Annales de l'I. H. P., section C

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Annales de l'I. H. P., section C, tome 1, nº 5 (1984), p. 401-412

<http://www.numdam.org/item?id=AIHPC_1984__1_5_401_0>

Ann. Inst. Henri Poincaré,
Vol. 1, n° 5, 1984, p. 401-412. *Analyse non linéaire*

Closed geodesics for the Jacobi metric and periodic solutions of prescribed energy of natural Hamiltonian systems

by

V. BENCI (*)

ABSTRACT. — We prove that the Hamiltonian system

$$\begin{cases} \dot{p} = -\dfrac{\partial V}{\partial q} \\ \dot{q} = p \end{cases} \qquad p, q \in \mathbb{R}^n ; \quad V \in C^2(\mathbb{R}^n)$$

has at least one periodic solution of energy h, provided that the set $\{ q \in \mathbb{R}^n \mid V(q) \leqslant h \}$ is compact.

Key-words: Hamiltonian systems, periodic orbit, closed geodesics, Jacobi metric.

RÉSUMÉ. — Nous démontrons que le système hamiltonien

$$\dot{p} = -\frac{\partial V}{\partial q} \qquad p, q \in \mathbb{R}^n ; V \in C^2(\mathbb{R}^n)$$

$$\dot{q} = p$$

a au moins une solution périodique d'énergie h, pourvu que l'ensemble $\{ q \in \mathbb{R}^n \mid V(q) \leqslant h \}$ soit compact.

AMS (MOS) Subject Classifications: 58 E 10, 58 F 22, 70 J 10, 34 C 25.
Work Unit Number 1 (Applied Analysis).
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1. INTRODUCTION AND MAIN RESULTS

We consider a natural Hamiltonian function $H \in C^2(R^{2n})$ i. e. function of the form

$$H(p, q) = 1/2 \mid p \mid^2 + V(q) \qquad p, q \in R^n \tag{1.1}$$

and the corresponding system of differential equations

$$\dot{p} = -\frac{\partial H}{\partial q}; \dot{q} = \frac{\partial H}{\partial p} \tag{1.2}$$

where « · » denotes d/dt.

It is well known that the function H itself is an integral of the system (1.2). In fact it represents the energy of the dynamical system described by (1.2). It is a natural problem to ask if the equation (1.2) has periodic solutions of a prescribed energy h. The main result of this paper is the following theorem:

THEOREM 1.1. — *Suppose that*

$$\Omega = \{ q \in R^n \mid V(q) < h \} \tag{1.3}$$

is bounded and not empty. Then the Hamiltonian system (1.2) *has at least one periodic solution of energy h.*

Remark I. — The assumption (1.3) is necessary. In fact the Hamiltonian $H(p, q) = 1/2 \mid p \mid^2 + q$ has no periodic solution.

Remark II. — If there is $q_0 \in \partial\Omega$ such that $\nabla V(q_0) = 0$, then $q \equiv q_0$ and $p \equiv 0$ is a periodic solution of (1.3) of energy h. If we want to have nonconstant periodic solutions of energy h, we need to add the following assumption

$$V(q) \neq 0 \qquad \text{for every} \quad q \in \partial\Omega . \tag{1.4}$$

If (1.4) is violated, then it may be that (1.3) has no nonconstant periodic solution as the following example shows:

$$H(p, q) = 1/2 \mid p \mid^2 + q^4 - q^3 \qquad (p, q) \in R ; h = 0 .$$

Remark III. — As it will be clear by the proof, Theorem 1.1 applies also to Hamiltonians of the form

$$H(p, q) = 1/2\Sigma_{ij}a_{ij}(q)p_ip_j + V(q) \tag{1.5}$$

where $\{ a_{ij}(q) \}$ is a positive definite matrix for every $q \in \Omega$. However, since our proof is based on the variational principle of Maupertuis-Jacobi, it cannot be applied to Hamiltonians whose « kinetic energy » term is not a positive definite quadratic form.

The search of periodic solutions of prescribed energy is a problem which has a long history. We refer to [R_1] and [Br] for recent surveys and we restrict ourselves to mention only some of the more recent results. Weinstein and Moser [W] [M] have studied the existence of periodic solutions near an equilibrium. In this case, under suitable assumptions, the existence of n periodic orbits can be proved. However, far from an equilibrium, the existence of n-periodic orbits can be proved only under more restrictive assumptions on the energy surface $H(p, q) = h$. Ekeland and Lasry [EL] have proved this fact when such surface is convex and contained in the set $A_R = \{ (p, q) \mid R \leqslant |p|^2 + |q|^2 \leqslant R\sqrt{2} \}$ for some $R > 0$ (see also Ambrosetti and Marcini for another proof [AM]). A result of Berestycki, Lasry, Mancini and Ruf [BLMR] is the last result in this direction as far as I know; it includes both the theorem of Weinstein and the theorem of Ekeland and Lasry.

If the existence of at least one periodic orbit is required more general Hamiltonians are allowed. Seifert, in a pioneering work [S], has proved that the Hamiltonian (1-5) has at least one periodic solution provided that Ω is diffeomorphic to a ball. The theorem of Seifert has been generalized in many ways (cf. [R_1]). The last results in this direction is due to Rabinowitz [R_2]. He considers a Hamiltonian of the form

$$H(p, q) = K(p, q) + V(q)$$

where $\frac{\partial K}{\partial q} \cdot p > 0$ for $|p| > 0$ and Ω is diffeomorphic to a ball.

Under these assumptions he has proved the existence of at least one periodic orbit. The result of Rabinowitz, compared with Theorem 1.1, allows a more general « kinetic energy » term but still has to impose that Ω is diffeomorphic to a ball.

After I finished writing the preprint of this paper, two papers [GZ] and [H] appeared in which Theorem 1.1 has been proved. Our proof of Theorem 1.1 is quite different from the proofs given in [GZ] and [H]. We do not use much of algebraic topology or differential geometry but rather functional analysis. Also our approximation scheme is not geometrical (shortening of geodesics) but rather dynamical (we use a singular potential well). Moreover, given another result of the author [B], our proof is very short.

Our method of proving Theorem 1.1 is based on the least action principle of Maupertuis-Jacobi (cf. e. g. [A] page 245 or [G] for Hamiltonians of the form (1-5)) which leads our problem to a problem of differential geometry which will be explained below.

Let Ω be an open set in R^n with smooth (say C^2) boundary and let $a \in C^2(\overline{\Omega}, R^n)$ be a nonnegative function. We consider the metric

$$d\rho = \sqrt{a(x)}ds \qquad x \in \overline{\Omega} \tag{1.6}$$

where $ds = \sqrt{\Sigma_i (dx_i)^2}$ is the Euclidean metric. If $a(x) = h - V(x)$, $(h \in \mathbb{R})$ the metric (1.6) is the « Jacobi metric » associated to the Hamiltonian (1.1). The Maupertius-Jacobi principle states that the closed geodesics of the « Jacobi metric » are the periodic orbits of (1.1) of energy h.

To be more precise we give the following definition.

DEFINITION 1.2. — A continuous function $\gamma : S^1 \to \bar{\Omega}$ $(S^1 = [0, 1]/\{0, 1\})$ is a closed geodesic with respect to the metric (1.6) if it satisfies the following assumptions:

i) $\gamma(t) \in \Omega$ except may be for $t = 0$ and $t = 1/2$

ii) $\gamma \in C^2(I, \Omega)$ where $I = \gamma^{-1}(\Omega)$

iii) $\frac{d}{dt}[a(\gamma)\dot{\gamma}] = 1/2 |\dot{\gamma}|^2 \nabla a(\gamma)$ for every $t \in I$.

Remark IV. — The closed geodesic as defined by the above definition are of two different type:

$$\text{interior geodesics: } \gamma(S^1) \cap \partial\Omega = \phi \tag{1.7}$$

$$\text{brake geodesics: } \gamma(S^1) \cap \partial\Omega = \{0, 1/2\}. \tag{1.8}$$

The interior geodesics are just smooth curves contained in Ω, while the brake geodesics satisfy the relation

$$\gamma(t) = \gamma(1 - t) \tag{1.9}$$

(1.9) is an easy consequence of the Maupertuis-Jacobi principle (cf. Remark V). The precise statement of the Maupertuis-Jacobi principle is the following

THEOREM 1.3. — *Suppose that*

$$a(x) = h - V(x) \tag{1.10}$$

and that (1.4) *is satisfied. Then to every closed geodesic, by a suitable reparametrization of the independent variable* (*time*), *corresponds a periodic solution of* (1.1) *of energy* h.

Proof. — For the convenience of the reader we shall give the proof of the Maupertuis-Jacobi principle. Let γ be a closed geodesic and let I_0 denote S^1 if γ is an interior geodesic or $(0, 1/2)$ if γ is a brake geodesic. As we can check easily $1/2\, a(\gamma) |\dot{\gamma}|^2$ is an integral of equation (*iii*). Then

$$1/2\, a(\gamma) |\dot{\gamma}|^2 = \lambda \qquad \text{for every} \quad t \in I_0 \tag{1.11}$$

where $\lambda > 0$ is the integration constant. By (1.10) and (1.11) and equation (*iii*) we get

$$a \frac{d}{dt}[a(\gamma)\dot{\gamma}] = -\lambda \nabla V(\gamma) \qquad \text{for} \quad t \in (0, 1). \tag{1.12}$$

Now we define the following function

$$(1.13) \qquad s(t) = \int_0^t \frac{\sqrt{\lambda}}{a(\gamma(\tau))} dt \qquad t \in \mathrm{I}_0 .$$

If $\gamma(t)$ is an interior geodesic $\dfrac{1}{a(\gamma(t))}$ is a bounded function. If $\gamma(t)$ is a brake geodesic we have to prove that the integral (1.13) converges.

By (1.11) we get the following inequality

$$\left| \frac{d}{dt} \frac{1}{a(\gamma(t))} \right| = \left| \frac{\nabla a(\gamma)\dot{\gamma}}{a(\gamma)^2} \right| \leqslant \frac{\sqrt{2\lambda} \, | \nabla a(\gamma) |}{a(\gamma)^{5/2}} \qquad t \in (0, 1/2) .$$

Since we have supposed $\mathrm{V} \in \mathrm{C}^2(\overline{\Omega})$, $| \nabla a(x) |$ is bounded for $x \in \overline{\Omega}$, so we have

$$\left| \frac{d}{dt} \frac{1}{a(\gamma(t))} \right| \leqslant \mathrm{M}_1 \left(\frac{1}{a(\gamma(t))} \right)^{5/2} \qquad t \in (0, 1/2)$$

where M_1 is a suitable constant.

The above inequality and standard estimates for ordinary differential equations give the following inequality near $t = 0$ and $t = 1/2$

$$\frac{1}{a(\gamma(t))} \leqslant \frac{\mathrm{M}_2}{(t - t_0)^{2/3}} \quad \text{with } t_0 = 0 \text{ or } 1/2 \text{ and } \mathrm{M}_1 \text{ is a suitable constant.}$$

Thus in every case the function (1.13) is well defined for $t \in \overline{\mathrm{I}}_0$. Since it is a continuous increasing function it is invertible; $t(s)$ will denote its inverse.

We now set

$$(1.14) \qquad q(s) = \gamma(t(s)) .$$

By (1.12) we have

$$(1.15) \qquad \frac{dt}{ds} = \lambda^{-1/2} a(\gamma) \qquad \text{for} \quad t \in s(\mathrm{I}_0) .$$

Then

$$\frac{d^2}{ds^2} q(s) = \frac{d}{ds}\left[\dot{\gamma} \frac{dt}{ds} \right] = \lambda^{-1/2} \frac{d}{ds} [a(\gamma)\dot{\gamma}]$$
$$= \lambda^{-1/2} \frac{d}{dt} [a(\gamma)\dot{\gamma}] \frac{dt}{ds} = \lambda^{-1} a(\gamma) \frac{d}{dt} [a(\gamma)\dot{\gamma}] .$$

Replacing the above inequality in (1.12) we get

$$(1.16) \qquad \frac{d^2}{ds^2} q(s) = - \nabla \mathrm{V}(q) .$$

The above inequality holds for every $s \in t(\mathrm{I}_0)$. If $\mathrm{I}_0 = (0, 1/2)$ arguing in the same way we can prove (1.16) for (1/2, 1). Thus (1.16) holds for every $s \in t(S^1)$. Moreover by (1.11), (1.15) and (1.10) we get

$$(1.17)\quad 1/2\left(\frac{d\dot{q}}{ds}\right)^2 + V(q) = 1/2\,|\dot{\gamma}|^2\left(\frac{dt}{ds}\right)^2 + V(q) = \frac{\lambda}{a(\gamma)}\left(\frac{a(\gamma)}{\lambda^{-1/2}}\right)^2 + h - a(\gamma) = h\,.$$

Finally setting $p(s) = \dfrac{dq(s)}{ds}$, we obtain a periodic solution $(q(s), p(s))$ of (1.1) of energy h.

Remark V. — By the proof of theorem we see that a brake geodesic generates a solution of (1.16) such that $q(s(0)) = \gamma(0)$ and $q(s(1/2)) = \gamma(1/2)$. Moreover, by the uniqueness of the solution of equation (1.16) it follows that $q(s(1/2) - s_0) = q(s(1/2) + s_0)$ for $s_0 \in (0, s(1/2))$. Therefore the brake geodesic satisfy (1.9). Also we have the following formula for the period of $q(s)$:

$$T = \int_0^1 \frac{\sqrt{\lambda}}{a(\gamma(\tau))}\,d\tau\,.$$

The problem with the metric (1.6) is that it degenerates for $x \in \partial\Omega$ so that the standard techniques of the Riemannian geometry cannot be applied without many troubles.

The main purpose of the next section is to prove the following theorem.

THEOREM 1.4. — *Let Ω be an open bounded set in R^n with boundary of class C^2 and let $a \in C^2(\overline{\Omega})$ be a nonnegative function. Then if*

$$(1.18)\qquad a(x) = 0 \quad \textit{if and only if} \quad x \in \partial\Omega$$

$$(1.19)\qquad \nabla a(x) \neq 0 \quad \textit{for} \quad x \in \partial\Omega$$

there exists at least one closed geodesic for the metric (1.6).

Clearly Theorem 1.1 is an immediate consequence of Theorems 1.3 and 1.4.

2. PROOF OF THEOREM 1.4

The geodesics are the critical points of the « length » functional

$$(2.1)\quad J(\gamma) = \int a(\gamma)\,|\dot{\gamma}|^2 dt \quad \gamma \in C^2(S^1, \overline{\Omega}) \quad \text{where} \quad S^1 = [0, 1]/\{0, 1\}\,.$$

However, since $a(x)$ degenerates for $x \to \partial\Omega$, it is difficult to study directly the functional (2.1) and an approximation scheme seems to make life easier.

Let $\chi \in C^\infty(R)$ be a function such that

$$\chi(t) = 0 \quad \text{for} \quad 0 \leqslant t \leqslant 1$$
$$\chi(t) = 2 \quad \text{for} \quad t \geqslant 2$$
$$\chi'(t) \geqslant 0$$

and for every $\varepsilon > 0$ we set

$$U_\varepsilon(x) = \chi\left(\frac{\varepsilon}{a(x)}\right)\frac{1}{a(x)} \tag{2.2}$$

and

$$J_\varepsilon(\gamma) = \int \{ 1/2 a(\gamma) \mid \dot{\gamma} \mid^2 - U_\varepsilon(\gamma) \} \, dt \qquad \gamma \in C^2(S^1, \Omega). \tag{2.3}$$

Clearly for every $\gamma \in C^2(S^1, \Omega)$, $J_\varepsilon(\gamma) \to J(\gamma)$ for $\varepsilon \to 0$. The critical points of J_ε satisfy the equation $dJ_\varepsilon(\gamma)[\delta\gamma] = 0$ i. e.

$$\int \left\{ a(\gamma_\varepsilon)\dot{\gamma}_\varepsilon \cdot \delta\dot{\gamma} + 1/2(\nabla a(\gamma_\varepsilon) \cdot \delta\gamma) \mid \dot{\gamma}_\varepsilon^2 \mid \right. \\ \left. + \left[\chi\left(\frac{\varepsilon}{a(\gamma_\varepsilon)}\right)\frac{1}{a(\gamma_\varepsilon)^2} + \varepsilon\chi'\left(\frac{\varepsilon}{a(\gamma_\varepsilon)}\right)\frac{1}{a(\gamma_\varepsilon)^3} \right] \nabla a(\gamma_\varepsilon) \cdot \delta\gamma \right\} dt = 0 \tag{2.4}$$

which gives the Euler-Lagrange equation for the functional (2.3)

$$\frac{d}{dt}[a(\gamma)\dot{\gamma}] = 1/2 \mid \dot{\gamma} \mid^2 \nabla a(\gamma) - \nabla U_\varepsilon(\gamma). \tag{2.5}$$

Of course equation (2.5) is an approximation of the geodesic equation (*iii*) of Definition 1.2. However equation (2.5) is easier to deal with. In fact we have the following result.

THEOREM 2.1. — *For every* $\varepsilon \in (0, \varepsilon_0)$ (*where* ε_0 *is small enough*) *there exists a function* $\gamma_\varepsilon \in C^2(S^1, R^n)$ *solution of* (2.5). *Moreover* γ_ε *can be chosen in such a way that the following estimate holds*

$$\alpha \leqslant J_\varepsilon(\gamma_\varepsilon) \leqslant \beta$$

where α *and* β *are constants which depend only on* Ω (*and not on* ε).

The proof of the above theorem is contained in [B], Theorem 1.1.

Our aim is to prove that $\{ \gamma_\varepsilon \}_{\varepsilon > 0}$ has a subsequence converging to a closed geodesic for our Jacobi metric. To carry out this program some estimates are necessary. We set

$$\begin{cases} S_\varepsilon = J_\varepsilon(\gamma_\varepsilon) = \displaystyle\int \{ 1/2 a(\gamma_\varepsilon) \mid \dot{\gamma}_\varepsilon \mid^2 - U_\varepsilon(\gamma_\varepsilon) \} \, dt \\ L_\varepsilon = \displaystyle\int 1/2 \, a(\gamma_\varepsilon) \mid \dot{\gamma}_\varepsilon \mid^2 dt \end{cases}$$

The interpretation of S_ε and L_ε are obvious: L_ε is the square of the length of the curve γ_ε in the Jacobi metric; S_ε can be regarded as the action functional of the trajectory γ_ε with respect to the Lagrangian function

$$L_\varepsilon(x, \xi) = 1/2\, a(x) \,|\, \xi \,|^2 - U_\varepsilon(x) \qquad (x \in \Omega,\ \xi \in T\Omega).$$

Notice that $L_\varepsilon(x, \xi)$ is not the Lagrangian function associated with the Hamiltonian (1.1).

LEMMA 2.2. — *There exists a sequence* $\varepsilon_k \to 0$ *and a constant* $L_0 > 0$ *such that*

a) $\qquad S_{\varepsilon_k} \to L_0 \quad$ *for* $\quad k \to +\infty$

b) $\qquad L_{\varepsilon_k} \to L_0 \quad$ *for* $\quad k \to +\infty$.

Proof. — By theorem 2.1 we have

$$\alpha \leqslant S_\varepsilon \leqslant \beta .$$

Then (*a*) follows straightforward.

By (1.18) and (1.19), for every $M > 0$ there exists $\varepsilon > 0$ such that

$$\frac{|\,\nabla a(x)\,|^2}{a(x)^2} \geqslant M \frac{1}{a(x)} \qquad \text{for every } x \text{ such that} \quad a(x) < \varepsilon/2 .$$

By the above inequality we get

$$\chi\left(\frac{\varepsilon}{a(x)}\right)\frac{|\,\nabla a(x)\,|^2}{a(x)^2} \geqslant M\chi\left(\frac{\varepsilon}{a(x)}\right)\frac{1}{a(x)} = MU_\varepsilon(x) . \tag{2.6}$$

So we can select sequences $\varepsilon_k \to 0$ and $M_k \to +\infty$ such that

$$\begin{cases} S_{\varepsilon k} \to L_0 \\ U_{\varepsilon k}(x) \leqslant \dfrac{1}{M_k}\chi\left(\dfrac{\varepsilon_k}{a(x)}\right)\dfrac{|\,\nabla a(x)\,|^2}{a(x)^2} \qquad \text{for every} \quad x \in \Omega . \end{cases} \tag{2.7}$$

By the equation (2.4) with $\delta\gamma(t) = \nabla a(\gamma_\varepsilon(t))$ we get

$$\int \Big\{ a(\gamma_\varepsilon) d^2 a(\gamma_\varepsilon)[\dot\gamma_\varepsilon]^2 + 1/2\,|\,\dot\gamma_\varepsilon\,|^2\,|\,\nabla a(\gamma_\varepsilon)\,|^2 + \chi\left(\frac{\varepsilon}{a(\gamma_\varepsilon)}\right)\frac{|\,\nabla a(\gamma_\varepsilon)\,|^2}{a(\gamma_\varepsilon)^2} + \varepsilon\chi'\left(\frac{\varepsilon}{a(\gamma_\varepsilon)}\right)\frac{|\,\nabla a(\gamma_\varepsilon)\,|^2}{a(\gamma_\varepsilon)^3} \Big\} dt = 0 \tag{2.8}$$

where $d^2 a(x)[\xi^2]$ denotes the second differential of $a(\cdot)$. Since the second and the fourth term in the above integral are nonnegative, we get the following inequality

$$\int \chi\left(\frac{\varepsilon}{a(\gamma_\varepsilon)}\right)\frac{|\,\nabla a(\gamma_{\varepsilon k})\,|^2}{a(\gamma_\varepsilon)^2} dt \leqslant \int a(\gamma_\varepsilon) d^2 a(\gamma_\varepsilon)[\dot\gamma_\varepsilon]^2 dt \leqslant \|\, d^2 a \,\| \int a(\gamma_\varepsilon)\,|\,\dot\gamma_\varepsilon\,|^2 dt \tag{2.9}$$

where we have set

$$\| d^2 a \| = \max \{ d^2 a(x)[\xi]^2 \mid x \in \overline{\Omega},\ \xi \in \mathrm{T}\Omega,\ |\xi| = 1 \}$$

So we have

$$\begin{aligned} \mathrm{L}_{\varepsilon_k} &= 1/2 \int a(\gamma_{\varepsilon_k}) \,|\, \dot{\gamma}_{\varepsilon_k} |^2 dt = \mathrm{S}_{\varepsilon_k} + \int \mathrm{U}_{\varepsilon_k}(\gamma_{\varepsilon_k}) dt \\ &\leqslant \mathrm{S}_{\varepsilon_k} + \frac{1}{\mathrm{M}_k} \int \chi_{\varepsilon_k}\left(\frac{\varepsilon_k}{a(\gamma_{\varepsilon_k})}\right) \frac{|\nabla a(\gamma_{\varepsilon_k})|^2}{a(\gamma_{\varepsilon_k})^2} dt \qquad \text{[by (2.7)]} \\ &\leqslant \mathrm{S}_{\varepsilon_k} + \frac{\| d^2 a \|}{\mathrm{M}_k} \int a(\gamma_{\varepsilon_k}) \,|\, \dot{\gamma}_{\varepsilon_k} |^2 dt \qquad \text{[by (2.9)]} \\ &= \mathrm{S}_{\varepsilon_k} + \frac{\| d^2 a \|}{\mathrm{M}_k} \mathrm{L}_{\varepsilon_k}. \end{aligned}$$

Thus by the above formula and the definition of S_ε and L_ε we get

$$\left(1 - \frac{\| d^2 a \|}{\mathrm{M}_k}\right) \mathrm{L}_{\varepsilon_k} \leqslant \mathrm{S}_{\varepsilon_k} \leqslant \mathrm{L}_{\varepsilon_k}.$$

Thus, since $\mathrm{M}_k \to +\infty$ and $\mathrm{S}_{\varepsilon_k} \to \mathrm{L}_0$ for $k \to +\infty$, the second assertion of the lemma follows.

COROLLARY 2.3. — If we set

$$\mathrm{E}_\varepsilon = \int_0^1 1/2 a(\gamma_\varepsilon) \,|\, \dot{\gamma}_\varepsilon |^2 + \mathrm{U}_\varepsilon(\gamma_\varepsilon) dt$$

we have

a) $1/2 a(\gamma_\varepsilon(t)) \,|\, \dot{\gamma}_\varepsilon(t) |^2 + \mathrm{U}_\varepsilon(\gamma_\varepsilon(t)) = \mathrm{E}_\varepsilon$ for every $t \in [0, 1]$

b) $\mathrm{E}_{\varepsilon_k} \to \mathrm{L}_0$ for $k \to +\infty$.

Proof. — *a)* by direct computation, it is easy to see that the left hand side of equation (*a*) is an integral of equation (2.5). More exactly if J_ε is interpreted as the Hamilton functional for the Lagrangian $\mathrm{L}_\varepsilon(x, \xi)$, then E_ε can be interpreted as the energy.

b) follows by the fact that we have the identity

$$\mathrm{E}_\varepsilon = 2\mathrm{L}_\varepsilon - \mathrm{S}_\varepsilon$$

and by lemma 2.2. ■

Now let $\mathrm{H}^1(\mathrm{S}^1)$ denote the Sobolev space obtained as the closure of $\mathrm{C}^\infty(\mathrm{S}^1, \mathrm{R}^n)$ with respect to the norm

$$\| \gamma \|_{\mathrm{H}^1} = \left[\int_0^1 |\, \dot{\gamma} |^2 + |\, \gamma |^2 dt \right]^{1/2}.$$

LEMMA 2.4. — There exists a sequence $\varepsilon_k \to 0$ and $\gamma \in \mathrm{H}^1(\mathrm{S}^1)$ such that $\gamma_{\varepsilon_k} \to \gamma$ weakly in $\mathrm{H}^1(\mathrm{S}^1)$.

Proof. — Consider the equality (2.8). Since the third and the fourth terms are nonnegative we get

$$(2.10)\quad 1/2\int |\dot\gamma_\varepsilon|\,|\nabla a(\gamma_\varepsilon)|^2 dt \leqslant -\int a(\gamma_\varepsilon) d^2a(\gamma_\varepsilon)[\dot\gamma_\varepsilon^2] \leqslant \|d^2a\| \int a(\gamma_\varepsilon)|\dot\gamma_\varepsilon|^2 dt$$

By (1.18), (1.19) and the compactness of $\overline{\Omega}$, there exists constants ν, $M > 0$ such that

$$(2.11)\qquad \mathrm{M}a(x) + |\nabla a(x)|^2 \geqslant \nu \qquad \text{for every} \quad x\in\Omega\,.$$

By Corollary 2.3 (*b*), we have that

$$(2.12)\qquad 1/2\int a(\gamma_{\varepsilon_k})\,|\dot\gamma_{\varepsilon_k}|^2 \leqslant \mathrm{L}_0 + 1 \qquad \text{for } k \text{ large enough}.$$

So by the above formula and (2.10) we get

$$1/2\int |\nabla a(\gamma_{\varepsilon_k})|^2\,|\dot\gamma_{\varepsilon_k}|^2 \leqslant (2\mathrm{L}_0 + 2)\,\|d^2a\| \qquad \text{for } k \text{ large}.$$

Adding the above formula with M times (2.12) we get

$$1/2\int\{\mathrm{M}a(\gamma_{\varepsilon_k})+|\nabla a(\gamma_{\varepsilon_k})|^2\}\,|\dot\gamma_\varepsilon|^2 \leqslant \mathrm{C} \quad \text{with} \quad \mathrm{C}=(2\mathrm{L}_0+2)\|da\|+(\mathrm{L}_0+1)\mathrm{M}.$$

Now, using (2.11) and the above formula we get

$$\frac{\nu}{2}\int|\dot\gamma_{\varepsilon_k}|^2 \leqslant \mathrm{C}_2\,.$$

The above inequality, and the fact that Ω is bounded imply that $\|\gamma_{k_\varepsilon}\|_{\mathrm{H}^1}$ is bounded. Then the conclusion follows, may be taking a new subsequence of ε_k's. ■

Finally we can prove Theorem 1.4.

Proof of Theorem 1.4. — By lemma 2.4 we have that

$$(2.13)\qquad \gamma_{\varepsilon_k} \to \gamma \quad \text{weakly in } \mathrm{H}^1(\mathrm{S}^1) \quad \text{and uniformly}.$$

We want to prove that there exists $t_0 \in \mathrm{S}^1$ and $d > 0$

$$(2.14)\qquad \operatorname{dist}(\gamma_{\varepsilon_k}(t_0), \partial\Omega) \geqslant d > 0 \quad \text{for every } k\,.$$

We argue indirectly and suppose that for every $t\in\mathrm{S}^1$ there exists a sequence $d_k \to 0$ such that

$$\operatorname{dist}(\gamma_{\varepsilon_k}(t), \partial\Omega) \leqslant d_k\,.$$

Then we have

$$\mathrm{L}_{\varepsilon_k} = \int 1/2\, a(\gamma_{\varepsilon_k})\,|\dot\gamma_{\varepsilon_k}|^2 dt \leqslant \max\{a(x)\mid \operatorname{dist}(x,\partial\Omega)\leqslant d_k\}\cdot\|\gamma_k\|^2_{\mathrm{H}^1}\,.$$

Thus by lemma 2.4 and (1.18), $L_{\varepsilon_k} \to 0$. But this fact contradicts lemma 2.2. So (2.14) holds. Therefore the set $\{ t \mid \gamma(t) \in \Omega \}$ is not empty. Let Δ be one of its connected components.

Now let $\phi \in C_0^\infty(\Delta, R^n)$ (i. e. a smooth function with support contained in Δ). By equation (2.4) with $\delta\gamma = \phi$ we get

$$(2.15) \quad \int_\Delta a(\gamma_{\varepsilon_k})\dot{\gamma}_{\varepsilon_k}\phi + 1/2 \mid \dot{\gamma}_{\varepsilon_k} \mid^2 (\nabla a(\gamma_{\varepsilon_k}) \cdot \phi) - \{ \nabla U_{\varepsilon_k}(\gamma_{\varepsilon_k})\phi \} \, dt = 0 .$$

Since $\gamma_{\varepsilon_k} \to \gamma$ uniformly then

$$(2.16) \quad \left\{ \begin{array}{l} a(\gamma_k) \to a(\gamma) \\ \nabla a(\gamma_k) \to \nabla a(\gamma) \end{array} \right. \quad \text{uniformly} .$$

Moreover by (2.2)

$$(2.17) \quad \nabla U_{\varepsilon_k}(\gamma_k(t)) = 0 \quad \text{for } k \text{ large enough and} \quad t \in \operatorname{supp} \phi .$$

Now (2.13), (2.16) and (2.17) allow us to take the limit in (2.15) and we get

$$\int_\Delta a(\gamma)\dot{\gamma}\phi + 1/2 \mid \dot{\gamma} \mid^2 (\nabla a(\gamma), \phi) dt = 0 \quad \text{for every} \quad \phi \in C_0^\infty(\Delta, R^n) .$$

Therefore γ satisfy equation (*iii*) of Definition 1.2 for every $t \in \Delta$. Thus if $\Delta = S^1$ we have obtained an interior geodesic and we are finished. If $\Delta \neq S^1$, we consider the affine transformation

$$\tau : \Delta \to (0, 1/2) .$$

Since equation (*iii*) is invariant for affine transformation, the function

$$\tilde{\gamma}(t) = \left\{ \begin{array}{ll} \gamma(\tau^{-1}(t)) & \text{if} \quad t \in (0, 1/2) \\ \gamma(\tau^{-1}(1 - t)) & \text{if} \quad t \in (1/2, 1) \end{array} \right.$$

provides a brake geodesic according to Definition 1.2. ■

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(Manuscrit reçu le 25 mai 1984)