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Asymptotic behaviour of minimal graphs over exterior domains

by

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ABSTRACT. — It is proved that if u is a $C^2(\mathbb{R}^n \sim \Omega)$ solution of the minimal surface equation, if Ω is bounded, and if $n \leq 7$, then $Du(x)$ has a limit (in \mathbb{R}^n) as $|x| \rightarrow \infty$. This extends a result of L. Bers for the case $n = 2$. The result here is actually obtained as a special application of a more general result valid for all n .

Key-words: Minimal surface, Tangent cone at ∞ .

RÉSUMÉ. — On démontre que si u est une solution $C^2(\mathbb{R}^n \sim \Omega)$ de l'équation de la surface minimale, si Ω est borné $n \leq 7$, alors $Du(x)$ a une limite (dans \mathbb{R}^n) telle que $|x| \rightarrow \infty$. Ceci étend un résultat de L. Bers dans le cas $n = 2$. Notre résultat est en fait un corollaire d'un résultat plus général, valable quel que soit n .

A well known result of L. Bers [BL] says that if u is a C^2 solution of the minimal surface equation over $\mathbb{R}^2 \sim \Omega$, where Ω is a bounded open subset of \mathbb{R}^2 , then $Du(x)$ has a limit $a \in \mathbb{R}^2$ as $|x| \rightarrow \infty$. A more geometric proof, valid for a solution u of any equation of minimal surface type, was given in [SL5].

Here we want to show that Bers' original result is also valid in dimension n , $3 \leq n \leq 7$; specifically, we shall prove

THEOREM 1. — *If u is a C^2 solution of the minimal surface equation over $\mathbb{R}^n \sim \Omega$, Ω bounded open in \mathbb{R}^n , $3 \leq n \leq 7$, then $Du(x)$ is bounded and has a limit as $|x| \rightarrow \infty$.*

(Since $n \geq 3$ it in fact follows from this that there is a constant b such that

$$\lim_{\rho \uparrow \infty} |u - l|_{C^2(\mathbb{R}^n \sim B_\rho)} = 0,$$

where $l(x) = a \cdot x + b$, $a = \lim_{|x| \rightarrow \infty} Du(x)$, and $B_\rho = \{x \in \mathbb{R}^n : |x| < \rho\}$.)

Of course in case $\Omega = \phi$, the fact that Du is bounded implies that it is constant (so that u is linear + constant), because each partial derivative $D_i u$ satisfies a uniformly elliptic divergence-form equation. (Cf. [MJ] [BDM].) Thus Theorem 1 may be viewed as an extension of this « Bernstein » result (for $\Omega = \phi$, $3 \leq n \leq 7$), which was due originally to Bernstein, Fleming, De Giorgi, Almgren, and J. Simons (see [SJ]).

We actually here derive Theorem 1 as a special consequence of a more general result, valid in all dimensions $n \geq 3$. Specifically we shall prove (in § 2, 3 below):

THEOREM 2. — *If u is a C^2 solution of the minimal surface equation on $\mathbb{R}^n \sim \Omega$, Ω bounded, then either $Du(x)$ is bounded and has a limit as $|x| \rightarrow \infty$ or else all tangent cones of graph u at ∞ are cylinders of the form $C \times \mathbb{R}$, where C is an $(n - 1)$ -dimensional minimizing cone in \mathbb{R}^n with $\partial C = 0$ and with $0 \in \text{sing } C$. (In particular $\text{spt } C$ is not a hyperplane in this latter case.)*

For the meaning of « tangent cone at ∞ », we refer to § 1 below. Of course here $\text{spt } C$ is the support of C and $\text{sing } C$ (the singular set of C) is the set of points $\xi \in \text{spt } C$ such that $\text{spt } C \cap B_\sigma(\xi)$ fails to be an embedded C^2 submanifold for each $\sigma > 0$. It will also be shown in § 2, 3 that C has the form $C = \partial[V]$, with V an open conical domain in \mathbb{R}^n . (That is, V is open in \mathbb{R}^n and $V = \{\lambda y : y \in V\}$ for each $\lambda > 0$.)

Notice that Theorem 1 follows immediately from Theorem 2 because there are no $(n - 1)$ -dimensional minimizing cones C in \mathbb{R}^n with $\partial C = 0$ and $0 \in \text{sing } C$ for $3 \leq n \leq 7$. (Indeed the regularity theory for minimizing currents guarantees that $\text{sing } T = \phi$ whenever T is an $(n - 1)$ -dimensional mass minimizing current with $\partial T = 0$ and $n \leq 7$; see e. g. [FH, 5.3.18] or [SL1, § 37].)

§ 1. PRELIMINARIES, TANGENT CONES AT ∞

In this section $n \geq 3$ is arbitrary and throughout we assume that u is a $C^2(\mathbb{R}^n \sim \Omega)$ solution of the minimal surface equation

$$(*) \quad \sum_{i,j=1}^n \left(\delta_{ij} - (1 + |\text{grad } u|^2)^{-1} (D_i u)(D_j u) \right) D_i D_j u = 0,$$

with Ω a bounded open subset of \mathbb{R}^n .

As a preliminary result, we establish the following lemma.

1.1. LEMMA. — *Either $|Du|$ is bounded on $\mathbb{R}^n \sim \Omega$ or else*

$$\lim_{j \rightarrow \infty} (\rho_j^{-1} \sup_{B_{\rho_j} \sim \Omega} |u|) = \infty$$

for each sequence $\{\rho_j\} \uparrow \infty$.

(Here, and subsequently, B_ρ is the open ball of radius ρ and centre 0 in \mathbb{R}^n .)

Proof. — Suppose there is a sequence $\{\rho_j\} \uparrow \infty$ with

$$\sup_{j \geq 1} (\rho_j^{-1} \sup_{B_{\rho_j} \sim \Omega} |u|) < \infty.$$

By the standard gradient estimates for solutions of the minimal surface equation (the version of [SL6; Theorem 1] is particularly convenient here, because $\sup_{\partial\Omega} |Du| < \infty$ by the assumption that u is C^2 on $\mathbb{R}^n \sim \Omega$), we have

$$\sup_{j \geq 1} \sup_{B_{\rho_j/2} \sim \Omega} |Du| < \infty;$$

that is, $\sup |Du| < \infty$ as required. ■

Next we note that (since (*) asserts exactly that $G = \text{graph } u$ has zero mean curvature) we have the formula (see [SL2] or [MS] or [AW] for discussion)

$$1.2 \quad \int_G \sum_{i=1}^{n+1} \nabla_i \phi^i d\mathcal{H}^n = 0,$$

where $\nabla_i = e_i \cdot \nabla$, $\nabla = \text{gradient operator on } G$, $\phi^1, \dots, \phi^{n+1} \in C_c^1(\mathbb{R}^{n+1} \sim \bar{\Omega} \times \mathbb{R})$. Notice that if ν is the upward unit normal for G and if f is C^1 in some neighbourhood of G , then

$$1.3 \quad \nabla_i f(x) = \sum_{j=1}^{n+1} (\delta_{ij} - \nu_i(x)\nu_j(x)) D_j f(x), \quad x \in G,$$

where $D_j f = \partial f / \partial x^j$ are the usual partial derivatives of f taken in \mathbb{R}^{n+1} .

We also have the standard fact (see e.g. [SL2, § 3]) that

$$1.4 \quad G \text{ is mass minimizing in } \mathbb{R}^{n+1} \sim (\bar{\Omega} \times \mathbb{R}),$$

in the sense that if we equip G with a smooth orientation, so that it becomes a multiplicity 1 current, then

$$1.5 \quad \underline{\underline{M}}(G \llcorner W) \leq \underline{\underline{M}}(T \llcorner W)$$

for any open $W \subset \subset \mathbb{R}^{n+1} \sim (\bar{\Omega} \times \mathbb{R})$ and for any integer multiplicity locally rectifiable current T in \mathbb{R}^{n+1} with $(\partial T) \llcorner W = 0$ and $\text{spt}(T - G) \subset \subset W$.

Next we recall that (from 1.2—see e.g. [GT, Ch. 16] and note

that the arguments easily modify to take account of the fact that we need $\text{spt } \phi^i \cap (\bar{\Omega} \times \mathbb{R}) = \emptyset$ in 1.2) there are the volume bounds.

$$1.6 \quad \mathcal{H}^n(G \cap B_\rho(y)) \leq c\rho^n, \quad 1 \leq \rho < \infty,$$

for suitable constant c , where $B_\rho(y)$ is the ball of radius ρ and centre y in \mathbb{R}^{n+1} (ρ, y arbitrary).

Recall also that one of the versions of the monotonicity formula can be written

$$1.7 \quad n \mathcal{H}^n(G \cap B_\rho \sim B_\sigma) = \rho b_\rho - \sigma b_\sigma, \quad R_0 \leq \sigma < \rho < \infty,$$

where R_0 is large enough to ensure $\partial G \equiv \text{graph}(u | \partial\Omega) \subset B_{R_0}$ (all balls have centre 0 unless explicitly indicated otherwise), and where

$$b_\rho = \frac{d}{d\rho} \int_{G \cap B_\rho} |\nabla r|^2 d\mathcal{H}^n = \int_{G \cap \partial B_\rho} |\nabla r| d\mathcal{H}^{n-1},$$

with $r(x) \equiv |x|$. (The last equality follows from the co-area formula.) The identity 1.7 follows from 1.2 simply by substituting $\phi^i(x) = \psi(r)x^i$ in 1.2, and then letting ψ approach the characteristic function of the interval (σ, ρ) . Notice that 1.7 (with $\sigma = R_0$) can be written (since $|\nabla r|^2 = 1 - (x \cdot v)^2/r^2$)

$$\frac{d}{d\rho} (\rho^{-n} \mathcal{H}^n(G \cap B_\rho \sim B_{R_0})) = \frac{d}{d\rho} \int_{G \cap B_\rho} \frac{(x \cdot v)^2}{r^{n+2}} d\mathcal{H}^n - \rho^{-n-1} R_0 b_{R_0}$$

(in the sense of distributions) for $\rho > R_0$, so that by integration we have, for $\rho > R \geq R_0$,

$$1.8 \quad \rho^{-n} \mathcal{H}^n(G \cap B_\rho \sim B_{R_0}) - R^{-n} \mathcal{H}^n(G \cap B_R \sim B_{R_0}) = \int_{G \cap B_\rho \sim B_R} \frac{(x \cdot v)^2}{r^{n+2}} d\mathcal{H}^n + E(\rho, R),$$

where

$$|E(\rho, R)| \leq cR^{-n}.$$

c independent of ρ, R . (Cf. the standard monotonicity identities of [AW], [MS], [SL1].)

Now for $\lambda > 0$ we let u_λ be the scaled function $u_\lambda(x) = \lambda u(\lambda^{-1}x)$, $x \in \mathbb{R}^n \sim \Omega_\lambda$, $\Omega_\lambda = \{\lambda y : y \in \Omega\}$, and let G_λ be the graph of u_λ , so that viewing G_λ (equipped with an appropriate orientation) as a current, we may write

$$G_\lambda = \partial \llbracket U_\lambda \rrbracket \llcorner (\mathbb{R}^{n+1} \sim (\Omega_\lambda \times \mathbb{R})),$$

where $U_\lambda = \{(x, y) : y > u_\lambda(x), x \in \mathbb{R}^n \sim \Omega_\lambda\}$, and where $\llbracket U_\lambda \rrbracket$ denotes the current obtained by integration of $(n+1)$ -forms $\omega \in \mathcal{D}^{n+1}(\mathbb{R}^{n+1})$ over U_λ .

By virtue of 1.5, 1.6 we can conclude from standard compactness

results (see e. g. [FH] or [SL1, Ch. 7]) that for any sequence $\{\lambda_j\} \downarrow 0$ there is a subsequence $\{\lambda_{j'}\}$ and a current $T = \partial[U]$ such that

$$1.9 \quad T = \lim G_{\lambda_{j'}}$$

in the weak sense of currents, and in the sense that $U_{\lambda_{j'}}$ converges to U in the $L^1_{loc}(\mathbb{R}^{n+1})$ sense,

$$1.10 \quad \mathcal{H}^n \llcorner G_{\lambda_{j'}} \rightarrow \mathcal{H}^n \llcorner S$$

in the sense of Radon measures on \mathbb{R}^{n+1} , where

$$S = \{x \in \mathbb{R}^{n+1} : \limsup_{\rho \downarrow 0} \rho^{-n} \underline{M}(T \llcorner B_\rho(x)) > 0\},$$

$$1.11 \quad G_{\lambda_{j'}} \rightarrow \text{spt } T$$

locally in the Hausdorff distance sense in $\mathbb{R}^{n+1} \sim (\{0\} \times \mathbb{R})$,

$$1.12 \quad T \text{ is minimizing in } \mathbb{R}^{n+1} \sim (\{0\} \times \mathbb{R}),$$

$$1.13 \quad \underline{M}(T \llcorner B_\rho(y)) \leq c\rho^n, \quad \rho > 0, \quad y \in \mathbb{R}^{n+1}$$

(notice that this includes $y \in \{0\} \times \mathbb{R}$).

Since $n \geq 3$ it is easy to check that 1.12 and 1.13 imply

$$1.14 \quad T \text{ is minimizing in } \mathbb{R}^{n+1}.$$

It is also standard that then (since $T = \partial[U]$) implies that T has multiplicity 1 \mathcal{H}^n -a. e. in S (S as in 1.10), and since the density function of a minimizing current is upper semi-continuous—see e. g. [FH] or [SL1, Ch. 7]) $\liminf_{\rho \downarrow 0} \omega_n^{-1} \rho^{-n} \underline{M}(T \llcorner B_\rho(y)) \geq 1$ at each point of $\text{spt } T$, and hence

$$1.15 \quad S = \text{spt } T \quad (S \text{ as in 1.10})$$

and we can (and shall) take U to be open with

$$1.16 \quad \text{spt } T = \partial U.$$

From the De Giorgi regularity theorem (see e. g. [SL1, § 24] or [G]) we have furthermore that for each $y \in \text{spt } T$ with $\lim_{\rho \downarrow 0} \omega_n^{-1} \rho^{-n} \underline{M}(T \llcorner B_\rho(y)) = 1$ there is $\sigma > 0$ such that

$$1.17 \quad \text{spt } T \cap B_\sigma(y) (= \partial U \cap B_\sigma(y)) \text{ is an embedded } C^\infty \text{ submanifold of } \mathbb{R}^{n+1}.$$

This guarantees in particular that the points of $\text{sing } T$ (i. e. the points $y \in \text{spt } T$ such that 1.17 fails for each $\sigma > 0$) form a closed set of \mathcal{H}^n -measure zero.

Finally we note that T is a cone; that is, if η is any homothety $x \mapsto \lambda x$ ($\lambda > 0$ fixed), then $\eta_\# T = T$. Indeed using 1.8, 1.10, 1.15 it is easy to see that

$$\rho^{-n} \mu(B_\rho) = \sigma^{-n} \mu(B_\sigma), \quad 0 < \sigma < \rho < \infty,$$

where $\mu = \mathcal{H}^n \llcorner \text{spt } T$, and then (since T is minimizing) that an identity like 1.8 holds for T with $R_0 = 0$ and $E \equiv 0$, thus giving

$$\int_{\text{spt } T} (v \cdot x)^2 d\mu(x) = 0,$$

where v is the unit normal of ∂U (which is well defined on $\text{reg } T \equiv \text{spt } T \sim \text{sing } T$). The fact that $\eta \# T = T$ for any homothety η now readily follows from this and the homotopy formula for currents. (See for example [SL1, Ch. 7] or [G] for similar arguments.)

Subsequently, any T obtained as described above will be called a *tangent cone for graph u at infinity*. In case $|Du|$ is bounded we can prove that there is a unique such T and it is a hyperplane. In fact we have the following result:

1.18. LEMMA. — *If Du is bounded (see Lemma 1.1) then it has a limit at infinity.*

Proof. — Since $|Du|$ is bounded, every tangent cone T of graph u at ∞ (obtained as above) is the graph of a Lipschitz weak solution of the minimal surface equation. From standard elliptic regularity theory, such solutions are smooth (see e. g. [GT, Ch. 13]). Hence since the graphs of these solutions are cones, they must all be linear functions.

It follows that for any given $\varepsilon > 0$ there is $R(\varepsilon) \geq 1$ such that if $R \geq R(\varepsilon)$ then there is a linear function l (possibly depending on R) such that

$$\sup_{B_{2R} \setminus B_R} |u - l| \leq \varepsilon R.$$

Combining this with the Schauder theory [(GT, Ch. 6)], applied to $u - l$ (which we may do since $u - l$ satisfies a linear elliptic equation with coefficients having finite C^1 norm—this follows from the fact that the C^2 norm of u is finite), we deduce

$$\sup_{B_R \setminus B_{R/2}} |D^2 u| = \sup_{B_R \setminus B_{R/2}} |D^2(u - l)| \leq c\varepsilon/R.$$

Hence, by integration along paths in $B_R \setminus B_{R/2}$, we have

$$(1) \quad |Du(x) - Du(y)| \leq c\varepsilon \quad \forall x, y \in B_R \setminus B_{R/2}.$$

On the other hand each component $\phi = D_j u$ of Du satisfies an equation of the form

$$(2) \quad D_i(a_{ik} D_k \phi) = 0,$$

where

$$a_{ik} = (1 + |Du|^2)^{-\frac{1}{2}}(\delta_{ik} - (1 + |Du|^2)^{-1}(D_i u)(D_k u))$$

(as we see by differentiating the divergence form version

$$\sum_{i=1}^n D_i(D_i u / \sqrt{1 + |Du|^2}) = 0$$

of the minimal surface equation). In particular $D_j u$ satisfies a maximum/minimum principle on bounded domains in $\mathbb{R}^n \sim \bar{\Omega}$; then in view of the arbitrariness of ε in (1), it follows that $\lim_{j \rightarrow \infty} D_j u$ exists. ■

§ 2. TANGENT CYLINDERS AT ∞

In this section we show that, unless Du is bounded as $|x| \rightarrow \infty$, every tangent cone T of graph u at ∞ (obtained as described in § 1) is a vertical cylinder:

$$2.1 \quad T = C \times \mathbb{R},$$

where $C = \partial[V]$, V open in \mathbb{R}^n with $\partial V = \text{spt } C$, and where C is minimizing in \mathbb{R}^n .

In case $\Omega = \emptyset$ this was already known (a proof appears in [MM] for example). The extension here to case $\Omega \neq \emptyset$ is given mainly for the reader's convenience, since no really new ideas are involved. Note however that the fact (observed by Fleming in case $\Omega = \emptyset$) that $\text{sing } C \neq \emptyset$ is not so easy to prove in case $\Omega \neq \emptyset$; this will be done in § 3.

We first note that (in the notation of § 1) by 1.9 and 1.15

$$2.2 \quad \lim_{j \rightarrow \infty} \int_{G_{\lambda_j}} v_j \cdot \phi d \mathcal{H}^n = \int_{\text{spt } T} v \cdot \phi d \mathcal{H}^n$$

for each fixed $\phi = (\phi^1, \dots, \phi^{n+1}) \in C_c^\infty(\mathbb{R}^{n+1}; \mathbb{R}^{n+1})$, where v_j is the upward pointing unit normal for G_{λ_j} and where v is the outward pointing normal of ∂U at regular points of $\partial U = \text{spt } T$. (U as in 1.16.) Thus in particular (since $v_j \cdot e_{n+1} > 0$ on G_{λ_j}) we have

$$2.3 \quad v \cdot e_{n+1} \geq 0 \quad \text{on } \text{reg } T.$$

We already remarked in § 1 that $\mathcal{H}^n(\text{sing } T) = 0$. We also need to recall the further regularity theory for minimizing currents $T = \partial[U]$:

$$2.4 \quad \begin{cases} \text{sing } T = \emptyset, & 3 \leq n \leq 6 \\ \text{sing } T \text{ is discrete,} & n = 7 \\ \mathcal{H}^{n-7+\alpha}(\text{sing } T) = 0 & \forall \alpha > 0 \text{ in case } n \geq 8 \end{cases}$$

(See e. g. [G] or [SL1, Ch. 7]), so that, since T is a cone, in particular

$$2.5 \quad \begin{cases} T \text{ is a hyperplane,} & n \leq 6 \\ \text{sing } T \subset \{0\}, & n = 7. \end{cases}$$

Notice that in case $n = 7$ we then also trivially have that $\text{reg } T$ is connected; otherwise $\text{reg } T \cap S^n$ would contain smooth compact *disjoint* embedded minimal surfaces Σ_1, Σ_2 , and we could rotate Σ_1 until it touched Σ_2 , thus contradicting the Hopf maximum principle. (Actually $\text{reg } T$ is connected for *all* n , by a result Bombieri and Giusti [BG].)

Next we claim, under the present assumption that $|Du|$ is not bounded, that

$$2.6 \quad L \subset \text{spt } T,$$

where L is a vertical ray from 0: either $L = \{ \lambda e_{n+1} : \lambda > 0 \}$ or $\{ \lambda e_{n+1} : \lambda < 0 \}$. To see this we note that if we let $u_j = u_{\lambda_j}$, then for each fixed $\sigma > 0$,

$$2.7 \quad \lim_{j \rightarrow \infty} \sup_{|x|=\sigma} |u_j| = \infty,$$

because otherwise Lemma 1.1 tells us that $|Du|$ is bounded on $\mathbb{R}^n \sim \Omega$, contrary to hypothesis. Thus 2.7 is established, and 2.6 clearly follows from this due to 1.11 and the fact that $\lim_{j \rightarrow \infty} \inf_{|x|>j} |u| < \infty$ by a standard barrier argument involving the catenoid.

Now we use the standard fact that $\Delta v \cdot e_{n+1} + |A|^2 v \cdot e_{n+1} = 0$ on $\text{reg } T$, where A is the second fundamental form of $\text{reg } T$, so that by 2.3

$$2.8 \quad \Delta v \cdot e_{n+1} \leq 0 \quad \text{on } \text{reg } T.$$

In case $n \leq 7$ we can use connectedness of $\text{reg } T$, 2.3, 2.5, 2.8 and 2.6 (which guarantees that $v \cdot e_{n+1} = 0$ at some points of $\text{reg } T$) to deduce by the Hopf maximum principle that $v \cdot e_{n+1} \equiv 0$ on $\text{reg } T$. Hence, again using 2.5, we have 2.1 as required.

In case $n \geq 8$, the argument is only slightly more complicated: by [BG] and 2.8 we have

$$(*) \quad \inf_{\text{reg } T \cap B_{\rho/2}(y)} e_{n+1} \cdot v \geq \rho^{-n} \int_{\text{reg } T \cap B_{\rho}(y)} e_{n+1} \cdot v \, d\mathcal{H}^n$$

for any $\rho > 0$ and $y \in \text{spt } T$. However we showed above that $\text{spt } T$ contains a vertical $\frac{1}{2}$ -line, and evidently $\inf_{\text{reg } T \cap B_{\rho}(y)} e_{n+1} \cdot v = 0$ for any y in this $\frac{1}{2}$ -line and any $\rho > 0$, thus by (*) $e_{n+1} \cdot v \equiv 0$.

Thus we have established $v \cdot e_{n+1} \equiv 0$ on $\text{reg } T$. Since $\partial T = 0$ it then easily follows (e. g. by using the homotopy formula for currents), that T is invariant under translations parallel to e_{n+1} . Thus (with U as in 1.16)

$U = V \times \mathbb{R}, T = C \times \mathbb{R}, C = \partial[V]$ with V open in \mathbb{R}^{n+1} , V invariant under homotheties. Of course C is minimizing in \mathbb{R}^n , because T is minimizing in \mathbb{R}^{n+1} . This completes the proof of 2.1.

§ 3. PROOF OF THEOREM 2

In view of Lemmas 1.1, 1.17 and the fact 2.1, Theorem 2 of the introduction will be proved if we can establish that

3.1 $\text{sing } C \neq \emptyset$ (i. e. C is not a hyperplane)

for any C as in 2.1.

Suppose for contradiction that $C \times \mathbb{R}$, as in 2.1, is indeed a hyperplane H in \mathbb{R}^{n+1} . We first claim that in this case H is the unique tangent cone for graph u at ∞ and that in fact, if η is a unit normal for H , there exist $R_2 > R_1$ such that

3.2 $G \sim B_{R_2} = \{x + h(x)\eta : x \in H \sim B_{R_1}\} \sim B_{R_2}$,

with $h \in C^2(H \sim B_{R_1})$ satisfying

3.3 $|h(x)| + |x| |Dh(x)| \leq c|x|^{1-\alpha}, \quad x \in H \sim B_{R_1}$,

for some constant $\alpha > 0$. This is actually a special case of the general unique tangent cone result of [AA]. For a somewhat simpler proof, see [SL3, II, § 6].

Now suppose without loss of generality that $e_n = (0, 0, \dots, 0, 1, 0)$ is normal to H (so that we can take $\eta = e_n$ in 3.2), and introduce new coordinates (y^1, \dots, y^{n+1}) for \mathbb{R}^{n+1} according to the transformation Q given by

$$\begin{cases} y' = x' & (y' = (y^1, \dots, y^{n-1}), \quad x' = (x^1, \dots, x^{n-1})) \\ y^n = x^{n+1} \\ y^{n+1} = x^n. \end{cases}$$

Then for suitable compact K and suitable R we have

$$G \sim K = Q(\text{graph } h | \mathbb{R}^n \sim B_R),$$

so that we have a diffeomorphism $\psi: \mathbb{R}^n \sim \tilde{K} \rightarrow \mathbb{R}^n \sim B_R^n$,

$$x = (x^1, \dots, x^n) \mapsto y = (y^1, \dots, y^n),$$

defined by $y' = x', y^n = u(x', x^n)$, where $\tilde{K} = \mathbb{R}^n \sim \pi(G \sim K)$, π the projection taking $z \in \mathbb{R}^{n+1}$ onto its first n -coordinates. The inverse is given

by $x' = y', x^n = h(y', y^n)$, so in particular we have $\frac{\partial u(x)}{\partial x^n} \frac{\partial h(y)}{\partial y^n} \equiv 1$ for $x \in \mathbb{R}^n \sim \tilde{K}$,

$y = \psi(x)$. Assuming without loss of generality that $u(te_n) \rightarrow \infty$ (rather than $-\infty$) as $t \rightarrow \infty$, we thus deduce

$$3.4 \quad \frac{\partial h(y)}{\partial y^n} > 0, \quad y \in \mathbb{R}^n \sim B_R^n.$$

Similarly if $G^- = \text{graph}(-u)$, then we have, for suitable compact K_1 and $R > 0$, that

$$3.5 \quad G^- \sim K_1 = Q(\text{graph } h^-) \quad \text{where} \quad h^- \in C^2(H \sim B_R).$$

Notice that, for suitably large $\rho > R$ and any $c \in \mathbb{R}$,

$\{(y', h(y', c)) : |(y', c)| > \rho\}$ and $\{(y', h^-(y', c)) : |(y', c)| > \rho\}$ coincide with $\{x : |(x', c)| > \rho, u(x) = c\}$ and $\{x : |(x', c)| > \rho, u(x) = -c\}$ respectively, so that

$$3.6 \quad h(y', y^n) = h^-(y', -y^n), \quad |y| > \rho.$$

Writing

$$w(y) = h(y) - h^-(y), \quad |y| > \rho,$$

we see that (using 3.3, 3.6 and the fact that h, h^- satisfy the minimal surface equation)

$$3.7 \quad \Delta w = \text{div}(A \cdot Dw), \quad |y| > \rho,$$

where the matrix A is smooth and

$$3.8 \quad |A| + |DA| |y| \leq c |y|^{-\alpha}, \quad |y| > \rho.$$

Also by 3.4, 3.6 we have

$$3.9 \quad \partial w(y) / \partial y^n > 0, \quad |y| > \rho$$

and

$$3.10 \quad w(y', y^n) = -w(y', -y^n).$$

Now let $\{t_j\} \uparrow \infty$ be arbitrary, and define

$$w_j(y) = \frac{h(t_j y)}{h(t_j e_n)}.$$

Since $w(y) > 0$ for $y^n > 0$ (by 3.9, 3.10), in view of 3.7, 3.8, 3.9, 3.10 we can use Harnack's inequality and Schauder estimates in order to deduce that there is a subsequence $\{t_{j'}\}$ such that

$$w_{j'} \rightarrow w_* \quad \text{locally in } C^1 \text{ on } \mathbb{R}^n \sim \{0\},$$

where w_* is harmonic on $\mathbb{R}^n \sim \{0\}$, $\partial w_* / \partial y^n \geq 0$, and $w_*(y', y^n) = -w_*(y', -y^n)$, $y \neq 0$. Thus w_* is bounded on $B_1(0) \sim \{0\}$, and hence the singularity at 0 is removeable; that is, w_* extends to a harmonic function on \mathbb{R}^n . But then

$\partial w_*/\partial y^n$ extends to be a non-negative harmonic function on all of \mathbb{R}^n and hence must be constant by Liouville's theorem. Thus $w_*(y', y^n) \equiv cy^n$ for some constant c . Since $w_*(e_n) = 1$ (by construction) we then have $w_*(y', y^n) \equiv y^n$.

In view of the arbitrariness of the sequence $\{t_j\}$ in the above argument, it follows that for each given $\varepsilon > 0$ there is a $T = T(\varepsilon) \geq \rho$ such that

$$w(2te_n) \geq 2(1 - \varepsilon)w(te_n)$$

for each $t \geq T$. Taking $t = 2^j T(\varepsilon)$ for ε small, and iterating, we then deduce that for any given $\beta \in (0, 1)$ there is $c = c(\beta)$ such that

$$w(te_n) \geq ct^{1-\beta}, \quad t \geq T.$$

However, taking $\beta < \alpha$ (α as in 3.3), this contradicts 3.3. This completes the proof of Theorem 2.

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