots<i_{l-q}\in\{1,\ldots,n\}\setminus\{j_1,\ldots,j_q\}} E\left(Y_{j_1,\ldots,j_q,i_1,\ldots,i_{l-q}}(X_{j_1},\ldots,X_{j_q},X_{i_1},\ldots,X_{i_{l-q}}) \mid X_{j_1},\ldots,X_{j_q}\right)\right)$$

for all $f \in D$ and all $U$-statistics $\sum_{1\leqslant i_1<\cdots<i_l\leqslant n} Y_{i_1,\ldots,i_l}(X_{i_1},\ldots,X_{i_l})$ with nonnegative kernels $Y_{i_1,\ldots,i_l}:\mathbf{R}^l \to \mathbf{R}_+$, $1 \leqslant i_k \leqslant n$; $i_r \neq i_s$, $r \neq s$; $k,r,s = 1,\ldots,l$; $l = 0,\ldots,m$, in independent r.v.'s $X_1,\ldots,X_n$. Similar inequality holds for sums of decoupled $U$-statistics. The class $D$ is quite wide and includes all nonnegative twice differentiable functions $f$ such that the function $f''(x)$ is nonincreasing in $x > 0$, and, in particular, the power functions $f(x) = x^t$, $1 < t \leqslant 2$; the power functions multiplied by logarithm $f(x) = (x+x_0)^t \ln(x+x_0)$, $1 < t < 2$, $x_0 \geqslant \max(\mathrm{e}^{(3t^2-6t+2)/(t(t-1)(2-t))}, 1)$; and the entropy-type functions $f(x) = (x+x_0)\ln(x+x_0)$, $x_0 \geqslant 1$. As an application of the results, we determine the best constants in Burkholder–Rosenthal-type inequalities for sums of $U$-statistics and prove new decoupling inequalities for

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[1] Supported in part by a NSF grant DMS/99/72237.

[2] The author thanks Victor de la Peña and the Department of Statistics, Columbia University, for their hospitality during his visits in 1999–2000.

those objects. The results obtained in the paper are, to our knowledge, the first known results on the best constants in sharp moment estimates for $U$-statistics of a general type.


*MSC:* primary 60E15; secondary 60F25

*Keywords:* Infinite degree $U$-statistics; Burkholder–Rosenthal-type inequalities; Decoupling inequalities

RÉSUMÉ. – Dans ce travail nous présentons une méthode permettant d'obtenir certaines inégalités fines pour les espérances mathématiques de fonctions de $U$-statistiques de degré infini. En particulier, nous démonstrons le résultat suivant : Soit $D$ la classe des fonctions $f:\mathbf{R}_+ \to \mathbf{R}_+$ telles que $f(x+z)-f(x)$ est une fonction concave de $x \in \mathbf{R}_+$ pour chaque $z \in \mathbf{R}_+$. Alors, nous avons :

$$Ef\left(\sum_{l=0}^{m} \sum_{1 \leqslant i_1 < \cdots < i_l \leqslant n} Y_{i_1,\ldots,i_l}(X_{i_1},\ldots,X_{i_l})\right)$$
$$\leqslant \sum_{q=0}^{m} \sum_{1 \leqslant j_1 < \cdots < j_q \leqslant n} Ef\left(\sum_{l=q}^{m} \sum_{i_1 < \cdots < i_{l-q} \in \{1,\ldots,n\} \setminus \{j_1,\ldots,j_q\}} E\left(Y_{j_1,\ldots,j_q,i_1,\ldots,i_{l-q}}(X_{j_1},\ldots,X_{j_q},X_{i_1},\ldots,X_{i_{l-q}}) \mid X_{j_1},\ldots,X_{j_q}\right)\right)$$

pour toute $f \in D$ et toute $U$-statistique $\sum_{1 \leqslant i_1 < \cdots < i_l \leqslant n} Y_{i_1,\ldots,i_l}(X_{i_1},\ldots,X_{i_l})$ avec noyaux non-négatifs $Y_{i_1,\ldots,i_l}:\mathbf{R}^l \to \mathbf{R}_+$, $1 \leqslant i_k \leqslant n$ ; $i_r \neq i_s$, $r \neq s$ ; $k,r,s = 1,\ldots,l$ ; $l = 0,\ldots,m$, et variables aléatoires $X_1,\ldots,X_n$ indépendantes. Une inégalité analogue est vraie pour les sommes de $U$-statistiques découplées. La classe $D$ est assez étendue et contient toutes les fonctions $f$ qui sont deux fois différentiables et telles que $f''(x)$ est décroissante au sens large sur $x > 0$. En particulier, $D$ contient les fonctions puissance $f(x) = x^t$, $1 < t \leqslant 2$ ; les fonctions puissance fois le logarithme $f(x) = (x+x_0)^t \ln(x+x_0)$, $1 < t < 2$, $x_0 \geqslant \max(\mathrm{e}^{(3t^2-6t+2)/(t(t-1)(2-t))}, 1)$ ; et les fonctions de type entropie $f(x) = (x+x_0)\ln(x+x_0)$, $x_0 \geqslant 1$. Comme application de ces résultats, nous détérminons les meilleures constantes dans les inégalités de type Burkholder–Rosenthal pour les sommes des $U$-statistiques et nous prouvons des nouvelles inégalités de découplage pour ces mêmes objets. Les résultats de ce travail sont, à notre connaissance, les premiers sur les meilleurs constantes dans les inégalités fines pour les $U$-statistiques de type général.


*Mots Clés :* $U$-statistiques de degré infini ; Inégalités de type Burkholder–Rosenthal ; Inégalités de découplage

## 1. Introduction

Recently, Klass and Nowicki [11], Ibragimov and Sharakhmetov [6–8] (see also [5] and [9]) and Giné et al. [3] obtained Burkholder–Rosenthal-type inequalities for $U$-statistics with nonnegative and degenerate kernels. Ibragimov and Sharakhmetov [6] also showed the significance of each term in the Burkholder–Rosenthal-type bounds for $U$-statistics of arbitrary order and obtained results concerning the rate of growth of the best constants in these bounds. Giné et al. [3] proved the Burkholder–Rosenthal-type inequalities for the $t$th moment of $U$-statistics of order $m$ with the constants

$L_m^t(t/\ln t)^{mt}$, where $L_m$ is a constant depending only on $m$, and obtained Bernstein-type exponential inequalities for $U$-statistics (for further discussion of the order of constants in the Burkholder–Rosenthal-type estimates for $U$-statistics see [6]). Ibragimov et al. [10] found the best constants in Burkholder–Rosenthal-type inequalities for bilinear forms in the case of the fixed number of random variables (r.v.'s). de la Peña et al. [2] determined the best constants in Burkholder–Rosenthal-type inequalities for sums of multilinear forms in independent nonnegative and symmetric r.v.'s.

In this paper, we present a method that allows one to obtain sharp inequalities for expectations of sums of $U$-statistics with nonnegative kernels, which represent an important case of infinite-degree $U$-statistics (see [4]). Using the approach, we prove, in particular, the following Burkholder–Rosenthal-type inequality:

$$Ef\left(\sum_{l=0}^{m}\sum_{1\leqslant i_1<\cdots<i_l\leqslant n} Y_{i_1,\ldots,i_l}(X_{i_1},\ldots,X_{i_l})\right)$$
$$\leqslant \sum_{q=0}^{m}\sum_{1\leqslant j_1<\cdots<j_q\leqslant n} Ef\left(\sum_{l=q}^{m}\sum_{i_1<\cdots<i_{l-q}\in\{1,\ldots,n\}\setminus\{j_1,\ldots,j_q\}}\right.$$
$$\left. E\left(Y_{j_1,\ldots,j_q,i_1,\ldots,i_{l-q}}(X_{j_1},\ldots,X_{j_q},X_{i_1},\ldots,X_{i_{l-q}})\mid X_{j_1},\ldots,X_{j_q}\right)\right)$$

for all $U$-statistics $\sum_{1\leqslant i_1<\cdots<i_l\leqslant n} Y_{i_1,\ldots,i_l}(X_{i_1},\ldots,X_{i_l})$ with nonnegative kernels $Y_{i_1,\ldots,i_l}:\mathbf{R}^l\to\mathbf{R}_+$, $1\leqslant i_k\leqslant n$; $i_r\neq i_s$, $r\neq s$; $k,r,s=1,\ldots,l$; $l=0,\ldots,m$ ($Y_{i_1,\ldots,i_l}\equiv\text{const}\geqslant 0$ for $l=0$) in independent r.v.'s $X_1,\ldots,X_n$ and all functions $f:\mathbf{R}_+\to\mathbf{R}_+$ such that the function $f(x+z)-f(x)$ is concave in $x\in\mathbf{R}_+$ for all $z\in\mathbf{R}_+$. A similar inequality holds for sums of decoupled $U$-statistics. The above condition is satisfied for all twice differentiable functions $f$ such that the function $f''(x)$ is nonincreasing in $x>0$, and, in particular, for the power functions $f(x)=x^t$, $1<t\leqslant 2$; the power functions multiplied by logarithm $f(x)=(x+x_0)^t\ln(x+x_0)$, $1<t<2$, $x_0\geqslant\max(\mathrm{e}^{(3t^2-6t+2)/(t(t-1)(2-t))},1)$; and the entropy-type functions $f(x)=(x+x_0)\ln(x+x_0)$, $x_0\geqslant 1$. As an application of the results, we determine the best constants in Burkholder–Rosenthal-type inequalities for sums of regular and decoupled $U$-statistics with nonnegative kernels and prove new decoupling inequalities for sums of $U$-statistics. We show, for instance, that the constant in the following Burkholder–Rosenthal-type inequality is sharp:

$$E\left(\sum_{l=0}^{m}\sum_{1\leqslant i_1<\cdots<i_l\leqslant n} Y_{i_1,\ldots,i_l}(X_{i_1},\ldots,X_{i_l})\right)^t$$
$$\leqslant \sum_{q=0}^{m}\sum_{1\leqslant j_1<\cdots<j_q\leqslant n} E\left(\sum_{l=q}^{m}\sum_{i_1<\cdots<i_{l-q}\in\{1,\ldots,n\}\setminus\{j_1,\ldots,j_q\}}\right.$$
$$\left. E\left(Y_{j_1,\ldots,j_q,i_1,\ldots,i_{l-q}}(X_{j_1},\ldots,X_{j_q},X_{i_1},\ldots,X_{i_{l-q}})\mid X_{j_1},\ldots,X_{j_q}\right)\right)^t,$$

$1<t\leqslant 2$, for all $U$-statistics $\sum_{1\leqslant i_1<\cdots<i_l\leqslant n} Y_{i_1,\ldots,i_l}(X_{i_1},\ldots,X_{i_l})$ with nonnegative kernels $Y_{i_1,\ldots,i_l}:\mathbf{R}^l\to\mathbf{R}_+$, $1\leqslant i_k\leqslant n$; $i_r\neq i_s$, $r\neq s$; $k,r,s=1,\ldots,l$; $l=0,\ldots,m$, in

independent r.v.'s $X_1, \ldots, X_n$. A similar result holds for sums of decoupled $U$-statistics. To our knowledge, the results obtained in the paper are the first known results on the best constants in sharp two-sided moment estimates for $U$-statistics of a general type.

## 2. Sharp estimates for expectations of functions of sums of $U$-statistics

Let $\mathbf{R}_+ = [0, \infty)$, $1 \leqslant m \leqslant n$, $X_1, \ldots, X_n, X_{p1}, \ldots, X_{pn}$, $p = 1, \ldots, m$, be independent r.v.'s and let $Y_{i_1,\ldots,i_l} : \mathbf{R}^l \to \mathbf{R}_+$, $1 \leqslant i_k \leqslant n$; $i_r \neq i_s$, $r \neq s$; $k, r, s = 1, \ldots, l$; $l = 0, \ldots, m$, be functions having the property that $Y_{i_1,\ldots,i_l}(x_1, \ldots, x_l) = Y_{i_{\pi(1)},\ldots,i_{\pi(l)}}(x_{\pi(1)}, \ldots, x_{\pi(l)})$, $x_k \in \mathbf{R}$, $k = 1, \ldots, l$, $1 \leqslant i_1 < \cdots < i_l \leqslant n$, for all permutations $\pi : \{1, \ldots, l\} \to \{1, \ldots, l\}$, $l = 2, \ldots, m$ (we assume that $Y_{i_1,\ldots,i_l} \equiv \text{const} \geqslant 0$ for $l = 0$). Consider the sums of regular $U$-statistics (symmetric statistics)

$$\sum_{l=0}^{m} \sum_{1 \leqslant i_1 < \cdots < i_l \leqslant n} Y_{i_1,\ldots,i_l}(X_{i_1}, \ldots, X_{i_l})$$

and decoupled $U$-statistics (symmetric statistics)

$$\sum_{l=0}^{m} \sum_{1 \leqslant i_1 < \cdots < i_l \leqslant n} Y_{i_1,\ldots,i_l}(X_{1,i_1}, \ldots, X_{l,i_l}).$$

In what follows, write

$$Y^{\text{reg}}(i_1, \ldots, i_l) = Y_{i_1,\ldots,i_l}(X_{i_1}, \ldots, X_{i_l}),$$
$$Y^{\text{dec}}(i_1, \ldots, i_l) = Y_{i_1,\ldots,i_l}(X_{1,i_1}, \ldots, X_{l,i_l}).$$

Denote by $D$ the class of functions $f : \mathbf{R}_+ \to \mathbf{R}_+$ such that the function $f(x + z) - f(x)$ is concave in $x \in \mathbf{R}_+$ for all $z \in \mathbf{R}_+$. The class $D$ is quite wide and includes all nonnegative twice differentiable functions $f$ such that the function $f''(x)$ is nonincreasing in $x > 0$, and, in particular, the power functions $f_1(x) = x^t$, $1 < t \leqslant 2$; the power functions multiplied by logarithm $f_2(x) = (x + x_0)^t \ln(x + x_0)$, $1 < t < 2$, $x_0 \geqslant \max(\mathrm{e}^{(3t^2-6t+2)/(t(t-1)(2-t))}, 1)$, and the entropy-type functions $f_3(x) = (x + x_0)\ln(x + x_0)$, $x_0 \geqslant 1$. Indeed, if the function $f''(x)$ is nonincreasing in $x > 0$, then we have $f''(x + z) \leqslant f''(x)$ for all $x > 0$, $z \geqslant 0$, and, therefore, $f(x + z) - f(x)$ is concave in $x \in \mathbf{R}_+$ for all $z \in \mathbf{R}_+$. It is obvious that $f_1''(x)$ is nonincreasing in $x > 0$ and, therefore, $f_1 \in D$. In addition to that, $f_2'''(x) = (x + x_0)^{t-3}(t(t-1)(t-2)\ln(x + x_0) + 3t^2 - 6t + 2) \leqslant 0$, $x > 0$, and, therefore, $f_2''(x)$ is nonincreasing in $x > 0$, and $f_2 \in D$. Since $f_3''(x) = 1/(x + x_0)$ is nonincreasing in $x > 0$, we have $f_3 \in D$.

In the inequalities throughout the paper, the extremal cases of the estimates such as $+\infty \leqslant +\infty$ are considered to be valid inequalities; we, therefore, do not include assumptions on finiteness of moments of the summand r.v.'s that ensure finiteness of moments of sums of $U$-statistics into formulations of the results.

The following theorems give sharp Burkholder–Rosenthal-type inequalities for sums of $U$-statistics. In what follows, $E(\cdot \mid X_{j_1}, \ldots, X_{j_q}) = E(\cdot \mid X_{j_1,i_{j_1}}, \ldots, X_{j_q,i_{j_q}}) = E(\cdot)$, the unconditional expectation operator, for $q = 0$.

THEOREM 1. – *For $f \in D$,*

$$
\begin{aligned}
&Ef\left(\sum_{l=0}^{m}\sum_{1\leqslant i_1<\cdots<i_l\leqslant n} Y^{\mathrm{reg}}(i_1,\ldots,i_l)\right)\\
&\quad\leqslant \sum_{q=0}^{m}\sum_{1\leqslant j_1<\cdots<j_q\leqslant n} Ef\left(\sum_{l=q}^{m}\sum_{i_1<\cdots<i_{l-q}\in\{1,\ldots,n\}\setminus\{j_1,\ldots,j_q\}}\right.\\
&\qquad\left. E\big(Y^{\mathrm{reg}}(j_1,\ldots,j_q,i_1,\ldots,i_{l-q})\mid X_{j_1},\ldots,X_{j_q}\big)\right). \qquad (1)
\end{aligned}
$$

THEOREM 2. – *For $f \in D$,*

$$
\begin{aligned}
&Ef\left(\sum_{l=0}^{m}\sum_{1\leqslant i_1<\cdots<i_l\leqslant n} Y^{\mathrm{dec}}(i_1,\ldots,i_l)\right)\\
&\quad\leqslant \sum_{q=0}^{m}\sum_{1\leqslant j_1<\cdots<j_q\leqslant m}\sum_{1\leqslant i_{j_1}<\cdots<i_{j_q}\leqslant n} Ef\left(\sum_{l=j_q}^{m}\sum_{\substack{1\leqslant i_p\leqslant n, i_{p_1}<i_{p_2},\ p_1<p_2,\\ p,p_1,p_2=1,\ldots,l,\ p\neq j_1,\ldots,j_q}}\right.\\
&\qquad\left. E\big(Y^{\mathrm{dec}}(i_1,\ldots,i_l)|X_{j_1,i_{j_1}},\ldots,X_{j_q,i_{j_q}}\big)\right). \qquad (2)
\end{aligned}
$$

COROLLARY 1. – *For a twice differentiable function $f:\mathbf{R}_+\to\mathbf{R}_+$ such that the function $f''(x)$ is nonincreasing on $x>0$, inequalities* (1) *and* (2) *hold.*

THEOREM 3. – *The constants in the following inequalities are sharp*:

$$
\begin{aligned}
&E\left(\sum_{l=0}^{m}\sum_{1\leqslant i_1<\cdots<i_l\leqslant n} Y^{\mathrm{reg}}(i_1,\ldots,i_l)\right)^t\\
&\quad\leqslant \sum_{q=0}^{m}\sum_{1\leqslant j_1<\cdots<j_q\leqslant n} E\left(\sum_{l=q}^{m}\sum_{i_1<\cdots<i_{l-q}\in\{1,\ldots,n\}\setminus\{j_1,\ldots,j_q\}}\right.\\
&\qquad\left. E\big(Y^{\mathrm{reg}}(j_1,\ldots,j_q,i_1,\ldots,i_{l-q})|X_{j_1},\ldots,X_{j_q}\big)\right)^t, \qquad (3)
\end{aligned}
$$

$$
\begin{aligned}
&E\left(\sum_{l=0}^{m}\sum_{1\leqslant i_1<\cdots<i_l\leqslant n} Y^{\mathrm{dec}}(i_1,\ldots,i_l)\right)^t\\
&\quad\leqslant \sum_{q=0}^{m}\sum_{1\leqslant j_1<\cdots<j_q\leqslant m}\sum_{1\leqslant i_{j_1}<\cdots<i_{j_q}\leqslant n} E\left(\sum_{l=j_q}^{m}\sum_{\substack{1\leqslant i_p\leqslant n,\ i_{p_1}<i_{p_2},\ p_1<p_2,\\ p,p_1,p_2=1,\ldots,l,\ p\neq j_1,\ldots,j_q}}\right.\\
&\qquad\left. E\big(Y^{\mathrm{dec}}(i_1,\ldots,i_l)|X_{j_1,i_{j_1}},\ldots,X_{j_q,i_{j_q}}\big)\right)^t, \quad 1<t\leqslant 2. \qquad (4)
\end{aligned}
$$

*Remark* 1. – It is not difficult to see that moment inequalities (2) and (4) for sums of decoupled $U$-statistics follow from their counter-parts (1) and (3) for sums of regular $U$-statistics, using the fact that any decoupled $U$-statistic can be represented as an undecoupled $U$-statistic with many zero kernels (it suffices to consider new r.v.'s $\tilde{X}_{(p-1)n+i} = X_{pi}$, $p = 1, \ldots, m$, $i = 1, \ldots, n$, and new kernels $\tilde{Y}_{i_1, n+i_2, \ldots, (l-1)n+i_l} = Y_{i_1, i_2, \ldots, i_l}$, $1 \leqslant i_1 < i_2 < \cdots < i_l \leqslant n$, $l = 0, \ldots, m$; $\tilde{Y}_{j_1, j_2, \ldots, j_l} = 0$, $1 \leqslant j_1 < j_2 < \cdots < j_l \leqslant mn$, $(j_1, j_2, \ldots, j_l) \neq (i_1, n + i_2, \ldots, (l-1)n + i_l)$ for $1 \leqslant i_1 < i_2 < \cdots < i_l \leqslant n$, $l = 1, \ldots, m$; $Y_{j_{\pi(1)}, \ldots, j_{\pi(l)}}(x_1, \ldots, x_l) = Y_{j_1, \ldots, j_l}(x_{\pi^{-1}(1)}, \ldots, x_{\pi^{-1}(l)})$, $x_k \in \mathbf{R}$, $k = 1, \ldots, l$, $1 \leqslant j_1 < \cdots < j_l \leqslant mn$, for all permutations $\pi : \{1, \ldots, l\} \to \{1, \ldots, l\}$, $l = 2, \ldots, m$, where $\pi^{-1}$ is the inverse of $\pi$).

*Remark* 2. – The essence of the Burkholder–Rosenthal-type bounds for sums of $U$-statistics given by Theorems 1–3 is that they give (sharp) estimates for moments of the sums in terms of expressions that do not contain moments of *sums* of r.v.'s. The bounds contain only directly computable expressions. For example, in the case of regular $U$-statistics of order $m$ in identically distributed r.v.'s the bounds consist of terms equivalent to $n^{(m-k)t+k} E(E(Y^{\text{reg}}(X_1, \ldots, X_m) \mid X_1, \ldots, X_k))^t$, $k = 0, 1, \ldots, m$ (and each of the terms is significant, as it was shown in [6], see also [5]). In the case of, let us say, sums of multilinear forms the terms in the bounds depend only on the moments of individual variables (see also [2]).

*Remark* 3. – From the results obtained in [5–9,11] (see also [3]) it follows that the following non-sharp (in the sense of constants) Burkholder–Rosenthal-type inequality holds for regular $U$-statistics of second order with nonnegative kernels (below, $C_i(t)$, $C_i^{\text{reg}}(t)$ and $C_i^{\text{dec}}(t)$ are constants depending on $t$ only):

$$
\begin{aligned}
E\Bigg(\sum_{1 \leqslant i<j \leqslant n} Y_{ij}^{\text{reg}}(X_i, X_j)\Bigg)^t \leqslant{}& C_1(t) \sum_{1 \leqslant i<j \leqslant n} E\big(Y_{ij}^{\text{reg}}(X_i, X_j)\big)^t \\
&+ C_2(t) \sum_{i=1}^{n-1} E\Bigg(\sum_{j=i+1}^{n} E\big(Y_{ij}^{\text{reg}}(X_i, X_j) \mid X_i\big)\Bigg)^t \\
&+ C_3(t) \sum_{j=2}^{n} E\Bigg(\sum_{i=1}^{j-1} E\big(Y_{ij}^{\text{reg}}(X_i, X_j) \mid X_j\big)\Bigg)^t \\
&+ C_4(t)\Bigg(\sum_{1 \leqslant i<j \leqslant n} E Y_{ij}^{\text{reg}}(X_i, X_j)\Bigg)^t, \quad t > 1.
\end{aligned}
$$

From (3) it follows that a "natural" form of Burkholder–Rosenthal-type inequality for regular $U$-statistics of second order with nonnegative kernels contains three, but not four terms and is given by

$$
\begin{aligned}
E\Bigg(\sum_{1 \leqslant i<j \leqslant n} Y_{ij}^{\text{reg}}(X_i, X_j)\Bigg)^t \leqslant{}& C_1^{\text{reg}}(t) \sum_{1 \leqslant i<j \leqslant n} E\big(Y_{ij}^{\text{reg}}(X_i, X_j)\big)^t \\
&+ C_2^{\text{reg}}(t) \sum_{i=1}^{n} E\Bigg(\sum_{j \neq i} E\big(Y_{ij}^{\text{reg}}(X_i, X_j) \mid X_i\big)\Bigg)^t \\
&+ C_3^{\text{reg}}(t)\Bigg(\sum_{1 \leqslant i<j \leqslant n} E Y_{ij}^{\text{reg}}(X_i, X_j)\Bigg)^t.
\end{aligned}
$$

Moreover, the best constants in the inequality are given by $C_i^{\mathrm{reg}}(t) = 1$, $i = 1, 2, 3$, for $1 < t \leqslant 2$. Similarly, from (4) it follows that a "natural" form of Burkholder–Rosenthal-type inequality for decoupled $U$-statistics of second order with nonnegative kernels contains four terms similar to those in [8], namely,

$$
\begin{aligned}
E\Biggl(\sum_{1\leqslant i<j\leqslant n} Y_{ij}^{\mathrm{dec}}(X_{1i}, X_{2j})\Biggr)^t \leqslant{}& C_1^{\mathrm{dec}}(t) \sum_{1\leqslant i<j\leqslant n} E\bigl(Y_{ij}^{\mathrm{dec}}(X_{1i}, X_{2j})\bigr)^t \\
&+ C_2^{\mathrm{dec}}(t) \sum_{i=1}^{n-1} E\Biggl(\sum_{j=i+1}^{n} E\bigl(Y_{ij}^{\mathrm{dec}}(X_{1i}, X_{2j}) \mid X_{1i}\bigr)\Biggr)^t \\
&+ C_3^{\mathrm{dec}}(t) \sum_{j=2}^{n} E\Biggl(\sum_{i=1}^{j-1} E\bigl(Y_{ij}^{\mathrm{dec}}(X_{1i}, X_{2j}) | X_{2j}\bigr)\Biggr)^t \\
&+ C_4^{\mathrm{dec}}(t)\Biggl(\sum_{1\leqslant i<j\leqslant n} EY_{ij}^{\mathrm{reg}}(X_{1i}, X_{2j})\Biggr)^t,
\end{aligned}
$$

and, moreover, the best constants in the above inequality are given by $C_i^{\mathrm{dec}}(t) = 1$, $i = 1, 2, 3, 4$, for $1 < t \leqslant 2$.

*Remark* 4. – Similarly to Remark 3, from moment inequalities for sums of multilinear forms obtained by Peña et al. [2] and Theorems 1–3 it follows that a "natural" form of Burkholder–Rosenthal-type inequalities for expectations of functions of sums of regular $U$-statistics of order not greater than $m$ with nonnegative kernels contains $m + 1$ terms and a "natural" form of Burkholder–Rosenthal-type inequalities for expectations of functions of sums of decoupled $U$-statistics of order not greater than $m$ with nonnegative kernels contains $2^m$ terms. Moreover, those theorems imply the following inequalities:

$$
\begin{aligned}
&E\Biggl(\sum_{l=0}^{m} \sum_{1\leqslant i_1<\cdots<i_l\leqslant n} Y^{\mathrm{reg}}(i_1, \ldots, i_l)\Biggr)^t \\
&\quad\leqslant (m+1) \max_{q=0,\ldots,m} \sum_{1\leqslant j_1<\cdots<j_q\leqslant n} E\Biggl(\sum_{l=q}^{m} \sum_{i_1<\cdots<i_{l-q}\in\{1,\ldots,n\}\setminus\{j_1,\ldots,j_q\}} \\
&\qquad E\bigl(Y^{\mathrm{reg}}(j_1, \ldots, j_q, i_1, \ldots, i_{l-q}) \mid X_{j_1}, \ldots, X_{j_q}\bigr)\Biggr)^t,
\end{aligned}
$$

$$
\begin{aligned}
&E\Biggl(\sum_{l=0}^{m} \sum_{1\leqslant i_1<\cdots<i_l\leqslant n} Y^{\mathrm{dec}}(i_1, \ldots, i_l)\Biggr)^t \\
&\quad\leqslant 2^m \max_{q=0,\ldots,m} \max_{1\leqslant j_1<\cdots<j_q\leqslant m} \sum_{1\leqslant i_{j_1}<\cdots<i_{j_q}\leqslant n} E\Biggl(\sum_{l=j_q}^{m} \sum_{\substack{1\leqslant i_p\leqslant n,\ i_{p_1}<i_{p_2},\ p_1<p_2,\\ p,p_1,p_2=1,\ldots,l,\ p\neq j_1,\ldots,j_q}} \\
&\qquad E\bigl(Y^{\mathrm{dec}}(i_1, \ldots, i_l) \mid X_{j_1,i_{j_1}}, \ldots, X_{j_q,i_{j_q}}\bigr)\Biggr)^t,
\end{aligned}
$$

$1 < t \leqslant 2$.

From the estimate

$$\sum_{k=1}^{N} z_k^t \leqslant \left(\sum_{k=1}^{N} z_k\right)^t, \quad z_1,\ldots,z_N \geqslant 0,\ t>1, \tag{5}$$

and Jensen's inequality it follows that

$$\begin{aligned} &E\left(\sum_{l=0}^{m}\sum_{1\leqslant i_1<\cdots<i_l\leqslant n} Y^{\mathrm{reg}}(i_1,\ldots,i_l)\right)^t \\ &\quad\geqslant \max_{q=0,\ldots,m}\sum_{1\leqslant j_1<\cdots<j_q\leqslant n} E\Biggl(\sum_{l=q}^{m}\sum_{i_1<\cdots<i_{l-q}\in\{1,\ldots,n\}\setminus\{j_1,\ldots,j_q\}} \\ &\qquad E\bigl(Y^{\mathrm{reg}}(j_1,\ldots,j_q,i_1,\ldots,i_{l-q})\mid X_{j_1},\ldots,X_{j_q}\bigr)\Biggr)^t, \end{aligned} \tag{6}$$

$$\begin{aligned} &E\left(\sum_{l=0}^{m}\sum_{1\leqslant i_1<\cdots<i_l\leqslant n} Y^{\mathrm{dec}}(i_1,\ldots,i_l)\right)^t \\ &\quad\geqslant \max_{q=0,\ldots,m}\max_{1\leqslant j_1<\cdots<j_q\leqslant m}\sum_{1\leqslant i_{j_1}<\cdots<i_{j_q}\leqslant n} E\Biggl(\sum_{l=j_q}^{m}\sum_{\substack{1\leqslant i_p\leqslant n,\ i_{p_1}<i_{p_2},\ p_1<p_2,\\ p,p_1,p_2=1,\ldots,l,\ p\neq j_1,\ldots,j_q}} \\ &\qquad E\bigl(Y^{\mathrm{dec}}(i_1,\ldots,i_l)\mid X_{j_1,i_{j_1}},\ldots,X_{j_q,i_{j_q}}\bigr)\Biggr)^t, \end{aligned} \tag{7}$$

$1<t\leqslant 2$. Assume that $X'_{p1},\ldots,X'_{pn}$, $p=1,\ldots,m$, are independent copies of the r.v.'s $X_1,\ldots,X_n$ (the primes are used to remind us about the independence between the sequences). From estimate (5), the inequality $(\sum_{k=1}^N z_k)^t \leqslant N^{t-1}\sum_{k=1}^N z_k^t$, $z_1,\ldots,z_N\geqslant 0$, $t>1$, and estimates (3), (4), (6) and (7) it follows that the following theorem holds ($C_m^k = m!/(k!(m-k)!)$, $0\leqslant k\leqslant m$).

THEOREM 4. – *The following decoupling inequalities hold*:

$$\begin{aligned} &(m+1)^{-1}E\left(\sum_{l=0}^{m}\sum_{1\leqslant i_1<\cdots<i_l\leqslant n} Y_{i_1,\ldots,i_l}(X'_{1,i_1},\ldots,X'_{l,i_l})\right)^t \\ &\leqslant E\left(\sum_{l=0}^{m}\sum_{1\leqslant i_1<\cdots<i_l\leqslant n} Y_{i_1,\ldots,i_l}(X_{i_1},\ldots,X_{i_l})\right)^t \\ &\leqslant \left(\sum_{k=0}^{m}(C_m^k)^t\right)E\left(\sum_{l=0}^{m}\sum_{1\leqslant i_1<\cdots<i_l\leqslant n} Y_{i_1,\ldots,i_l}(X'_{1,i_1},\ldots,X'_{l,i_l})\right)^t, \quad 1<t\leqslant 2. \end{aligned}$$

Note that the constant in the upper decoupling inequality given by Theorem 4 satisfies the inequality $\sum_{k=0}^m (C_m^k)^t \leqslant 2^{mt}$. As far as we know, the constants in the estimates in Theorem 4 are the best available so far, and it is likely that they are the sharp ones.

Similarly, the estimate

$$\sum_{k=1}^{N} f(z_k) \leqslant f\left(\sum_{k=1}^{N} z_k\right), \quad z_1, \ldots, z_N \geqslant 0 \tag{8}$$

for all convex functions $f : \mathbf{R}_+ \to \mathbf{R}_+$ with $f(0) = 0$ and Jensen's inequality imply that

$$\begin{aligned} &Ef\left(\sum_{l=0}^{m} \sum_{1 \leqslant i_1 < \cdots < i_l \leqslant n} Y^{\mathrm{reg}}(i_1, \ldots, i_l)\right) \\ &\quad \geqslant \max_{q=0,\ldots,m} \sum_{1 \leqslant j_1 < \cdots < j_q \leqslant n} Ef\left(\sum_{l=q}^{m} \sum_{i_1 < \cdots < i_{l-q} \in \{1,\ldots,n\} \setminus \{j_1,\ldots,j_q\}} \right. \\ &\qquad \left. E\left(Y^{\mathrm{reg}}(j_1, \ldots, j_q, i_1, \ldots, i_{l-q}) \mid X_{j_1}, \ldots, X_{j_q}\right)\right), \end{aligned} \tag{9}$$

$$\begin{aligned} &Ef\left(\sum_{l=0}^{m} \sum_{1 \leqslant i_1 < \cdots < i_l \leqslant n} Y^{\mathrm{dec}}(i_1, \ldots, i_l)\right) \\ &\quad \geqslant \max_{q=0,\ldots,m} \max_{1 \leqslant j_1 < \cdots < j_q \leqslant m} \sum_{1 \leqslant i_{j_1} < \cdots < i_{j_q} \leqslant n} Ef\left(\sum_{l=j_q}^{m} \sum_{\substack{1 \leqslant i_p \leqslant n,\ i_{p_1} < i_{p_2},\ p_1 < p_2, \\ p, p_1, p_2 = 1,\ldots,l,\ p \neq j_1,\ldots,j_q}} \right. \\ &\qquad \left. E\left(Y^{\mathrm{dec}}(i_1, \ldots, i_l) \mid X_{j_1, i_{j_1}}, \ldots, X_{j_q, i_{j_q}}\right)\right) \end{aligned} \tag{10}$$

for all convex functions $f : \mathbf{R}_+ \to \mathbf{R}_+$ with $f(0) = 0$. From (8), the inequality $f(\sum_{k=1}^{N} z_k) \leqslant N^{-1} \sum_{k=1}^{N} f(N z_k)$, $z_1, \ldots, z_N \geqslant 0$, for all convex functions $f : \mathbf{R}_+ \to \mathbf{R}_+$, and estimates (1), (2), (9) and (10) it follows that the following more general results hold.

THEOREM 5. – *The following decoupling inequalities hold*:

$$\begin{aligned} &(m+1)^{-1} Ef\left(\sum_{l=0}^{m} \sum_{1 \leqslant i_1 < \cdots < i_l \leqslant n} Y_{i_1,\ldots,i_l}(X'_{1,i_1}, \ldots, X'_{l,i_l})\right) \\ &\quad \leqslant Ef\left(\sum_{l=0}^{m} \sum_{1 \leqslant i_1 < \cdots < i_l \leqslant n} Y_{i_1,\ldots,i_l}(X_{i_1}, \ldots, X_{i_l})\right) \\ &\quad \leqslant \sum_{k=0}^{m} Ef\left(C_m^k \sum_{l=0}^{m} \sum_{1 \leqslant i_1 < \cdots < i_l \leqslant n} Y_{i_1,\ldots,i_l}(X'_{1,i_1}, \ldots, X'_{l,i_l})\right) \end{aligned}$$

*for all convex functions* $f \in D$ *with* $f(0) = 0$.

*Remark* 5. – It is easy to see, using the derivations at the beginning of the section, that the class of convex functions $f \in D$ with $f(0) = 0$ includes the functions $f(x) = x^t$, $1 < t \leqslant 2$; $f(x) = (x + x_0)^t \ln(x + x_0) - x_0^t \ln x_0$, $1 < t < 2$, $x_0 \geqslant \max(\mathrm{e}^{(3t^2-6t+2)/(t(t-1)(2-t))}, 1)$; and $f(x) = (x + x_0)\ln(x + x_0) - x_0 \ln x_0$, $x_0 \geqslant 1$.

*Remark* 6. – From Khintchine–Marcinkiewicz–Zygmund inequalities for $U$-statistics (e.g., [1,5–9]) it follows that analogues of inequalities (3) and (4) with appropriately adjusted constants hold for sums of $U$-statistics with degenerate kernels. Moreover, by Hoeffding's expansion, this implies corresponding inequalities for sums of $U$-statistics with not necessarily degenerate kernels.

### 3. Proof of the theorems

Let us prove Theorem 1. Let us use induction on the number of r.v.'s $X_1, \ldots, X_n$. Let us first demonstrate the argument in the case $m = 2$. Suppose that $f \in D$, $c_0 \geqslant 0$, and $Y_i : \mathbf{R} \to \mathbf{R}_+$, $Y_{ij} : \mathbf{R}^2 \to \mathbf{R}_+$, $1 \leqslant i, j \leqslant n$, $i \neq j$, are functions such that $Y_{ij}(x_i, x_j) = Y_{ji}(x_j, x_i)$, $x_i, x_j \in \mathbf{R}$, $1 \leqslant i < j \leqslant n$. Let $Y^{\text{reg}}(i) = Y_i(X_i)$, $Y^{\text{reg}}(i, j) = Y_{ij}(X_i, X_j)$, $E_j(\cdot) = E(\cdot \mid X_1, \ldots, X_{j-1}, X_{j+1}, \ldots, X_n)$, $1 \leqslant i, j \leqslant n$, $i \neq j$, and let $E(\cdot)$ be the unconditional expectation operator. Let us show that

$$
\begin{aligned}
&Ef\left(c_0 + \sum_{i=1}^{n} Y^{\text{reg}}(i) + \sum_{1 \leqslant i < j \leqslant n} Y^{\text{reg}}(i, j)\right) \\
&\quad \leqslant \sum_{1 \leqslant i < j \leqslant n} Ef\big(Y^{\text{reg}}(i, j)\big) + \sum_{i=1}^{n} Ef\left(Y^{\text{reg}}(i) + \sum_{j=1, j \neq i}^{n} E_j Y^{\text{reg}}(i, j)\right) \\
&\qquad + f\left(c_0 + \sum_{i=1}^{n} EY^{\text{reg}}(i) + \sum_{1 \leqslant i < j \leqslant n} EY^{\text{reg}}(i, j)\right). \qquad (11)
\end{aligned}
$$

Suppose that it is already known that estimate (11) holds in the case of $n - 1$ r.v.'s $X_1, \ldots, X_{n-1}$. Let us prove that this implies that the inequality is valid in the case of $n$ r.v.'s $X_1, \ldots, X_n$. From the inequality

$$
Ef(X + z) - Ef(X) \leqslant f(EX + z) - f(EX) \qquad (12)
$$

for $f \in D$ and for an arbitrary nonnegative r.v. $X$ and all $z \in \mathbf{R}_+$ (implied by Jensen's inequality) we have, letting $X = Y^{\text{reg}}(n) + \sum_{i=1}^{n-1} Y^{\text{reg}}(i, n)$ and $z = c_0 + \sum_{i=1}^{n-1} Y^{\text{reg}}(i) + \sum_{1 \leqslant i < j \leqslant n-1} Y^{\text{reg}}(i, j)$,

$$
\begin{aligned}
&Ef\left(c_0 + \sum_{i=1}^{n} Y^{\text{reg}}(i) + \sum_{1 \leqslant i < j \leqslant n} Y^{\text{reg}}(i, j)\right) \\
&\quad = Ef\left(Y^{\text{reg}}(n) + \sum_{i=1}^{n-1} Y^{\text{reg}}(i, n) + \left(c_0 + \sum_{i=1}^{n-1} Y^{\text{reg}}(i) + \sum_{1 \leqslant i < j \leqslant n-1} Y^{\text{reg}}(i, j)\right)\right) \\
&\quad \leqslant Ef\left(Y^{\text{reg}}(n) + \sum_{i=1}^{n-1} Y^{\text{reg}}(i, n)\right) \\
&\qquad + Ef\left(EY^{\text{reg}}(n) + c_0 + \sum_{i=1}^{n-1}\big(Y^{\text{reg}}(i) + E_n Y^{\text{reg}}(i, n)\big) + \sum_{1 \leqslant i < j \leqslant n-1} Y^{\text{reg}}(i, j)\right).
\end{aligned}
$$

Conditioning on $X_n$ and using the induction hypothesis, we obtain

$$Ef\left(Y^{\text{reg}}(n)+\sum_{i=1}^{n-1}Y^{\text{reg}}(i,n)\right) \leqslant \sum_{i=1}^{n-1}Ef\left(Y^{\text{reg}}(i,n)\right)+Ef\left(Y^{\text{reg}}(n)+\sum_{i=1}^{n-1}E_iY^{\text{reg}}(i,n)\right).$$

In addition to that (also by the induction hypothesis),

$$\begin{aligned}
&Ef\left(EY^{\text{reg}}(n)+c_0+\sum_{i=1}^{n-1}\left(Y^{\text{reg}}(i)+E_nY^{\text{reg}}(i,n)\right)+\sum_{1\leqslant i<j\leqslant n-1}Y^{\text{reg}}(i,j)\right)\\
&\quad\leqslant \sum_{1\leqslant i<j\leqslant n-1}Ef\left(Y^{\text{reg}}(i,j)\right)+\sum_{i=1}^{n-1}Ef\left(Y^{\text{reg}}(i)+\sum_{j=1,j\neq i}^{n}E_jY^{\text{reg}}(i,j)\right)\\
&\qquad+f\left(EY^{\text{reg}}(n)+c_0+\sum_{i=1}^{n-1}\left(EY^{\text{reg}}(i)+EY^{\text{reg}}(i,n)\right)+\sum_{1\leqslant i<j\leqslant n-1}EY^{\text{reg}}(i,j)\right).
\end{aligned}$$

From the latter relations it follows that

$$\begin{aligned}
&Ef\left(c_0+\sum_{i=1}^{n}Y^{\text{reg}}(i)+\sum_{1\leqslant i<j\leqslant n}Y^{\text{reg}}(i,j)\right)\\
&\quad\leqslant \sum_{i=1}^{n-1}Ef\left(Y^{\text{reg}}(i,n)\right)+Ef\left(Y^{\text{reg}}(n)+\sum_{i=1}^{n-1}E_iY^{\text{reg}}(i,n)\right)\\
&\qquad+\sum_{1\leqslant i<j\leqslant n-1}Ef\left(Y^{\text{reg}}(i,j)\right)+\sum_{i=1}^{n-1}Ef\left(Y^{\text{reg}}(i)+\sum_{j=1,j\neq i}^{n}E_jY^{\text{reg}}(i,j)\right)\\
&\qquad+f\left(EY^{\text{reg}}(n)+c_0+\sum_{i=1}^{n-1}\left(EY^{\text{reg}}(i)+EY^{\text{reg}}(i,n)\right)+\sum_{1\leqslant i<j\leqslant n-1}EY^{\text{reg}}(i,j)\right)\\
&\quad=\sum_{1\leqslant i<j\leqslant n}Ef\left(Y^{\text{reg}}(i,j)\right)+\sum_{i=1}^{n}Ef\left(Y^{\text{reg}}(i)+\sum_{j=1,j\neq i}^{n}E_jY^{\text{reg}}(i,j)\right)\\
&\qquad+f\left(c_0+\sum_{i=1}^{n}EY^{\text{reg}}(i)+\sum_{1\leqslant i<j\leqslant n}EY^{\text{reg}}(i,j)\right).
\end{aligned}$$

The fact that by (12)

$$Ef\left(c_0+Y_1(X_1)\right)\leqslant Ef\left(Y_1(X_1)\right)+f\left(c_0+EY_1(X_1)\right) \tag{13}$$

for all $f\in D$ and $c_0\geqslant 0$, that is, (11) is valid in the case $n=1$, completes the proof. Let us follow the same approach in the case of arbitrary $m$. Suppose that $f\in D$, and $Y_{i_1,\dots,i_l}:\mathbf{R}^l\to\mathbf{R}_+$, $1\leqslant i_k\leqslant n$; $i_r\neq i_s$, $r\neq s$; $k,r,s=1,\dots,l$; $l=0,\dots,m$, are functions such that $Y_{i_1,\dots,i_l}(x_1,\dots,x_l)=Y_{i_{\pi(1)},\dots,i_{\pi(l)}}(x_{\pi(1)},\dots,x_{\pi(l)})$, $x_k\in\mathbf{R}$, $k=1,\dots,l$, $1\leqslant i_1<\dots<i_l\leqslant n$, for all permutations $\pi:\{1,\dots,l\}\to\{1,\dots,l\}$, $l=2,\dots,m$.

Let $Y^{\mathrm{reg}}(i_1,\ldots,i_l)=Y_{i_1,\ldots,i_l}(X_{i_1},\ldots,X_{i_l})$, $E_{i_1,\ldots,i_l}(\cdot)=E(\cdot\mid X_k,k=1,\ldots,n;\ k\neq i_1,\ldots,i_l)$, $1\leqslant i_k\leqslant n$; $i_r\neq i_s,\ r\neq s$; $k,r,s=1,\ldots,l$; $l=0,\ldots,m$, and let $E(\cdot)$ be the unconditional expectation operator. Suppose that we already have the bound

$$Ef\left(\sum_{l=0}^{m}\sum_{1\leqslant i_1<\cdots<i_l\leqslant n-1}Y^{\mathrm{reg}}(i_1,\ldots,i_l)\right)$$
$$\leqslant\sum_{q=0}^{m}\sum_{1\leqslant j_1<\cdots<j_q\leqslant n-1}Ef\left(\sum_{l=q}^{m}\sum_{i_1<\cdots<i_{l-q}\in\{1,\ldots,n-1\}\setminus\{j_1,\ldots,j_q\}}\right.$$
$$\left.E_{i_1,\ldots,i_{l-q}}Y^{\mathrm{reg}}(j_1,\ldots,j_q,i_1,\ldots,i_{l-q})\right)$$

for all $f\in D$. From inequality (12) we obtain, letting $X=\sum_{l=0}^{m-1}\sum_{1\leqslant i_1<\cdots<i_l\leqslant n-1}Y^{\mathrm{reg}}(i_1,\ldots,i_l,n)$ and $z=\sum_{l=0}^{m}\sum_{1\leqslant i_1<\cdots<i_l\leqslant n-1}Y^{\mathrm{reg}}(i_1,\ldots,i_l)$,

$$Ef\left(\sum_{l=0}^{m}\sum_{1\leqslant i_1<\cdots<i_l\leqslant n}Y^{\mathrm{reg}}(i_1,\ldots,i_l)\right)$$
$$\leqslant Ef\left(\sum_{l=0}^{m}\sum_{1\leqslant i_1<\cdots<i_l\leqslant n}E_nY^{\mathrm{reg}}(i_1,\ldots,i_l)\right)$$
$$+Ef\left(\sum_{l=0}^{m-1}\sum_{1\leqslant i_1<\cdots<i_l\leqslant n-1}Y^{\mathrm{reg}}(i_1,\ldots,i_l,n)\right). \tag{14}$$

From the induction hypothesis we get (we assume $Y^{\mathrm{reg}}(i_1,\ldots,i_m,n)=0$ for all $1\leqslant i_k\leqslant n-1$; $i_r\neq i_s,\ r\neq s$; $k,r,s=1,\ldots,m$)

$$Ef\left(\sum_{l=0}^{m}\sum_{1\leqslant i_1<\cdots<i_l\leqslant n}E_nY^{\mathrm{reg}}(i_1,\ldots,i_l)\right)$$
$$=Ef\left(\sum_{l=0}^{m}\sum_{1\leqslant i_1<\cdots<i_l\leqslant n-1}\big(Y^{\mathrm{reg}}(i_1,\ldots,i_l)+E_nY^{\mathrm{reg}}(i_1,\ldots,i_l,n)\big)\right)$$
$$\leqslant\sum_{q=0}^{m}\sum_{1\leqslant j_1<\cdots<j_q\leqslant n-1}Ef\left(\sum_{l=q}^{m}\sum_{i_1<\cdots<i_{l-q}\in\{1,\ldots,n-1\}\setminus\{j_1,\ldots,j_q\}}\right.$$
$$E_{i_1,\ldots,i_{l-q}}Y^{\mathrm{reg}}(j_1,\ldots,j_q,i_1,\ldots,i_{l-q})$$
$$\left.+E_{i_1,\ldots,i_{l-q},n}Y^{\mathrm{reg}}(j_1,\ldots,j_q,i_1,\ldots,i_{l-q},n)\right)$$
$$=\sum_{q=0}^{m}\sum_{1\leqslant j_1<\cdots<j_q\leqslant n-1}Ef\left(\sum_{l=q}^{m}\sum_{i_1<\cdots<i_{l-q}\in\{1,\ldots,n\}\setminus\{j_1,\ldots,j_q\}}\right.$$
$$\left.E_{i_1,\ldots,i_{l-q}}Y^{\mathrm{reg}}(j_1,\ldots,j_q,i_1,\ldots,i_{l-q})\right). \tag{15}$$

Conditioning on the variable $X_n$ we also get by the induction assumptions

$$Ef\left(\sum_{l=0}^{m-1}\sum_{1\leqslant i_1<\cdots<i_l\leqslant n-1}Y^{\mathrm{reg}}(i_1,\ldots,i_l,n)\right)$$
$$\leqslant\sum_{q=0}^{m-1}\sum_{1\leqslant j_1<\cdots<j_q\leqslant n-1}Ef\left(\sum_{l=q}^{m-1}\sum_{i_1<\cdots<i_{l-q}\in\{1,\ldots,n-1\}\setminus\{j_1,\ldots,j_q\}}\right.$$
$$\left.E_{i_1,\ldots,i_{l-q}}Y^{\mathrm{reg}}(j_1,\ldots,j_q,i_1,\ldots,i_{l-q},n)\right). \tag{16}$$

From (14)–(16) it follows that

$$Ef\left(\sum_{l=0}^{m}\sum_{1\leqslant i_1<\cdots<i_l\leqslant n}Y^{\mathrm{reg}}(i_1,\ldots,i_l)\right)$$
$$\leqslant\sum_{q=0}^{m}\sum_{1\leqslant j_1<\cdots<j_q\leqslant n-1}Ef\left(\sum_{l=q}^{m}\sum_{i_1<\cdots<i_{l-q}\in\{1,\ldots,n\}\setminus\{j_1,\ldots,j_q\}}\right.$$
$$\left.E_{i_1,\ldots,i_{l-q}}Y^{\mathrm{reg}}(j_1,\ldots,j_q,i_1,\ldots,i_{l-q})\right)$$
$$+\sum_{q=0}^{m-1}\sum_{1\leqslant j_1<\cdots<j_q\leqslant n-1}Ef\left(\sum_{l=q}^{m-1}\sum_{i_1<\cdots<i_{l-q}\in\{1,\ldots,n-1\}\setminus\{j_1,\ldots,j_q\}}\right.$$
$$\left.E_{i_1,\ldots,i_{l-q}}Y^{\mathrm{reg}}(j_1,\ldots,j_q,i_1,\ldots,i_{l-q},n)\right)$$
$$=\sum_{q=0}^{m}\sum_{1\leqslant j_1<\cdots<j_q\leqslant n}Ef\left(\sum_{l=q}^{m}\sum_{i_1<\cdots<i_{l-q}\in\{1,\ldots,n\}\setminus\{j_1,\ldots,j_q\}}\right.$$
$$\left.E_{i_1,\ldots,i_{l-q}}Y^{\mathrm{reg}}(j_1,\ldots,j_q,i_1,\ldots,i_{l-q})\right).$$

The fact that by (13) inequality (1) holds in the case of one r.v. $X_1$ completes the proof of Theorem 1. Theorem 2 might be proven in a similar way (or deduced from Theorem 1, see Remark 1). Corollary 1 is an immediate consequence of Theorems 1 and 2. Applying Theorems 1 and 2 for $f(x)=x^t$, we obtain inequalities (3) and (4). Let $1<t\leqslant 2$, $a_k,b_k>0$, $a_k^t<b_k$, and let $c_{i_1,\ldots,i_l}\geqslant 0$, $1\leqslant i_k\leqslant n$; $i_r\neq i_s, r\neq s$; $k,r,s=1,\ldots,l$; $l=0,\ldots,m$; $c_{i_1,\ldots,i_l}=c_{i_{\pi(1)},\ldots,i_{\pi(l)}}$, $1\leqslant i_1<\cdots<i_l\leqslant n$, for all permutations $\pi:\{1,\ldots,l\}\to\{1,\ldots,l\}$, $l=2,\ldots,m$ (we assume that $c_{i_1,\ldots,i_l}=c_0\geqslant 0$ for $l=0$). Let us set $Y_{i_1,\ldots,i_l}(x_1,\ldots,x_l)=c_{i_1,\ldots,i_l}x_1\ldots x_l$, $1\leqslant i_k\leqslant n$; $i_r\neq i_s, r\neq s$; $k,r,s=1,\ldots,l$; $l=0,\ldots,m$ ($Y_{i_1,\ldots,i_l}(x_1,\ldots,x_l)=c_0$ for $l=0$). Consider, similarly to [12], independent nonnegative r.v.'s $X_1^{(s_1)},\ldots,X_n^{(s_n)}$, $s_k=1,2,\ldots$, $k=1,2,\ldots,n$, with the following distributions: $P(X_k^{(s_k)}=a_k)=1-1/s_k$, $P(X_k^{(s_k)}=b_k^{(s_k)})=a_k/(s_k b_k^{(s_k)})$, $P(X_k^{(s_k)}=0)=1/s_k-a_k/(s_k b_k^{(s_k)})$, where $b_k^{(s_k)}=(\frac{s_k b_k-a_k^t(s_k-1)}{a_k})^{1/(t-1)}$. It is not difficult

to see that $b_k^{(s_k)} \geqslant a_k$, $0 \leqslant a_k/(s_k b_k^{(s_k)}) \leqslant 1/s_k$, $b_k^{(s_k)} \to \infty$, $(b_k^{(s_k)})^{t-1} a_k/s_k = b_k - a_k^t(1 - 1/s_k) \to b_k - a_k^t$ as $s_k \to \infty$. We have that for all nonnegative r.v.'s $Z_1$ and $Z_2$ with finite $t$th moment independent of $X_k^{(s_k)}$,

$$\begin{aligned} E\big(Z_1 X_k^{(s_k)} + Z_2\big)^t = {} & E(Z_1 a_k + Z_2)^t (1 - 1/s_k) + E Z_2^t \big(1/s_k - a_k/\big(s_k b_k^{(s_k)}\big)\big) \\ & + \big(E\big(Z_1 b_k^{(s_k)} + Z_2\big)^t - E Z_1^t \big(b_k^{(s_k)}\big)^t\big) a_k/\big(s_k b_k^{(s_k)}\big) \\ & + E Z_1^t \big(b_k^{(s_k)}\big)^{t-1} a_k/s_k. \end{aligned} \tag{17}$$

It is not difficult to see that $(1+x)^t - 1 \leqslant t(x + x^t)$ for all $t \in (1,2]$ and all $x \geqslant 0$. Consequently,

$$0 \leqslant E\big(Z_1 + Z_2/b_k^{(s_k)}\big)^t - E Z_1^t \leqslant t\big(E Z_1^{t-1} Z_2/b_k^{(s_k)} + E Z_2^t/(b_k^{(s_k)})^t\big).$$

Therefore,

$$\begin{aligned} & \big(E\big(Z_1 b_k^{(s_k)} + Z_2\big)^t - E Z_1^t \big(b_k^{(s_k)}\big)^t\big) a_k/\big(s_k b_k^{(s_k)}\big) \\ & \quad = \big(E\big(Z_1 + Z_2/b_k^{(s_k)}\big)^t - E Z_1^t\big)\big(b_k^{(s_k)}\big)^{t-1} a_k/s_k \to 0 \end{aligned}$$

as $s_k \to \infty$, and from (17) we obtain

$$E\big(Z_1 X_k^{(s_k)} + Z_2\big)^t \to E Z_1^t \big(b_k - a_k^t\big) + E(Z_1 a_k + Z_2)^t \tag{18}$$

as $s_k \to \infty$, for all r.v.'s $Z_1$ and $Z_2$ defined above. Let us show that

$$\begin{aligned} & E\left(\sum_{l=0}^m \sum_{1 \leqslant i_1 < \cdots < i_l \leqslant n} Y_{i_1,\ldots,i_l}\big(X_{i_1}^{(s_{i_1})}, \ldots, X_{i_l}^{(s_{i_l})}\big)\right)^t \\ & \quad = E\left(\sum_{l=0}^m \sum_{1 \leqslant i_1 < \cdots < i_l \leqslant n} c_{i_1,\ldots,i_l} X_{i_1}^{(s_{i_1})} \ldots X_{i_l}^{(s_{i_l})}\right)^t \\ & \quad \to \sum_{q=0}^m \sum_{1 \leqslant j_1 < \cdots < j_q \leqslant n} \prod_{r=1}^q \big(b_{j_r} - a_{j_r}^t\big) \\ & \qquad \times \left(\sum_{l=q}^m \sum_{i_1 < \cdots < i_{l-q} \in \{1,\ldots,n\} \setminus \{j_1,\ldots,j_q\}} c_{j_1,\ldots,j_q,i_1,\ldots,i_{l-q}} a_{i_1} \ldots a_{i_{l-q}}\right)^t, \end{aligned} \tag{19}$$

as $s_1 \to \infty, \ldots, s_n \to \infty$. Let us use induction on the number of the r.v.'s $X_1^{(s_1)}, \ldots, X_n^{(s_n)}$. Suppose we have already proven relation (19) for all sums of multilinear forms of order not greater than $m$, $1 \leqslant m \leqslant n-1$, in the case of $n-1$ r.v.'s $X_1^{(s_1)}, \ldots, X_{n-1}^{(s_{n-1})}$, that is suppose that the relation

$$\begin{aligned} & E\left(\sum_{l=0}^m \sum_{1 \leqslant i_1 < \cdots < i_l \leqslant n-1} c_{i_1,\ldots,i_l} X_{i_1}^{(s_{i_1})} \ldots X_{i_l}^{(s_{i_l})}\right)^t \\ & \quad \to \sum_{q=0}^m \sum_{1 \leqslant j_1 < \cdots < j_q \leqslant n-1} \prod_{r=1}^q \big(b_{j_r} - a_{j_r}^t\big) \end{aligned}$$

$$\times\left(\sum_{l=q}^{m}\sum_{i_1<\cdots<i_{l-q}\in\{1,\ldots,n-1\}\setminus\{j_1,\ldots,j_q\}} c_{j_1,\ldots,j_q,i_1,\ldots,i_{l-q}}a_{i_1}\ldots a_{i_{l-q}}\right)^t,$$

as $s_1\to\infty,\ldots,s_{n-1}\to\infty$, is valid. Letting $k=n$,

$$Z_1=\sum_{l=0}^{m-1}\sum_{1\leqslant i_1<\cdots<i_l\leqslant n-1} c_{i_1,\ldots,i_l,n}X_{i_1}^{(s_{i_1})}\ldots X_{i_l}^{(s_{i_l})},$$

$$Z_2=\sum_{l=0}^{m}\sum_{1\leqslant i_1<\cdots<i_l\leqslant n-1} c_{i_1,\ldots,i_l}X_{i_1}^{(s_{i_1})}\ldots X_{i_l}^{(s_{i_l})},$$

from (18) we get

$$\begin{aligned}&E\left(\sum_{l=0}^{m}\sum_{1\leqslant i_1<\cdots<i_l\leqslant n} c_{i_1,\ldots,i_l}X_{i_1}^{(s_{i_1})}\ldots X_{i_l}^{(s_{i_l})}\right)^t\\&\quad\to\left(b_n-a_n^t\right)E\left(\sum_{l=0}^{m-1}\sum_{1\leqslant i_1<\cdots<i_l\leqslant n-1} c_{i_1,\ldots,i_l,n}X_{i_1}^{(s_{i_1})}\ldots X_{i_l}^{(s_{i_l})}\right)^t\\&\qquad+E\left(E\left(\sum_{l=0}^{m}\sum_{1\leqslant i_1<\cdots<i_l\leqslant n} c_{i_1,\ldots,i_l}X_{i_1}^{(s_{i_1})}\ldots X_{i_l}^{(s_{i_l})}\;\Big|\;X_n^{(s_n)}=a_n\right)\right)^t,\end{aligned}\tag{20}$$

as $s_n\to\infty$. From the induction hypothesis it follows that

$$\begin{aligned}&E\left(\sum_{l=0}^{m-1}\sum_{1\leqslant i_1<\cdots<i_l\leqslant n-1} c_{i_1,\ldots,i_l,n}X_{i_1}^{(s_{i_1})}\ldots X_{i_l}^{(s_{i_l})}\right)^t\\&\quad\to\sum_{q=0}^{m-1}\sum_{1\leqslant j_1<\cdots<j_q\leqslant n-1}\prod_{r=1}^{q}(b_{j_r}-a_{j_r}^t)\\&\qquad\times\left(\sum_{l=q}^{m-1}\sum_{i_1<\cdots<i_{l-q}\in\{1,\ldots,n-1\}\setminus\{j_1,\ldots,j_q\}} c_{j_1,\ldots,j_q,i_1,\ldots,i_{l-q},n}a_{i_1}\ldots a_{i_{l-q}}\right)^t,\end{aligned}\tag{21}$$

as $s_1\to\infty,\ldots,s_{n-1}\to\infty$. Moreover (we assume $c_{i_1,\ldots,i_m,n}=0$ for all $1\leqslant i_k\leqslant n-1$; $i_r\neq i_s$, $r\neq s$; $k,r,s=1,\ldots,m$)

$$\begin{aligned}&E\left(E\left(\sum_{l=0}^{m}\sum_{1\leqslant i_1<\cdots<i_l\leqslant n} c_{i_1,\ldots,i_l}X_{i_1}^{(s_{i_1})}\ldots X_{i_l}^{(s_{i_l})}\;\Big|\;X_n^{(s_n)}=a_n\right)\right)^t\\&\quad=E\left(\sum_{l=0}^{m}\sum_{1\leqslant i_1<\cdots<i_l\leqslant n-1}(c_{i_1,\ldots,i_l}+c_{i_1,\ldots,i_l,n}a_n)X_{i_1}^{(s_{i_1})}\ldots X_{i_l}^{(s_{i_l})}\right)^t\\&\quad\to\sum_{q=0}^{m}\sum_{1\leqslant j_1<\cdots<j_q\leqslant n-1}\prod_{r=1}^{q}(b_{j_r}-a_{j_r}^t)\left(\sum_{l=q}^{m}\sum_{i_1<\cdots<i_{l-q}\in\{1,\ldots,n-1\}\setminus\{j_1,\ldots,j_q\}}\right.\\&\qquad\left.(c_{j_1,\ldots,j_q,i_1,\ldots,i_{l-q}}+c_{j_1,\ldots,j_q,i_1,\ldots,i_{l-q},n}a_n)a_{i_1}\ldots a_{i_{l-q}}\right)^t\end{aligned}$$

$$= \sum_{q=0}^{m} \sum_{1\leqslant j_1<\cdots<j_q\leqslant n-1} \prod_{r=1}^{q} (b_{j_r} - a_{j_r}^t) \times \left( \sum_{l=q}^{m} \sum_{i_1<\cdots<i_{l-q}\in\{1,\ldots,n\}\setminus\{j_1,\ldots,j_q\}} c_{j_1,\ldots,j_q,i_1,\ldots,i_{l-q}} a_{i_1}\ldots a_{i_{l-q}} \right)^t \tag{22}$$

as $s_1 \to \infty, \ldots, s_{n-1} \to \infty$. From (20)–(22) it follows that

$$\begin{aligned} &E\left( \sum_{l=0}^{m} \sum_{1\leqslant i_1<\cdots<i_l\leqslant n} c_{i_1,\ldots,i_l,n} X_{i_1}^{(s_{i_1})}\ldots X_{i_l}^{(s_{i_l})} \right)^t \\ &\quad\to (b_n - a_n^t) \sum_{q=0}^{m-1} \sum_{1\leqslant j_1<\cdots<j_q\leqslant n-1} \prod_{r=1}^{q} (b_{j_r} - a_{j_r}^t) \\ &\qquad\times \left( \sum_{l=q}^{m-1} \sum_{i_1<\cdots<i_{l-q}\in\{1,\ldots,n-1\}\setminus\{j_1,\ldots,j_q\}} c_{j_1,\ldots,j_q,i_1,\ldots,i_{l-q},n} a_{i_1}\ldots a_{i_{l-q}} \right)^t \\ &\qquad+ \sum_{q=0}^{m} \sum_{1\leqslant j_1<\cdots<j_q\leqslant n-1} \prod_{r=1}^{q} (b_{j_r} - a_{j_r}^t) \\ &\qquad\times \left( \sum_{l=q}^{m} \sum_{i_1<\cdots<i_{l-q}\in\{1,\ldots,n\}\setminus\{j_1,\ldots,j_q\}} c_{j_1,\ldots,j_q,i_1,\ldots,i_{l-q}} a_{i_1}\ldots a_{i_{l-q}} \right)^t \\ &\quad= \sum_{q=0}^{m} \sum_{1\leqslant j_1<\cdots<j_q\leqslant n} \prod_{r=1}^{q} (b_{j_r} - a_{j_r}^t) \\ &\qquad\times \left( \sum_{l=q}^{m} \sum_{i_1<\cdots<i_{l-q}\in\{1,\ldots,n\}\setminus\{j_1,\ldots,j_q\}} c_{j_1,\ldots,j_q,i_1,\ldots,i_{l-q}} a_{i_1}\ldots a_{i_{l-q}} \right)^t, \end{aligned} \tag{23}$$

as $s_1 \to \infty, \ldots, s_n \to \infty$. Therefore, (19) is valid. The following constants satisfy the conditions stated before (17): $b_k = a_k = 1/n$, $k = 1, \ldots, n$; $c_{i_1,\ldots,i_l} = 0$, $1 \leqslant i_k \leqslant n$; $i_r \neq i_s$, $r \neq s$; $k, r, s = 1, \ldots, l$; $l = 0, \ldots, m-1$; $c_{i_1,\ldots,i_m} = (\sum_{q=0}^{m} \frac{1}{q!}(\frac{1}{(m-q)!})^t)^{-1/t}$, $1 \leqslant i_k \leqslant n$; $i_r \neq i_s, r \neq s$; $k, r, s = 1, \ldots, m$. For these parameters, we get

$$\begin{aligned} &\sum_{q=0}^{m} \sum_{1\leqslant j_1<\cdots<j_q\leqslant n} \prod_{r=1}^{q} (b_{j_r} - a_{j_r}^t) \\ &\quad\times \left( \sum_{l=q}^{m} \sum_{i_1<\cdots<i_{l-q}\in\{1,\ldots,n\}\setminus\{j_1,\ldots,j_q\}} c_{j_1,\ldots,j_q,i_1,\ldots,i_{l-q}} a_{i_1}\ldots a_{i_{l-q}} \right)^t \\ &\qquad= \sum_{q=0}^{m} C_n^q (n^{-1} - n^{-t})^q (C_{n-q}^{m-q} n^{-(m-q)} c_{1,\ldots,m})^t \\ &\qquad\to \sum_{q=0}^{m} \frac{1}{q!} \left( \frac{1}{(m-q)!} \right)^t c_{1,\ldots,m}^t = 1, \end{aligned} \tag{24}$$

as $n \to \infty$. Moreover, since $EX_k^{(s_k)} = a_k$, $E(X_k^{(s_k)})^t = b_k$, $s_k = 1, 2, \ldots,\ k = 1, \ldots, n$, we obtain

$$\sum_{q=0}^{m} \sum_{1 \leqslant j_1 < \cdots < j_q \leqslant n} E\Biggl(\sum_{l=q}^{m} \sum_{i_1 < \cdots < i_{l-q} \in \{1,\ldots,n\} \setminus \{j_1,\ldots,j_q\}}$$

$$E_{i_1,\ldots,i_{l-q}} Y_{j_1,\ldots,j_q,i_1,\ldots,i_{l-q}}\bigl(X_{j_1}^{(s_{j_1})}, \ldots, X_{j_q}^{(s_{j_q})}, X_{i_1}^{(s_{i_1})}, \ldots, X_{i_{l-q}}^{(s_{i_{l-q}})}\bigr)\Biggr)^t$$

$$= \sum_{q=0}^{m} \sum_{1 \leqslant j_1 < \cdots < j_q \leqslant n} \prod_{r=1}^{q} E\bigl(X_{j_r}^{(s_{j_r})}\bigr)^t$$

$$\times \Biggl(\sum_{l=q}^{m} \sum_{i_1 < \cdots < i_{l-q} \in \{1,\ldots,n\} \setminus \{j_1,\ldots,j_q\}} c_{j_1,\ldots,j_q,i_1,\ldots,i_{l-q}} EX_{i_1}^{(s_{i_1})} \ldots EX_{i_{l-q}}^{(s_{i_{l-q}})}\Biggr)^t$$

$$= \sum_{q=0}^{m} \sum_{1 \leqslant j_1 < \cdots < j_q \leqslant n} \prod_{r=1}^{q} b_{j_r}^t$$

$$\times \Biggl(\sum_{l=q}^{m} \sum_{i_1 < \cdots < i_{l-q} \in \{1,\ldots,n\} \setminus \{j_1,\ldots,j_q\}} c_{j_1,\ldots,j_q,i_1,\ldots,i_{l-q}} a_{i_1} \ldots a_{i_{l-q}}\Biggr)^t$$

$$= \sum_{q=0}^{m} C_n^q n^{-q} \bigl(C_{n-q}^{m-q} n^{-(m-q)} c_{1,\ldots,m}\bigr)^t \to 1, \tag{25}$$

as $n \to \infty$. Relations (19), (24) and (25) imply sharpness of the constants in inequality (3). Sharpness of the constants in inequality (4) might be proven in a similar way. The decoupling inequalities in Theorems 4 and 5 follow from inequalities (1)–(10), as explained before the theorems. The proof is complete.

## Acknowledgement

The authors are grateful to an anonymous referee for many useful suggestions.

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