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A description of self-similar Blow-up for dimensions $n \geqslant 3$

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ABSTRACT. — A precise description of the asymptotic behavior near the blowup singularity for solutions of $u_t - \Delta u = f(u)$ which blowups in finite time T is given.

Key words : Blowup, self similar, nonlinear parabolic equation, thermal runaway.

RÉSUMÉ. — On établie une description précise de la conduite asymptotique autour de la singularité de l'explosion totale pour la solution de l'équation $u_t - \Delta u = f(u)$.

Classification A.M.S. : 35 B 05, 35 K 55, 35 K 60, 34 C 15.

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0. INTRODUCTION

The purpose of this paper is to give a precise description of the asymptotic behavior for solutions $u(z, t)$ of

$$u_t = \Delta u + f(u) \tag{0.1}$$

which blow-up in finite positive time T. We assume $f(u)=u^p\,(p>1)$ or $f(u)=e^u$, and $z \in B_R = \{z \in \mathbb{R}^n : |z| < R\}$ where R is sufficiently large to guarantee blow-up.

Giga and Kohn ([8], [11]) recently characterized the asymptotic behavior of solutions $u(z, t)$ of (0.1) with $f(u)=u^p$ near a blow-up singularity assuming a suitable upper bound on the rate of blow-up and provided $n=1$, 2, or $n \geqq 3$ and $p \leqq \dfrac{n+2}{n-2}$. For $B_R \subseteqq \mathbb{R}^n$ using recent *a priori* bounds established by Friedman-McLeod [7], this implies that solutions $u(z, t)$ of (0.1) with suitable initial-boundary conditions satisfy

$$(T-t)^\beta u(z, t) \to \beta^\beta \quad \text{as } t \to T^- \tag{0.2}$$

provided $|z| \leqq C(T-t)^{1/2}$ for arbitrary $C \geqq 0$ and where $\beta = \dfrac{1}{p-1}$.

For $f(u)=e^u$ and $n=1$ or 2, Bebernes, Bressan, and Eberly [1] proved that solutions $u(z, t)$ of (0.1) satisfy

$$u(z, t) + \ln(T-t) \to 0 \quad \text{as } t \to T^- \tag{0.3}$$

provided $|z| \leqq C(T-t)^{1/2}$ for arbitrary $C \geqq 0$.

The real remaning difficulty in understanding how the single point blow-up occurs for (0.1) rests on determining the nonincreasing globally Lipschitz continuous solutions of an associated steady-state equation

$$y'' + \left(\frac{n-1}{x} - \frac{x}{2}\right) y' + F(y) = 0, \qquad 0 < x < \infty \tag{0.4}$$

where $F(y) = y^p - \beta y$ or $e^y - 1$ for $f(y) = y^p$ or e^y respectively and where $y(0) > 0$ and $y'(0) = 0$.

For $F(y) = y^p - \beta y$ in the cases $n=1$, 2, or $n \geqq 3$ and $p \leqq \dfrac{n}{n-2}$, we give a new proof of a special case of a known result ([8], Theorem 1) that the only such positive solution of (0.4) is $y(x) \equiv \beta^\beta$. For $F(y) = e^y - 1$ and $n=1$, Bebernes and Troy [3] proved that the only such solution is $y(x) \equiv 0$.

Eberly [5] gave a much simpler proof showing $y(x)\equiv 0$ is the only solution for the same nonlinearity valid for $n=1$ and 2.

For $3\leqq n\leqq 9$, Troy and Eberly [6] proved that (0.4) has infinitely many nonincreasing globally Lipschitz continuous solutions on $[0, \infty)$ for $F(y)=e^y-1$. Troy [10] proved a similar multiplicity result for (0.4) with $F(y)=y^p-\beta y$ for $3\leqq n\leqq 9$ and $p>\frac{n+2}{n-2}$.

This multiple existence of solutions complicates the stability analysis required to precisely describe the evolution of the time-dependent solutions $u(z, t)$ of (0.1) near the blow-up singularity.

In this paper we extend the results of Giga-Kohn [8] and Bebernes-Bressan-Eberly [1] to the dimensions $n\geqq 3$ by proving that, in spite of the multiple existence of solutions of (0.4), the asymptotic formulas (0.2) and (0.3) remain the same as in dimensions 1 and 2. The key to unraveling these problems is a precise understanding of the behavior of the nonconstant solutions relative to a singular solution of (0.4) given by

$$S_e(x)=\ln\frac{2(n-2)}{x^2} \tag{0.5}$$

for $f(u)=e^u$ and $n\geqq 3$, and

$$S_p(x)=\left\{-4\beta\left[\beta+\frac{1}{2}(2-n)\right]/x^2\right\}^{\beta} \tag{0.6}$$

for $f(u)=u^p$ and $\beta+\frac{1}{2}(2-n)<0$, $n\geqq 3$. This will be accomplished by counting how many times the graphs of a nonconstant self-similar solution crosses that of the singular solution.

1. STATEMENT OF THE RESULTS

We consider the initial value problem

$$\left.\begin{array}{ll} u_t-\Delta u=f(u), & (z, t)\in\Omega\times(0, T) \\ u(z, 0)=\varphi(z), & z\in\Omega \\ u(z, t)=0, & (z, t)\in\partial\Omega\times(0, T) \end{array}\right\} \tag{1.1}$$

where $\Omega = B_R = \{z \in \mathbb{R}^n : |z|z| < R\}$, φ is nonnegative, radially symmetric, nonincreasing ($\varphi(z) \geqq \varphi(x)$ for $|z| \leqq |x| \leqq R$), and $\Delta\varphi + f(\varphi) \geqq 0$ on Ω. The two nonlinearities considered are

$$f(u) = e^u \tag{1.2}$$

or

$$f(u) = u^p, \qquad u \geqq 0, \quad p > 1. \tag{1.3}$$

We assume $R > 0$ and $\varphi(z) \geqq 0$ are such that the radially symmetric solution $u(z, t)$ blows-up in finite positive time T. By the maximum principle, $u(., t)$ is radially decreasing for each $t \in [0, T)$ and $u_t(z, t) > 0$ for $(z, t) \in \Omega \times (0, T)$.

Friedman and McLeod [7] proved that blow-up occurs only at $z = 0$. The following arguments are essentially those used in [7] to obtain the needed *a priori* bounds.

Let $U(t) = u(0, t)$. Since $\Delta u(0, t) \leqq 0$ because u is radially symmetric and decreasing, from (1.1) it follows that $U'(t) \leqq f(U(t))$. Integrating, we have

$$-\ln(T - t) \leqq u(0, t), \qquad t \in [0, T) \tag{1.4}$$

for $f(u) = e^u$, and

$$\beta^\beta (T - t)^{-\beta} \leqq u(0, t), \qquad t \in [0, T) \tag{1.5}$$

for $F(u) = u^p$

Define the radially symmetric function $J(z, t) = u_t - \delta f(u)$ where $\delta > 0$ is to be determined. Then $J_t - \Delta J - f'(u) J \geqq 0$. For $0 < \eta < \min(R, T)$, let $\Omega_\eta = B_{R-\eta}$ be the ball of radius $R - \eta$ centered at $0 \in \mathbb{R}^n$. Let $\Pi_\eta = \Omega_\eta \times (\eta, T)$. Since blow-up occurs only at $z = 0$, $u(z, t)$ is bounded on the parabolic boundary of Π_η and $f(u) \leqq C_0 < \infty$ there. Since $u_t > 0$ on $\Omega \times (0, T)$, we have $u_t \geqq C > 0$ on the parabolic boundary of Π_η. Hence, for $\delta > 0$ sufficiently small, $J \geqq C - \delta C_0 > 0$ there. By the maximum principle, $J > 0$ on Π_η. An integration yields the following upper bound on $u(0, t)$:

$$u(0, t) \leqq -\ln[\delta(T - t)], \qquad t \in [\eta, T) \tag{1.6}$$

for $f(u) = e^u$, and

$$u(0, t) \leqq \left(\frac{\beta}{\delta}\right)^\beta (T - t)^{-\beta}, \qquad t \in [\eta, T) \tag{1.7}$$

for $f(u) = u^p$. In fact, since $u_t(., t) \geqq 0$ for $t \in [0, T)$, these bounds are true for all $t \in [0, T)$.

As in [7], we also have the existence of $\bar{t} < T$ such that

$$|\nabla u(z, t)| \leqq [2 e^{u(0, t)}]^{1/2}, \qquad (z, t) \in \bar{\Omega} \times [\bar{t}, T) \tag{1.8}$$

for $f(u)=e^u$, and

$$|\nabla u(z, t)| \leqq \left[\frac{2}{p+1}[u(0, t)]^{p+1}\right]^{1/2}, \qquad (z, t) \in \bar{\Omega} \times [\bar{t}, \mathrm{T}) \qquad (1.9)$$

for $f(u)=u^p$.

In this paper we prove the following two theorems which describe the asymptotic self-similar blow-up of $u(z, t)$.

THEOREM 1. — *(a) For* $n \geqq 3$, *the solution* $u(z, t)$ *of* (1.1)-(1.2) *satisfies* $u(z, t)+\ln(\mathrm{T}-t) \to 0$ *uniformly on* $\{(z, t): |z| \leqq \mathrm{C}(\mathrm{T}-t)^{1/2}\}$ *for arbitrary* $\mathrm{C} \geqq 0$ *as* $t \to \mathrm{T}^-$.

(b) For $n \geqq 3$ *and* $p > \frac{n}{n-2}$, *the solution* $u(z, t)$ *of* (1.1)-(1.3) *satisfies* $(\mathrm{T}-t)^\beta u(z, t) \to \beta^\beta$ *uniformly on* $\{(z, t): |z| \leqq \mathrm{C}(\mathrm{T}-t)^{1/2}\}$ *for arbitrary* $\mathrm{C} \geqq 0$ *as* $t \to \mathrm{T}^-$.

THEOREM 2. — *Let* $r=|z|$ *and* $v(r, t)=u(z, t)$. *There is a value* $r_1 \in (0, \mathrm{R})$ *such that the following properties hold.*

(a) $v(r_1, 0)=\mathrm{S}_*(r_1)$ *where* S_* *is the singular solution given in* (0.5) *or* (0.6).

(b) $v(r, 0)<\mathrm{S}_*(r)$ *for* $0<r<r_1$.

(c) For each $r \in (0, r_1)$ *there is a* $\bar{t}=\bar{t}(r) \in (0, \mathrm{T})$ *such that* $v(r, t)>\mathrm{S}_*(r)$ *for* $t \in (\bar{t}, \mathrm{T})$.

2. THE SELF-SIMILAR PROBLEM

Since the solution $u(z, t)$ of (1.1) is radially symmetric, the initial-boundary value problem can be reduced to a problem in one spatial dimension.

Let $\Pi'=\{(r, t): 0<r<\mathrm{R}, 0<t<\mathrm{T}\}$. If $r=|z|$, then $v(r, t)=u(z, t)$ is well-defined on Π' and satisfies

$$v_t=v_{rr}+\frac{n-1}{r}v_r+f(v), \qquad (r, t) \in \Pi' \qquad (2.1)$$

$$\begin{aligned} &v(r, 0)=\varphi(r), \qquad r \in (0, \mathrm{R}) \\ &v_r(0, t)=0, \qquad v(\mathrm{R}, t)=0, \qquad t \in (0, \mathrm{T}) \end{aligned} \qquad (2.2)$$

To analyze the behavior of v as $t \to T^-$, we make the following change of variables:

$$\sigma = \ln [T/(T-t)], \qquad x = r (T-t)^{-1/2}$$

Then Π' transforms into Π where

$$\Pi = \{(x, \sigma) : \sigma > 0,\ 0 < x < RT^{-1/2} e^{1/2\,\sigma}\}.$$

If $f(u) = e^u$, set

$$w(x, \sigma) = v(r, t) + \ln (T-t).$$

If $f(u) = u^p$, set

$$w(x, \sigma) = (T-t)^\beta v(r, t).$$

Then $w(x, \sigma)$ solves

$$w_\sigma = w_{xx} + c(x) w_x + F(w), \qquad (x, \sigma) \in \Pi \tag{2.3}$$

$$w_x(0, \sigma) = 0, \qquad \sigma \in (0, \infty) \tag{2.4}$$

where $c(x) = (n-1)/x - x/2$; if $f(u) = e^u$, then

$$F(w) = e^w - 1$$

$$w(RT^{-1/2} e^{1/2\,\sigma}, \sigma) = -\sigma + \ln T, \qquad \sigma \in (0, \infty) \tag{2.5}$$

$$w(x, 0) = \varphi(x T^{1/2}) + \ln T, \qquad x \in (0, RT^{-1/2})$$

and if $f(u) = u^p$, then

$$\left.\begin{aligned} &F(w) = w^p - \beta w \\ &w(RT^{-1/2} e^{1/2\,\sigma}, \sigma) = 0, \qquad \sigma \in (0, \infty) \\ &w(x, 0) = T^\beta \varphi(x T^{1/2}), \qquad x \in (0, RT^{-1/2}) \end{aligned}\right\} \tag{2.6}$$

Using the *a priori* bounds established in section I for $u(z, t)$ using the ideas of [7], we have the following *a priori* estimates for $w(x, \sigma)$. For $F(w) = e^w - 1$, from (1.4) and (1.6)

$$0 \leqq w(0, \sigma) \leqq -\ln \delta, \qquad \sigma \geqq 0. \tag{2.7}$$

For $F(w) = w^p - \beta w$, from (1.5) and (1.7)

$$\beta^\beta \leqq w(0, \sigma) \leqq (\beta/\delta)^\beta, \qquad \sigma \geqq 0. \tag{2.8}$$

The estimates (1.8) and (1.9) imply that

$$-\gamma \leqq w_x(x, \sigma) \leqq 0 \quad \text{on } \bar{\Pi} \tag{2.9}$$

for some positive constant γ, and combining this with (2.7) and (2.8) yields

$$-\gamma x \leqq w(x, \sigma) \leqq \mu \quad \text{on } \bar{\Pi} \tag{2.10}$$

where γ and μ are positive constants depending on δ. In fact, for $F(w)=w^p-\beta w$, $w(x,\sigma)=(T-t)^\beta v(r,t)\geqq 0$ since $v(r,0)\geqq 0$ and $v_t(r,t)\geqq 0$.

3. BEHAVIOR NEAR SINGULAR SOLUTIONS

The partial differential equation (2.3) has a time-independent solution fro certain choices of n and p. More precisely, if $n>2$ and $F(w)=e^w-1$, then

$$S_e(x)=\ln[2(n-2)/x^2] \tag{3.1}$$

is a singular solution of (2.3). If $F(w)=w^p-\beta w$, $n>2$ and $p>\dfrac{n}{n-2}$, then

$$S_p(x)=\left\{-4\beta\left[\beta+\frac{1}{2}(2-n)\right]/x^2\right\}^\beta \tag{3.2}$$

is a singular solution of (2.3). These solutions are in fact singular solutions of (2.1) because

$$1+\frac{1}{2}xS_e'=0, \qquad S_e''+\frac{n-1}{x}S_e'+\exp(S_e)=0 \tag{3.3}$$

and

$$\beta S_p+\frac{1}{2}xS_p'=0, \qquad S_p'=0, \qquad S_p''+\frac{n-1}{x}S_p'+(S_p)^p=0 \tag{3.4}$$

for $0<x<\infty$.

Consider first the singular solution $S_e(x)$ of (2.3) with $F(w)=e^w-1$. Then $S_e(0^+)=\infty>w(0,0)$ and

$$S_e(RT^{-1/2})=\ln[2(n-2)TR^{-2}]<\ln T=w(RT^{-1/2},0)$$

since $2(n-2)<R^2$ for blow-up in finite time (Lacey [9], Bellout [4]). This proves that $w(x,0)$ intersects $S_e(x)$ at least once for $0<x<RT^{-1/2}$.

Similarly for $F(w)=w^p-\beta w$ and $S_p(x)$, we can make the following observations: $S_p(0^+)=\infty>w(0,0)$ and $S_p(RT^{-1/2})>0=w(RT^{-1/2},0)$. If $w(x,0)\leqq S_p(x)$ on $[0, RT^{-1/2}]$, we conclude by the maximum principle that $w(x,\sigma)\leqq S_p(x)$ on $\bar{\Pi}$. By the result of Troy [10] (*see* part *b* of Lemma 4.4), any positive global nonincreasing time-independent solution $y(x)$ associated with (2.3) must interest $S_p(x)$ transversally at least once. By the argument given in Giga-Kohn [8] (or see our theorem 5.1),

$w(x, \sigma) \to 0$ as $\sigma \to \infty$ for each $x \geqq 0$. In particular, $w(0, \sigma) \to 0$, a contradiction to (2.8).

In either case, we can conclude that there exists a first $x_1 \in (0, RT^{-1/2})$ such that $w(x_1, 0) = S_*(x_1)$ and $w(x, 0) < S_*(x)$ on $(0, x_1)$.

LEMMA 3.1. – *There is a continuously differentiable function $x_1(\sigma)$ with domain $[0, \infty)$ such that $x_1(0) = x_1$ and $w(x_1(\sigma), \sigma) = S_*(x_1(\sigma))$ for all $\sigma \geqq 0$.*

Proof. – Define $D(x, \sigma) = w(x, \sigma) - S_*(x)$. We first claim that $\nabla D \neq (0, 0)$ whenever $D = 0$. We had $v_t(r, t) > 0$ on Π'. For $f(v) = e^v$,

$$v_t = (T - t)^{-1}\left(w_\sigma + 1 + \frac{1}{2} x w_x\right),$$

and for $f(v) = v^p$,

$$v_t = (T - t)^{-\beta - 1}\left(w_\sigma + \beta w + \frac{1}{2} x w_x\right).$$

If $\nabla D = (0, 0)$ at a point in Π where $D = 0$, then $D_\sigma = 0$ implies that $w_\sigma = 0$. For $f(v) = e^v$, $D_x = 0$ implies that $1 + \frac{1}{2} x w_x = 0$. For $f(v) = v^p$, $D_x = 0$ implies that $\beta w + \frac{1}{2} x w_x = 0$. In either case, $v_t = 0$ is forced at some point in Π', a contradiction.

Secondly, we claim that $D_x \neq 0$ at any value $(\bar{x}, \bar{\sigma}) \in \Pi$ where $D(\bar{x}, \bar{\sigma}) = 0$ and $D(x, \bar{\sigma}) < 0$ in a left neighborhood of $\bar{x}$.

If $D(\bar{x}, \bar{\sigma}) = 0$ and $D_x(\bar{x}, \bar{\sigma}) = 0$, then equations (2.3), (3.3), and (3.4) imply that $D_{xx}(\bar{x}, \bar{\sigma}) = D_\sigma(\bar{x}, \bar{\sigma})$. In addition, since $v_t > 0$ we have $D_\sigma(\bar{x}, \bar{\sigma}) > 0$. Thus $D_{xx}(\bar{x}, \bar{\sigma}) > 0$, which implies that $(\bar{x}, \bar{\sigma})$ is a local minimum point for D, a contradiction to $D < 0$ on a left neighborhood of $\bar{x}$. Thus, $D_x(\bar{x}, \bar{\sigma}) > 0$.

Recall that $v(r, 0) = \varphi(r)$ where $\Delta\varphi + f(\varphi) \geqq 0$. This implies

$$D_{xx}(x, 0) + \frac{n-1}{x} D_x(x, 0) + F(w(x, 0)) - F(S_*(x)) \geqq 0$$

for x in a left neighborhood of x_i. On a left neighborhood of x_1, this in turn yields $(x^{n-1} D_x(x, 0))_x \geqq 0$. An integration yields $D_x(x_1, 0) > 0$. By the implicit function theorem, there is a continuously differentiable function $x_1(\sigma)$ such that $x_1(0) = x_1$ and $D(x_1(\sigma), \sigma) = 0$ for some maximal interval $[0, \sigma_0)$. If $\sigma_0 < \infty$, then by continuity $D(x_1(\sigma_0), \sigma_0) = 0$.

But $D_x(x_1(\sigma_0), \sigma_0) > 0$, so the implicit function theorem allows an extension of the domain past σ_0, a contradiction to the maximality of $[0, \sigma_0)$. Thus, $\sigma_0 = \infty$. □

For $f(u) = u^p$, since $w(0, 0) < S_p(0^+)$, $w(RT^{-1/2}, 0) < S_p(RT^{-1/2})$, and $w(x_1, 0) = S_p(x_1)$ transversally, there must be a last point of intersection between $w(x, 0)$ and $S_p(x)$, say $x_L \in (x_1, RT^{-1/2})$. A construction similar to Lemma 3.1 leads to the existence of a continuously differentiable function $x_L(\sigma)$ with domain $[0, \infty)$ such that $x_L(0) = x_L$ and $w(x_L(\sigma), \sigma) = S_p(x_L(\sigma))$ for $\sigma \geqq 0$.

Let $\Pi_1 = \{(x, \sigma) : \sigma > 0, 0 < x < x_1(\sigma)\}$. We can now prove the following comparison result on this set.

LEMMA 3.2. — *$D(x, \sigma) < 0$ for $(x, \sigma) \in \Pi_1$.*

Proof. — By Lemma 3.1, we have shown that $D \leqq 0$ on the parabolic boundary of Π_1. Since $F(w)$ is a local one-sided Lipschitz continuous function, we can apply the Nagumo-Westphal comparison result to obtain $D \leqq 0$ on $\bar{\Pi}_1$.

If $D(x_0, \sigma_0) = 0$ for some $(x_0, \sigma_0) \in \Pi_1$, then $D_x(x_0, \sigma_0) = 0$, $D_{xx}(x_0, \sigma_0) \leqq 0$ and $D_\sigma(x_0, \sigma_0) \neq 0$ [since $\nabla D \neq (0, 0)$ when $D = 0$]. But $D_\sigma(x_0, \sigma_0) \neq 0$ implies $D(x_0, \sigma)$ is positive for some σ near σ_0. This contradicts $D \leqq 0$ on $\bar{\Pi}_1$.

Let $x_2 = \sup\{x \in (x_1, RT^{-1/2}] : D(s, 0) \geqq 0$ for $s \in [x_1, 0) = 0$ and $D_x(x_1, 0) > 0$, the supremum exists. For $f(u) = e^u$, $x_2 \leqq RT^{-1/2}$, and for $f(u) = u^p$, $x_2 \leqq x_L < RT^{-1/2}$. Define $x_2(\sigma) = x_2 e^{1/2\sigma}$ and $\Pi_2 = \{(x, \sigma) : \sigma > 0, x_1(\sigma) < x < x_2(\sigma)\}$.

LEMMA 3.3. — *$D(x_2(\sigma), \sigma) \geqq 0$ for all $\sigma \geqq 0$. Moreover, $D(x, \sigma) > 0$ for $(x, \sigma) \in \Pi_2$.*

Proof. — Let $E(\sigma) = D(x_2(\sigma), \sigma)$. By definition of x_2, $E(0) = D(x_2, 0) \geqq 0$. Also, $E'(\sigma) = D_\sigma(x_2(\sigma), \sigma) + \frac{1}{2} x_2(\sigma) D_x(x_2(\sigma), \sigma)$. We had earlier that $v_t(r, t) \geqq 0$ on $\bar{\Pi}'$. Via the change of variables $(r, t) \to (x, \sigma)$, this implies $E'(\sigma) \geqq 0$ in the case $f(v) = e^v$ and $e^{-\beta\sigma} \frac{d}{d\sigma}[e^{\beta\sigma} E(\sigma)] = E'(\sigma) + \beta E(\sigma) \geqq 0$ in the case $f(v) = v^p$. An integration yields $E(\sigma) \geqq 0$ for $\sigma \geqq 0$.

On the parabolic boundary of Π_2, we now have that $D \geqq 0$. By the Nagumo-Westphal comparison theorem, $D \geqq 0$ on $\bar{\Pi}_2$. A similar argument as in Lemma 3.2 shows that $D > 0$ on Π_2. □

COROLLARY 3.4. — *For each* $N>0$ *there is a* $\sigma_N>0$ *such that for each* $\sigma>\sigma_N$, $w(x,\sigma)$ *intersects* $S_*(x)$ *at most once for* $x\in[0, N]$.

Proof. — For each $N>0$ choose σ_N such that $N=x_2\exp\left(\frac{1}{2}\sigma_N\right)$. Lemma 3.2 guarantees that $D(x,\sigma)<0$ for $x\in[0, x_1(\sigma))$ and Lemma 3.3 guarantees that $D(x,\sigma)>0$ for $\in(x_1(\sigma), x_2(\sigma)]$. For $\sigma>\sigma_N$, $[0, N]\subseteqq[0, x_2(\sigma)]$ by definition of σ_N, so $D=0$ at most once on this interval. □

In section 5 we will see that $x_1(\sigma)\to l$ as $\sigma\to\infty$ where $S_e(l)=0$ or $S_p(l)=\beta^\beta$.

4. ANALYSIS OF THE STEADY-STATE PROBLEM

The time-independent solutions of (2.3)-(2.4) satisfy

$$y''+c(x)y'+F(y)=0, \qquad 0<x<\infty \tag{4.1}$$

$$y(0)=\alpha, \qquad y'(0)=0 \tag{4.2}$$

In this section we will analyze the behavior of a particular class of solutions of (4.1) which are possible members of the ω-limit set for the initial-boundary value problems (2.3)-(2.4)-(2.5) or (2.3)-(2.4)-(2.6).

By the *a priori* bounds stated in section 2, we have that $w(0,\sigma)$ is bounded for $\sigma\geqq 0$. More precisely for $F(w)=e^w-1$, $w(0,\sigma)\in[0, -\ln\delta]$, and for $F(w)=w^p-\beta w$, $w(0,\sigma)\in[\beta^\beta, (\beta/\delta)^\beta]$, for $\sigma\geqq 0$. We also had $-\gamma\leqq w_x(x,\sigma)\leqq 0$ on $\bar{\Pi}$ and, for $F(w)=w^p-\beta w$, $w\geqq 0$ on $\bar{\Pi}$.

If $F(w)=e^w-1$, we need to consider those solutions $y(x)$ of (4.1)-(4.2) which satisfy

$$y(0)=\alpha\geqq 0, \qquad y'(x)\leqq 0 \quad \text{for } x\geqq 0, \quad y'(x) \text{ bounded below.} \tag{4.3}$$

For $n=1$ or 2, (4.1)-(4.2)-(4.3) has only the solution $y(x)\equiv 0$ ([3], [5]). For $3\leqq n\leqq 9$, (4.1)-(4.2)-(4.3) has infinitely many nonconstant solutions [6]. In this section we prove that all nonconstant solutions of (4.1)-(4.2)-(4.3) must intersect the singular solution $S_e(x)$ at least twice. Hence, the only solution intersecting $S_e(x)$ exactly once is $y(x)\equiv 0$.

For $F(w)=w^p-\beta w$, we consider those solutions $y(x)$ of (4.1)-(4.2) which satisfy

$$y(0)=\alpha\geqq\beta^\beta, \qquad y'(x)\leqq 0 \quad \text{and} \quad y(x)>0 \quad \text{for } x\geqq 0. \tag{4.4}$$

For $n=1$, 2, or $n\geqq 3$ with $p\leqq\dfrac{n}{n-2}$ we prove a special case of the known result [8] that the only solution to (4.1)-(4.2)-(4.4) is $y(x)\equiv\beta^\beta$. Troy [10] showed that, for $n\geqq 3$ and $p>\dfrac{n+2}{n-2}$, (4.1)-(4.2)-(4.4) has infinitely many nonconstant solutions. In this section we show that any nonconstant solution $y(x)$ of (4.1)-(4.2)-(4.4) must intersect $S_p(x)$ at least twice. Hence, the only solution intersecting $S_p(x)$ *exactly once is* $y(x)\equiv\beta^\beta$.

LEMMA 4.1. — *Consider initial value problem* (4.1)-(4.2).
(*a*) *Any solution to* (4.1)-(4.2)-(4.3) *must satisfy* $y(\sqrt{2n})\leqq 0$.
(*b*) *Any solution to* (4.1)-(4.2)-(4.4) *must satisfy* $y(\sqrt{2n})\leqq\beta^\beta$.

Proof. — (*a*) In this case, $F(y)=e^y-1\geqq y$, so equation (4.1) implies that $y''+c(x)y'+y\leqq 0$. Let $u(x)=\alpha(1-x^2/2n)$. Then $u''+c(x)u'+u=0$, $u(0)=y(0)$, and $u'(0)=y'(0)$. Define $W(x)=u(x)y'(x)-u'(x)y(x)$. While $u(x)>0$, $W'+c(x)W\leqq 0$ and $W(0)=0$, so an integration yields that $W(x)\leqq 0$. But $(y/u)'(x)=W(x)/[u(x)]^2\leqq 0$, so integrating from 0 to $\sqrt{2n}$ yields $y(\sqrt{2n})\leqq u(\sqrt{2n})=0$.

Note that for $\alpha>0$, if $y(z)=0$, then $y'(z)<0$ by uniqueness to initial value problems, so $y(x)<0$ for $x>z$.

(*b*) The function $F(y)=y^p-\beta y$ in convex, so $F(y)\geqq y-\beta^\beta$ and equation (4.1) implies that $v''+c(x)v'+v\leqq 0$ where $v(x)=y(x)-\beta^\beta$. A similar argument as in part (*a*) shows that $v(\sqrt{2n})\leqq 0$, thus, $y(\sqrt{2n})\leqq\beta^\beta$.

Note that for $\alpha>\beta^\beta$, if $y(z)=\beta^\beta$, then $y'(z)<0$ by uniqueness to initial value problems, so $y(x)<\beta^\beta$ for $x>z$. □

Define $h(x)=y''+\dfrac{n-1}{x}y'$. For $F(y)=e^y-1$, define $g(x)=1+\dfrac{1}{2}xy'$ and for $F(y)=y^p-\beta y$, define $g(x)=\beta y+\dfrac{1}{2}xy'$. It can be shown that h and g satisfy the following equations:

$$g''+c(x)g'+[F'(y)-1]g=0, \qquad g(0)>0, \quad g'(0)=0. \tag{4.5}$$

$$h''+c(x)h'+[F'(y)-1]h=-F''(y)(y')^2, \qquad h(0)\leqq 0, \quad h'(0)=0. \tag{4.6}$$

For $F(y)=e^y-1$,

$$g'-\frac{1}{2}xg=-\frac{1}{2}xe^y+\frac{1}{2}(2-n)y'. \tag{4.7}$$

For $F(y)=y^p-\beta y$,

$$g'-\frac{1}{2}xg=-\frac{1}{2}xy^p+\left[\beta+\frac{1}{2}(2-n)\right]y'. \tag{4.8}$$

Also define $W(x)=g(x)h'(x)-g'(x)h(x)$. Then

$$W'+c(x)W=-F''(y)(y')^2 g,\ W(0)=0,$$

and

$$W(x)=-x^{1-n}e^{(1/4)x^2}\int_0^x s^{n-1}e^{-(1/4)s^2}F''[y(s)][y'(s)]^2 g(s)\,ds \qquad (10)$$

$$=:-x^{1-n}e^{(1/4)x^2}I(x)$$

where $I(x)\geqq 0$, while $g>0$ on $(0, x)$. Note that $\left(\frac{h}{g}\right)'(x)=W(x)/[g(x)]^2$, so while $g>0$ on $(0, x)$, we have

$$h(x)=\frac{h(0)}{g(0)}g(x)-g(x)\int_0^x t^{1-n}e^{(1/4)t^2}I(t)[g(t)]^{-2}\,dt \qquad (4.9)$$

LEMMA 4.2. — *Consider initial value problem* (4.1)-(4.2).

(*a*) *If* $y(x)$ *is a solution to* (4.1)-(4.2)-(4.3) *with* $\alpha>0$, *then* $g(x)$ *must have a zero.*

(*b*) *If* $y(x)$ *is a solution to* (4.1)-(4.2)-(4.4) *with* $\alpha>\beta^\beta$, *then* $g(x)$ *must have a zero.*

Proof. — Suppose that $g(x)\geqq\varepsilon>0$ for all $x\geqq 0$. Note that $h(0)<0$ because $\alpha>0$ [part (*a*)] or $\alpha>\beta^\beta$ [part (*b*)]. Then (4.9) implies that $h(x)\leqq[h(0)/g(0)]g(x)\leqq-\delta<0$ since $h(0)/g(0)<0$ and since $I(x)\geqq 0$. Multiplying by x^{n-1} and integrating yields $y'(x)\leqq-\frac{\delta}{n}x$. This contradicts the boundedness of y' in equation (4.3) and forces y to be negative eventually, contradicting equation (4.4). Thus, $g(x)$ cannot be bounded away from zero.

Suppose that $g(x)>0$ for $x\geqq 0$ and that g is not bounded away from zero. Suppose there is an increasing unbounded sequence $\{x_k\}_1^\infty$ such that $g'(x_k)=0$. Equation (4.5) implies that $g''(x_k)=[1-F'(y(x_k))]g(x_k)$. However, Lemma 4.1 implies that $1-F'(y(x_k))>0$ for k sufficiently large. This forces $g''(x_k)>0$ for k sufficiently large, a contradiction, since this would imply that g has two local minimums without a local maximum between. It must be the case that $g'(x)<0$ for x sufficiently large and $g(x)\to 0$ as $x\to\infty$.

Suppose there is an increasing unbounded sequence $\{x_k\}_1^\infty$ such that $g''(x_k)=0$ and $g'(x_k)\leqq-L<0$. Then equation (4.5) implies that $0=c(x_k)g'(x_k)+[F'(y(x_k))-1]g(x_k)$ where $c(x_k)\to-\infty$, $g'(x_k)\leqq-L$, $F'(y(x_k))-1$ is bounded, and $g(x_k)\to 0$. But then the right-hand side of the last equality must become infinite, a contradiction. Thus, $g'(x)<0$ for x large and $g'(x) \to 0$.

In equation (4.9), take the limit as $x \to \infty$ to obtain

$$\lim_{x \to \infty} h(x) = - \lim_{x \to \infty} g(x) \int_0^x t^{1-n} e^{(1/4)t^2} \mathrm{I}(t) [g(t)]^{-2} dt$$
$$= \lim_{x \to \infty} x^{1-n} e^{(1/4)x^2} \mathrm{I}(x) [g'(x)]^{-1} = -\infty$$

where we have used L'Hôpital's rule. This implies that $h(x) \leqq -\delta < 0$ for x sufficiently large. Multiplying by x^{n-1} and integrating yields $y'(x) \leqq \mathrm{K} - \frac{\delta}{n} x$ for x sufficiently large. As before, this contradicts the boundedness of y' in equation (4.3) and forces y to be negative eventually, contradicting equation (4.4).

In all of the above cases, we arrived at contradictions, so there must be a value x_0 such that $g(x_0)=0$, $g'(x_0)<0$, and $g(x)>0$ on $[0, x_0)$. □

LEMMA 4.3. — *Consider problem* (4.1)-(4.2)-(4.3).

(*a*) *If* $1 \leqq n \leqq 2$, *then the only solution is* $y(x) \equiv 0$.

(*b*) *If* $n>2$, *then the only solution which intersects* $\mathrm{S}_e(x)$ *exactly once is* $y(x) \equiv 0$.

Proof. — (*a*) Let $1 \leqq n \leqq 2$, then $\frac{1}{2}(2-n) \geqq 0$. Let x_0 be the first zero for $g(x)$. Suppose there is an $x_1 > x_0$ such that $g'(x_1)=0$ and $g(x)<0$ on $(x_0, x_1]$. Equation (4.7) implies that

$$0 < -\frac{1}{2} x_1 g(x_1) = g'(x_1) - \frac{1}{2} x_1 g(x_1) = -\frac{1}{2} x_1 e^{y(x_1)} + \frac{1}{2}(2-n) y'(x_1) < 0$$

which is a contradiction. Thus, $g'(x)<0$ for $x \geqq x_0$ and so $g(x) \leqq -\varepsilon < 0$ for $x \geqq \bar{x} > x_0$. But $h(x) = g(x) - e^{y(x)} \leqq g(x) \leqq -\varepsilon$. Multiplying by x^{n-1} and integrating yields $y'(x) \leqq \mathrm{K} - \frac{\varepsilon}{n} x$, contradicting equation (4.3). As a result, the only solution of (4.1)-(4.2)-(4.3) for these values of n is $y(x) \equiv 0$.

(*b*) Let $n>2$. Define $\mathrm{D}(x) = y(x) - \mathrm{S}_e(x)$ where S_e is the singular solution discussed in section 3. Then

$$\left.\begin{aligned} \mathrm{D}'' + c(x)\mathrm{D}' + \frac{2(n-2)}{x^2}(e^{\mathrm{D}} - 1) = 0, \qquad 0 < x < \infty, \\ \mathrm{D}(0^+) = -\infty, \qquad \mathrm{D}'(0^+) = \infty. \end{aligned}\right\} \qquad (4.10)$$

Note that $D'>0$ while $D<0$ on $(0, x]$. Suppose that $D(x)<0$ for all $x \geqq 0$. Then $e^D - 1 < 0$ and $D'' + c(x) D' \geqq 0$. Integrating this last equation yields

$$x^{n-1} e^{-(1/4) x^2} D'(x) \geqq \bar{x}^{n-1} e^{-(1/4) \bar{x}^2} D'(\bar{x}) =: p > 0.$$

Consequently,

$$D(x) \geqq D(\bar{x}) + \int_{\bar{x}}^{x} p t^{1-n} e^{(1/4) t^2} dt.$$

But the right-hand side of this inequality must be positive for x sufficiently large, contradicting our assumption. Thus, $D(x)$ must have a first zero x_1 and $D'(x)>0$ on $(0, x_1]$.

By Lemma 4.2, $g(x)$ must have a zero x_0. But then $D'(x_0) = \frac{2}{x_0} g(x_0) = 0$ and $x_0 > x_1$. If $D(x_0)<0$, then there must have been a second zero x_2 for D. Otherwise, $D(x)>0$ on $(x_1, x_0]$, Suppose that $D>0$ for all $x \geqq x_0$. Then there is an $\bar{x}$ sufficiently large such that $D(\bar{x})>0$, $D'(\bar{x})<0$, $D''(\bar{x})>0$, and $c(\bar{x})<0$. Evaluating equation (4.10) at $\bar{x}$ yields $0<(D''+c D'+e^D-1)(\bar{x})=0$, a contradiction. Thus, D must have a second zero x_2.

We have shown that there are at least two points of intersection between the graphs of $y(x)$ and $S_e(x)$ for $\alpha>0$. Thus, the only solution to (4.1)-(4.2)-(4.3) which intersects $S_e(x)$ exactly once is $y(x) \equiv 0$. □

LEMMA 4.4. – *Consider initial value problem* (4.1)-(4.2)-(4.4).

(a) *If* $1 \leqq n \leqq n \leqq 2$, *or if* $n>2$ *and* $\beta + \frac{1}{2}(2-n) \geqq 0$, *then the only solutions is* $y(x) \equiv \beta^\beta$.

(*b*) *If* $n>2$ *and* $\beta + \frac{1}{2}(2-n) < 0$, *then the only solution which intersects* $S_p(x)$ *exactly once is* $y(x) \equiv \beta^\beta$.

Proof. – (*a*) In this case, $\beta + \frac{1}{2}(2-n) \geqq 0$. Let x_0 be the first zero for $g(x)$. Suppose there is an $x_1 > x_0$ such that $g'(x_1)=0$ and $g(x)<0$ on $(x_0, x_1]$. Equation (4.8) implies that

$$0 < -\frac{1}{2} x_1 g(x_1) = g'(x_1) - \frac{1}{2} x_1 g(x_1)$$

$$= -\frac{1}{2} x_1 [y(x_1)]^p + \left[\beta + \frac{1}{2}(2-n)\right] y'(x_1) \leqq 0$$

which is a contradiction. Thus $g'(x_0)<0$ for $x \geqq x_0$ and so $g(x) \leqq -\varepsilon < 0$ for $x \geqq \bar{x} > x_0$. But $h(x)=g(x)-[y(x)]^p \leqq g(x) \leqq -\varepsilon$. Multiplying by x^{n-1} and integrating yields $y'(x) \leqq K - \frac{\varepsilon}{n} x$, which forces $y(x)$ to have a zero. This contradicts equation (4.4). As a result, the only solution for these cases is $y(x) \equiv \beta^\beta$.

(*b*) Let $n>2$ and $f+\frac{1}{2}(2-n)<0 \left(p>\frac{n}{n-2}\right)$. The result for the cases $p>\frac{n+2}{n-2}$ is proved by Troy [10]. For the larger range $p>\frac{n}{n-2}$ we have the following proof. Define $W(x)=y(x)S_p'(x)-y'(x)S_p(x)$ and $Q(u)=F(u)/u$. Then $W'+c(x)W=yS_p[Q(y)-Q(S_p)]$. Note that $Q(u)$ is an increasing function. Also note that $W(x)=-2Kx^{-2\beta-1}g(x)$ where $S_p(x)=Kx^{-2\beta}$. Thus, $x^{n-1}W(x)=-2Kx^{n-2-2\beta}g(x)$ where $n-2-2\beta>0$. As a result, $x^{n-1}W(x) \to 0$ as $x \to 0^+$. Integrating the equation for $W(x)$, we obtain

$$x^{n-1} e^{-(1/4)x^2} W(x) = \int_0^x t^{n-1} e^{-(1/4)t^2} y(t) S_p(t) [Q(y(t)) - Q(S_p(t))] \, dt.$$

If $0<y<S_p$ for all $x \geqq 0$, then since $Q(u)$ is increasing, $W(x)<0$ for all x. But then $g(x)>0$ for all x is forced, a contradiction to Lemma 4.2. Consequently, there must be a value z such that $y(z)=S_p(z)$.

Also, $W(x)<0$ for $x \in [0, x_0)$. At x_0, $0<W'(x_0)$ which implies that $y(x_0)>S_p(x_0)$. [Note that $W'(x_0)=0$ and $y(x_0)=S_p(x_0)$ imply that $y'(x_0)=S'(x_0)$ which in turn would imply, by uniqueness to initial value problems, that $y(x) \equiv S_p(x)$, a contradiction.] So $z<x_0$ is necessary.

Let $x_1>x_0$ be small enough so that $W(x_1)>0$. Suppose that $y>S_p$ for all $x>z$. Then integrating the equation for $W(x)$, we have $W'+c(x)W \geqq 0$ and

$$x^{n-1} e^{-(1/4)x^2} W(x) \geqq x_1^{n-1} e^{-(1/4)x_1^2} W(x_1) =: p > 0.$$

But $(S_p/y)'(x)=W(x)/[y(x)]^2$, so

$$(S_p/y)(x) \geqq (S_p/y)(x_1) + p \int_{x_1}^x t^{1-n} e^{(1/4)t^2} [y(t)]^{-2} \, dt.$$

For x sufficiently large, the right-hand side must become larger than 1, in which case $(S/y)(x) \geqq 1$. That is, there is another value q where $y(q)=S_p(q)$.

We have shown that there are at least two points of intersection between the graphs of $y(x)$ and $S_p(x)$ for $\alpha > \beta^\beta$. Thus, the only solution to (4.1)-(4.2)-(4.4) which intersects $S_p(x)$ exactly once is $y(x) \equiv \beta^\beta$. □

5. THE CONVERGENCE RESULTS

We are now able to precisely describe how the blowup asymptotically evolves in dimensions $n \geqq 3$. Let $w(x, \sigma)$ be the solution of (2.3)-(2.4)-(2.5) or (2.3)-(2.4)-(2.6) depending on the nonlinearity being considered. By Corollary 3.4 we know that for each $N > 0$ there is a $\sigma_N > 0$ such that $w(x, \sigma)$ intersects $S_*(x)$ at most once on $[0, N]$ for each $\sigma > \sigma_N$. By Lemmas 4.3 and 4.4, the only possible steady-state solution of (2.3) with $F(w) = e^w - 1$ which intersects $S_e(x)$ at most once is $y(x) \equiv 0$, and for $F(w) = w^p - \beta w$, the only possible steady-state solution of (2.3) intersecting $S_p(x)$ at most once is $y(x) \equiv \beta^\beta$.

Because of these observations we are now able to prove a convergence or stability result similar to those given in [8] and [1] which prove that the ω-limit set for (2.3)-(2.4)-(2.5) consists of the singleton critical point $y(x) \equiv 0$, and for (2.3)-(2.4)-(2.6), $y(x) \equiv \beta^\beta$.

For the sake of completeness, we include the proof of the following theorem which is influenced by the ones given in [1] and [8].

THEOREM 5.1. — *Let* $n \geqq 3$.

(*a*) *As* $\sigma \to \infty$, *the solution* $w(x, \sigma)$ *of* (2.3)-(2.4)-(2.5) *converges to* $y(x) \equiv 0$ *uniformly in x on compact subsets of* $[0, \infty)$.

(*b*) *As* $\sigma \to \infty$, *the solution* $w(x, \sigma)$ *of* (2.3)-(2.4)-(2.6) *converges to* $y(x) \equiv \beta^\beta$ *uniformly in x on compact subsets of* $[0, \infty)$.

Proof. — Define $w^\tau(x, \sigma) := w(x, \sigma + \tau)$ as the function obtained by shifting w in time by the amount τ. We will show that as $\tau \to \infty$, $w^\tau(x, \sigma)$ converges to the solution $y(x)$ uniformly on compact subsets of $\mathbb{R}^+ \times \mathbb{R}$. Provided that the limiting function is unique, it is equivalent to prove that given any unbounded increasing sequence $\{n_j\}$, there exists a subsequence $\{n_j\}$ such that w^{n_j} converges to $y(x)$ uniformly on compact subsets of $\mathbb{R}^+ \times \mathbb{R}$.

Let $N \in \mathbb{Z}^+$. For i sufficiently large, the rectangle given by $Q_{2N} = \{(x, \sigma): 0 \leqq x \leqq 2N, |\sigma| \leqq 2N\}$ lies in the domain of w^{n_i}. The radially symmetric function $\tilde{w}(\zeta, \sigma) = w^{n_i}(|\zeta|, \sigma)$ solves the parabolic equation

$$\tilde{w}_\sigma = \Delta \tilde{w} - \frac{1}{2}\langle \zeta, \nabla \tilde{w}\rangle + F(\tilde{w})$$

on the cylinder given by $\Gamma_{2N} = \{(\zeta, \sigma) \in \mathbb{R}^n \times \mathbb{R}: |\zeta| \leqq 2N, |\sigma| \leqq 2N\}$ with $-2N\gamma \leqq \tilde{w}(\zeta, \sigma) \leqq \mu$ using (2.10).

By Schauder's interior estimates, all partial derivatives of $\tilde{w}$ can be uniformly bounded on the subcylinder $\Gamma_N \subseteqq \Gamma_{2N}$. Consequently, w^{n_i}, $w^{n_i}_\sigma$, and $w^{n_i}_{xx}$ are uniformly Lipschitz continuous on $Q_N \subseteqq Q_{2N}$. Their Lipschitz constants depend on N but not on i. By the Arzela-Ascoli theorem, there is a subsequence $\{n_j\}_1^\infty$ and a function $\bar{w}$ such that w^{n_j}, $w^{n_j}_\sigma$, $w^{n_j}_{xx}$ converge to $\bar{w}$, $\bar{w}_\sigma$, and $\bar{w}_{xx}$, respectively, uniformly on Q_N.

Repeating the construction for all N and taking a diagonal subsequence, we can conclude that $w^{n_j} \to \bar{w}$, $w^{n_j}_\sigma \to \bar{w}$, and $w^{n_j}_{xx} \to \bar{w}_{xx}$ uniformly on every compact subset in $\mathbb{R}^+ \times \mathbb{R}$. Clearly $\bar{w}$ satisfies (2.3)-(2.4) with $-\gamma \leqq \bar{w}_x \leqq 0$. For $n \geqq 3$ and $F(w) = e^w - 1$, the limiting function $\bar{w}$ intersects $S_e(x)$ at most once since, by Corollary 3.4, $w^{n_j}(x, \sigma)$ intersects $S_e(x)$ at most once on [0, N] for each $\sigma > \sigma_N$, and $0 \leqq \bar{w}(0, \sigma) \leqq -\ln \delta$ for $\sigma \geqq 0$. For $n \geqq 3$, $\beta + \frac{1}{2}(2-n) < 0$, and

$$F(w) = w^p - \beta w,$$

Corollary 3.4 guarantees that $\bar{w}$ intersects $S_p(x)$ at most once. By (2.8) we have $\beta^\beta \leqq w(0, \sigma) \leqq (\beta/\delta)^\beta$ for $\sigma \geqq 0$.

We now prove that $\bar{w}$ is independent of σ. For the solution $w(x, \sigma)$ of (2.3)-(2.4)-(2.5) or (2.6), define the energy functional

$$\left.\begin{aligned} E(\sigma) &= \int_0^v \rho(x)\left[\frac{1}{2} w_x^2 - G(w)\right] dx, \\ v = RT^{-1/2} e^{1/2\,\sigma}, &\qquad \rho(x) = x^{n-1} e^{-(1/4)x^2} \end{aligned}\right\} \tag{5.1}$$

where $G(w) = e^w - w$ if $F(w) = e^w - 1$, and $G(w) = w^{p+1}/(p+1) - \frac{1}{2}\beta w^2$ if $F(w) = w^p - \beta w$.

Multiplying equation (2.3) by ρw_σ and integrating from 0 to v yields the equation

$$\int_0^v \rho w_\sigma^2 \, dx = \int_0^v w_\sigma (\rho w_x)_x \, dx + \int_0^v \frac{\partial}{\partial\sigma}[\rho G(w)] \, dx$$

$$= \int_0^v \frac{\partial}{\partial\sigma}\left[\rho G(w) - \frac{1}{2}\rho w_x^2\right] dx + \rho w_\sigma w_x \Big|_{x=0}^{x=v} \qquad (5.2)$$

Moreover,

$$E'(\sigma) = \int_0^v \frac{\partial}{\partial\sigma}\left[\frac{1}{2}\rho w_x^2 - \rho G(w)\right] dx$$

$$+ \frac{1}{2} v \left\{ \rho(v) \left[\frac{1}{2} w_x^2(v, \sigma) - G(w(v, \sigma))\right]\right\} \qquad (5.3)$$

Therefore, for all a, b with $0 \leqq a < b$, integrating (5.2) with respect to σ from a to b, and using (5.3), we have

$$\int_a^b \int_0^v \rho w_x \, dx \, d\sigma = -\int_a^b E'(\sigma) \, d\sigma + \int_a^b \rho(v) w_\sigma(v, \sigma) w_x(v, \sigma) \, d\sigma$$

$$+ \frac{1}{2} \int_a^b \rho(v) \left[\frac{1}{2} w_x^2(v, \sigma) - G(w(v, \sigma))\right] d\sigma$$

$$=: E(a) - E(b) + \psi(a, b) \qquad (5.4)$$

Recalling that $|w_x| \leqq \gamma$ and observing that

$$w_\sigma(v, \sigma) = -1 - R u_r(R, T(1 - e^{-\sigma}))$$

for $f(u) = e^u$, or $w_\sigma(v, \sigma) = -R u_r(R, T(1 - e^{-\sigma}))$ for $f(u) = u^p$, we see that in either case the quantity is uniformly bounded as $\sigma \to \infty$. We conclude that

$$\lim_{a \to \infty} \{ \sup_{b > a} \psi(a, b) \} = 0 \qquad (5.5)$$

For any fixed N, we shall prove that

$$\int_{Q_N}\int \rho \bar{w}_\sigma^2 \, dx \, d\sigma = \lim_{n_j \to \infty} \int_{Q_N}\int \rho (w_\sigma^{n_j})^2 \, dx \, d\sigma = 0.$$

Note that it is not a restriction to assume that $\lim_{j \to \infty} (n_{j+1} - n_j) = \infty$. For all j large enough, $N \leqq RT^{-1/2} \exp\left[\frac{1}{2}(n_j - N)\right]$ and $n_{j+1} - n_j \geqq 2N$. Hence,

$$\int_{-N}^{N} \int_{0}^{N} \rho (w_\sigma^{n_j})^2 \, dx \, d\sigma \leqq \int_{-N}^{-N+n_{j+1}-n_j} \int_{0}^{RT^{-1/2}\exp(1/2\, n_j)} \rho (w_\sigma^{n_j})^2 \, dx \, d\sigma$$

$$= E(n_j - N) - E(n_{j+1} - N) + \psi(n_j - N, n_{j+1} - N)$$

by (5.4). As a consequence of (5.5), we have

$$\int_{Q_N} \int \rho \, \bar{w}_\sigma^2 \, dx \, d\sigma \leqq \lim_{j \to \infty} \sup [E(n_j - N) - E(n_{j+1} - N)]. \tag{5.6}$$

Fix any K arbitrarily large. For j sufficiently large, we have

$$E(n_j - N) - E(n_{j+1} - N)$$

$$= \int_0^K \frac{1}{2} \rho \{ [w_x^{n_j}(x, -N)]^2 - [w_x^{n_{j+1}}(x, -N)]^2 \} \, dx$$

$$- \int_0^K \rho [G(w^{n_j}(x, -N) - G(w^{n_{j+1}}(x, -N)] \, dx$$

$$+ \int_K^{RT^{-1/2}\exp[1/2\,(n_j - N)]} \rho \left\{ \frac{1}{2} [w_x^{n_j}(x, -N)]^2 - G(w^{n_j}(x, -N)) \right\} dx$$

$$\int_K^{RT^{-1/2}\exp[1/2\,(n_j - N)]} \rho \left\{ \frac{1}{2} [w_x^{n_{j+1}}(x, -N)]^2 - G(w^{n_{j+1}}(x, -N)) \right\} dx \tag{5.7}$$

In (5.7), the first two integrals on the right-hand side converge to zero as $j \to \infty$. Recalling that $|w_x^{n_j}(x, -N)| \leqq \gamma$ and $-\gamma x \leqq w^{n_j}(x, -N) \leqq \mu$, we see that the sum of the absolute values of the last two integrals is bounded by $M \int_K^\infty x^{n-1} e^{-(1/4)x^2} \, dx$ where M is a positive constant. This integral can be made arbitrarily small by choosing K large enough.

This proves that $\int_{-N}^{N} \rho \, \bar{w}_\sigma^2 \, dx \, d\sigma = 0$ and hence $\bar{w}_\sigma = 0$. Thus, $\bar{w}(x, \sigma) = \bar{w}(x, 0) = y(x)$ where $y(x)$ is a nonincreasing globally Lipschitz continuous solution of (4.1)-(4.2) which intersects $S_*(x)$ at most once. If $f(u) = e^u$, then $y(0) \in [0, -\ln \delta]$ and so $y(x) \equiv 0$ is the only solution which intersects $S_e(x)$ exactly (and thus at most) once on $[0, \infty)$. Similarly for $f(u) = u^p$, $y(0) \in [\beta^\beta, (\beta/\delta)^\beta]$ and the only possible solution is $y(x) \equiv \beta^\beta$.

Since the limiting solution $y(x)$ is unique in either case, $\omega^\tau(x, \sigma) \to y(x)$ as $\tau \to \infty$ and we have the result asserted. □

Proof of Theorem 1. – The last theorem shows that $w(x, \sigma) \to y(x)$ uniformly in x on compact subsets of $[0, \infty)$ as $\sigma \to \infty$.

(*a*) In the case $f(u) = e^u$, changing back to the variables (r, t), we have that $v(r, t) + \ln(T-t) \to 0$ as $t \to T^-$ provided $r \leqq C(T-t)^{1/2}$ for arbitrary $C \geqq 0$.

In particular, $v(0, t) + \ln(T-t) \to 0$ as $t \to T^-$.

(*b*) In the case $f(u) = u^p$ we obtain $(T-t)^\beta v(r, t) \to \beta^\beta$ as $t \to T^-$ provided $r \leqq C(T-t)^{1/2}$ for arbitrary $C \geqq 0$. In particular, $(T-t)^\beta v(0, t) \to \beta^\beta$ as $t \to T^-$.

Proof of Theorem 2. – Theorem 5.1 guarantees that the first branch of zeros $x_1(\sigma)$ of $D(x, \sigma) = w(x, \sigma) - S_*^*(x)$ is bounded and converges to l where $S_e(l) = 0$ or $S_p(l) = \beta^\beta$.

Define $r_1 = x_1 T^{1/2}$. Then $D(x_1, 0) = 0$ implies that $v(r_1, 0) = S_*(r_1)$. In addition, $v(r, 0) < S_*(r)$ for $r \in (0, r_1)$.

Since $x_1(\sigma)$ is bounded and since $\dfrac{d}{d\sigma} D(r T^{-1/2} e^{1/2\sigma}, \sigma) \geqq 0$ for each $r \in (0, r_1)$, there is a value $\bar{\sigma} > 0$ such that

$$r T^{-1/2} e^{1/2\bar{\sigma}} = x_1(\bar{\sigma}) D(x_1(\bar{\sigma}), \bar{\sigma}) = 0,$$

and $D(r T^{-1/2} e^{1/2\sigma}, \sigma) > 0$ for $\sigma > \bar{\sigma}$. Changing back to the variables (r, t) with $\bar{\sigma} = \ln[T//(T-\bar{t})]$, we obtain $v(r, t) > S_*(r)$ for $t \in (\bar{t}, T)$.

Remark. – After this paper was completed we received the preprint [11] of Giga and Kohn. In the introduction there is a detailed discussion of self-similar solutions and their importance in describing the behavior of solutions near a blow up point. The referee pointed out a number of papers ([12] to [18]) which are related to the ideas used in this paper. Their relevance is discussed in [11]. The referee also pointed out a briefer proof of Lemma 4.1 which we have used.

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