

# ANNALES DE L'I. H. P., SECTION C

A. BENSOUSSAN

L. BOCCARDO

F. MURAT

## **On a non linear partial differential equation having natural growth terms and unbounded solution**

*Annales de l'I. H. P., section C*, tome 5, n° 4 (1988), p. 347-364

[http://www.numdam.org/item?id=AIHPC\\_1988\\_\\_5\\_4\\_347\\_0](http://www.numdam.org/item?id=AIHPC_1988__5_4_347_0)

© Gauthier-Villars, 1988, tous droits réservés.

L'accès aux archives de la revue « Annales de l'I. H. P., section C » (<http://www.elsevier.com/locate/anihpc>) implique l'accord avec les conditions générales d'utilisation (<http://www.numdam.org/conditions>). Toute utilisation commerciale ou impression systématique est constitutive d'une infraction pénale. Toute copie ou impression de ce fichier doit contenir la présente mention de copyright.

NUMDAM

Article numérisé dans le cadre du programme  
Numérisation de documents anciens mathématiques

<http://www.numdam.org/>

## On a non linear partial differential equation having natural growth terms and unbounded solution

by

**A. BENSOUSSAN**

University Paris-Dauphine and I.N.R.I.A.

**L. BOCCARDO**

University of Rome I

and

**F. MURAT**

University of Paris VI and C.N.R.S.

---

**ABSTRACT.** — We prove the existence of a solution of the nonlinear elliptic equation:  $A(u) + g(x, u, Du) = h(x)$ , where  $A$  is a Leray-Lions operator from  $W_0^{1,p}(\Omega)$  into  $W^{-1,p'}(\Omega)$  and  $g$  is a nonlinear term with “natural” growth with respect to  $Du$  [i.e. such that  $|g(x, u, \xi)| \leq b(|u|)(|\xi|^p + c(x))$ ], satisfying the sign condition  $g(x, u, \xi)u \geq 0$  but no growth condition with respect to  $u$ . Here  $h$  belongs to  $W^{-1,p'}(\Omega)$ , thus the solution  $u$  of the problem does not in general be more smooth than  $W_0^{1,p}(\Omega)$ . The existence of a solution is also proved for the corresponding obstacle problem.

**RÉSUMÉ.** — Nous démontrons l’existence d’une solution du problème elliptique non linéaire  $A(u) + g(x, u, Du) = h(x)$ , où  $A$  est un opérateur de Leray-Lions de  $W_0^{1,p}(\Omega)$  à valeurs dans  $W^{-1,p'}(\Omega)$  et où  $g$  est un

---

Classification A.M.S. : 35J20, 35J65, 35J85, 47H15, 49A29.

terme non linéaire à croissance « naturelle » en  $Du$  [i.e. tel que  $|g(x, u, \xi)| \leq b(|u|)(|\xi|^p + c(x))$ ], qui satisfait la condition de signe  $g(x, u, \xi)u \geq 0$  mais dont la croissance en  $u$  n'est pas limitée. Le second membre  $h$  appartient à  $W^{-1, p'}(\Omega)$ , et la solution  $u$  du problème n'est donc pas, en général, plus régulière que  $W_0^{1, p}(\Omega)$ . Nous démontrons également l'existence d'une solution pour l'inéquation variationnelle avec obstacle associée à ce problème.

---

## INTRODUCTION

In this paper we prove the existence of solutions of non linear elliptic equations of the type

$$A(u) + g(x, u, Du) = h,$$

where  $A$  is a Leray-Lions operator from  $W_0^{1, p}(\Omega)$  into  $W^{-1, p'}(\Omega)$ ,  $h \in W^{-1, p'}(\Omega)$ , and  $g$  is a non linear lower order term having natural growth (of order  $p$ ) with respect to  $|Du|$ . With respect to  $|u|$ , we do not assume any growth restrictions, but we assume the "sign-condition"

$$g(x, u, \xi)u \geq 0.$$

It will turn out that, for any solution  $u$ ,  $g(x, u, Du)$  will be in  $L^1(\Omega)$ , but, for each  $v \in W_0^{1, p}(\Omega)$ ,  $g(x, v, Dv)$  can be very odd, and does not necessarily belong to  $W^{-1, p'}(\Omega)$ .

In the present paper, the main features are the "sign-condition" and the non smoothness of the right hand side  $h$ .

Let us point out that, if  $h$  is sufficiently smooth, existence results of bounded solutions have been recently obtained in [1], [3], [2], [8], [9], [10], [19]. But, in general, it is well known that, even for the corresponding linear equation, one cannot expect an  $L^\infty$  solution: the solutions can be only in  $W_0^{1, p}(\Omega)$ . They are bounded if  $h \in W^{-1, q}(\Omega)$ ,  $q > N/(p-1)$ , see [5].

When  $g$  does not depend on  $Du$ , existence results for this type of problems have been proved in [15], [20], [11], [4]. When  $g$  depends on  $Du$

with natural growth, the case where  $A$  is linear has been solved in [7], [6]. This result was generalized to non linear  $A$ 's in [13], [16]. The proofs of the four last paper are based on the almost everywhere convergence of the gradients, a result which is due to J. Frehse [14] in the non-linear case.

In the present paper we present a proof which proceeds from different ideas. We consider  $u_\epsilon$  defined by

$$u_\epsilon \in W_0^{1,p}(\Omega) : A(u_\epsilon) + \frac{g(x, u_\epsilon, Du_\epsilon)}{1 + \epsilon |g(x, u_\epsilon, Du_\epsilon)|} = h.$$

Because of the "sign-condition" it is easy to obtain a  $W_0^{1,p}(\Omega)$ -estimate on  $u_\epsilon$ . Extracting a subsequence,  $u_\epsilon$  tends to  $u$  in  $W_0^{1,p}(\Omega)$  weakly. The problem will be solved whenever the convergence will be proved to be *strong* in  $W_0^{1,p}(\Omega)$ . We obtain this result proving that the positive part  $u_\epsilon^+$  of  $u_\epsilon$  strongly converges to  $u^+$  (and that the similar property holds for  $u_\epsilon^-$ ). The proof consists in two steps. In the first one, we prove that the "exceeding" part of  $u_\epsilon^+$ , defined as  $(u_\epsilon^+ - u_k^+)^+$  (where  $u_k^+$  is the truncation of  $u^+$  at level  $k$ ) is controlled in  $W_0^{1,p}(\Omega)$  in terms of  $(u^+ - u_k^+)$ . The second step is to prove that the "bounded" part  $(u_\epsilon^+ - u_k^+)^-$  of  $u_\epsilon^+$  strongly converges to zero in  $W_0^{1,p}(\Omega)$ . For this we use the technique of multiplying by a non linear test function  $\varphi((u_\epsilon^+ - u_k^+)^-)$  introduced in [8], [9], [10]; this is allowed because  $0 \leq (u_\epsilon^+ - u_k^+)^- \leq k$ .

Finally we give also an existence result for the corresponding obstacle problem.

## 1. STATEMENT OF THE PROBLEM

### 1.1. Assumptions

Let  $\Omega$  be a bounded open set of  $\mathbb{R}^N$ . Let  $1 < p < +\infty$  be fixed and  $A$  be a non linear operator from  $W_0^{1,p}(\Omega)$  into its dual  $W^{-1,p'}(\Omega)$   $\left(\frac{1}{p} + \frac{1}{p'} = 1\right)$  defined by

$$A(u) = -\operatorname{div} a(x, u, Du),$$

where  $a(x, s, \xi)$  is a Carathéodory function  $a: \Omega \times \mathbb{R} \times \mathbb{R}^N \rightarrow \mathbb{R}^N$  such that

$$\left. \begin{aligned} |a(x, s, \xi)| &\leq \beta [|s|^{p-1} + |\xi|^{p-1} + k(x)] \\ [a(x, s, \xi) - a(x, s, \eta)] \cdot (\xi - \eta) &> 0, \quad \forall \xi \neq \eta \\ a(x, s, \xi) \cdot \xi &\geq \alpha |\xi|^p, \end{aligned} \right\} \quad (1.1)$$

where  $k(x) \in L^{p'}(\Omega)$ ,  $k \geq 0$ ,  $\beta > 0$ ,  $\alpha > 0$ .

Let  $g(x, s, \xi)$  be a Carathéodory function such that

$$\left. \begin{aligned} g(x, s, \xi) s &\geq 0 \\ |g(x, s, \xi)| &\leq b(|s|)(|\xi|^p + c(x)), \end{aligned} \right\} \quad (1.2)$$

where  $b$  is a continuous and increasing function with (finite) values on  $\mathbb{R}^+$ , and  $c \in L^1(\Omega)$ ,  $c \geq 0$ .

We consider

$$h \in W^{-1, p'}(\Omega). \quad (1.3)$$

### 1.2. The main result

Consider the non linear elliptic problem with Dirichlet boundary conditions

$$\left. \begin{aligned} \Delta u + g(x, u, Du) &= h \quad \text{in } \mathcal{D}'(\Omega) \\ u \in W_0^{1, p}(\Omega), \quad g(x, u, Du) &\in L^1(\Omega), \quad u g(x, u, Du) \in L^1(\Omega) \end{aligned} \right\} (1.4)$$

Our objective is to prove the following

**THEOREM 1.1.** — *Under the assumptions (1.1), (1.2), (1.3) there exists a solution of (1.4).*

Before giving the proof of the theorem, let us emphasize that the main difficulty stems from the fact that  $u$  is unbounded. Since  $h$  is only in  $W^{-1, p'}(\Omega)$ , it is impossible to expect the existence of an  $L^\infty$  solution (see [5]).

Usually boundedness plays an important role in the study of equations of the type (1.4). Indeed, to overcome the difficulties due to the quadratic growth of the non linear term, non linear test functions (with respect to

the solution) are used in [2], [8], [9], [10] and it is important to know beforehand that such exponentials remain bounded. We shall see that it is possible to avoid this assertion, in the present framework.

**2. PROOF OF THEOREM 1.1**

**2.1. An approximation scheme**

Let us define

$$g_\varepsilon(x, s, \xi) = \frac{g(x, s, \xi)}{1 + \varepsilon |g(x, s, \xi)|}, \tag{2.1}$$

and let us consider the equation

$$\left. \begin{aligned} A(u_\varepsilon) + g_\varepsilon(x, u_\varepsilon, Du_\varepsilon) &= h \\ u_\varepsilon &\in W_0^{1,p}(\Omega) \end{aligned} \right\} \tag{2.2}$$

which has a solution by the classical result of J. Leray and J. L. Lions [17].

Multiplying (2.2) by  $u_\varepsilon$  and using (1.2) we get

$$\langle A(u_\varepsilon), u_\varepsilon \rangle \leq \langle h, u_\varepsilon \rangle$$

hence

$$\alpha \|u_\varepsilon\|^2 \leq \|h\| \cdot \|u_\varepsilon\|$$

from which we get

$$\|u_\varepsilon\|_{W_0^{1,p}} \leq c_1. \tag{2.3}$$

Hence we can extract a subsequence, still denoted by  $u_\varepsilon$ , with

$$u_\varepsilon \rightharpoonup u \text{ in } W_0^{1,p}(\Omega) \text{ weakly and a. e.} \tag{2.4}$$

**2.2. Convergence of the positive part of  $u_\varepsilon$**

Our objective in this paragraph is to prove that

$$u_\varepsilon^+ \rightarrow u^+ \text{ in } W_0^{1,p}(\Omega) \text{ strongly.} \tag{2.5}$$

Let  $k$  be a positive constant. Let us define

$$u_k^+ = u^+ \wedge k.$$

In a first stage we shall fix  $k$ , and use the notation

$$z_\varepsilon = u_\varepsilon^+ - u_k^+. \quad (2.6)$$

*First step: the behaviour of  $z_\varepsilon^+$*

Note that  $z_\varepsilon \in W_0^{1,p}(\Omega)$ , therefore  $z_\varepsilon^+ \in W_0^{1,p}(\Omega)$ . Multiplying (2.2) by  $z_\varepsilon^+$  yields

$$\langle A u_\varepsilon, z_\varepsilon^+ \rangle + \int_\Omega g_\varepsilon(x, u_\varepsilon, Du_\varepsilon) z_\varepsilon^+ dx = \langle h, z_\varepsilon^+ \rangle. \quad (2.7)$$

Note that where  $z_\varepsilon^+ > 0$ ,  $u_\varepsilon^+ > 0$  hence  $u_\varepsilon > 0$  and from (1.2)  $g_\varepsilon(x, u_\varepsilon, Du_\varepsilon) \geq 0$ . Therefore we deduce from (2.7)

$$\langle A(u_\varepsilon), z_\varepsilon^+ \rangle \leq \langle h, z_\varepsilon^+ \rangle.$$

Hence

$$\int_\Omega a(x, u_\varepsilon, Du_\varepsilon) Dz_\varepsilon^+ dx \leq \langle h, z_\varepsilon^+ \rangle.$$

Since  $u_\varepsilon = u_\varepsilon^+$  on the set  $\{x \in \Omega, z_\varepsilon^+(x) > 0\}$ , we can also write

$$\int_\Omega a(x, u_\varepsilon, Du_\varepsilon^+) Dz_\varepsilon^+ dx \leq \langle h, z_\varepsilon^+ \rangle,$$

which implies

$$\begin{aligned} & \int_\Omega [a(x, u_\varepsilon, Du_\varepsilon^+) - a(x, u_\varepsilon, Du_k^+)] D(u_\varepsilon^+ - u_k^+)^+ dx \\ & \leq - \int_\Omega a(x, u_\varepsilon, Du_k^+) D(u_\varepsilon^+ - u_k^+)^+ dx + \langle h, z_\varepsilon^+ \rangle \quad (2.8) \end{aligned}$$

As  $\varepsilon \rightarrow 0$ , we have

$$z_\varepsilon^+ \rightarrow (u^+ - u_k^+)^+ \quad \text{a. e.}$$

But  $z_\varepsilon^+$  is bounded in  $W_0^{1,p}(\Omega)$ , hence also

$$z_\varepsilon^+ \rightharpoonup (u^+ - u_k^+)^+ \text{ in } W_0^{1,p}(\Omega) \text{ weakly, as } \varepsilon \rightarrow 0 \text{ (} k \text{ fixed).}$$

Define

$$R_k = - \int_{\Omega} a(x, u, Du_k^+) D(u^+ - u_k^+)^+ dx + \langle h, (u^+ - u_k^+)^+ \rangle. \quad (2.9)$$

Since we have

$$(u^+ - u_k^+)^+ \rightarrow 0 \text{ in } W_0^{1,p}(\Omega), \text{ as } k \rightarrow +\infty,$$

we obtain that

$$R_k \rightarrow 0, \text{ as } k \rightarrow +\infty;$$

passing to the limit in  $\varepsilon$  (for fixed  $k$ ) in (2.8) yields

$$\overline{\lim}_{\varepsilon \rightarrow 0} \int_{\Omega} [a(x, u_\varepsilon, Du_\varepsilon^+) - a(x, u_\varepsilon, Du_k^+)] D(u_\varepsilon^+ - u_k^+)^+ dx \leq R_k. \quad (2.10)$$

*Second step: behaviour of  $z_\varepsilon^-$*

We shall use as a test function in (2.2) the function

$$v_\varepsilon = \varphi_\lambda(z_\varepsilon^-), \quad \varphi_\lambda(s) = s e^{\lambda s^2},$$

where  $\lambda$  will be chosen later. Note that

$$0 \leq z_\varepsilon^- \leq k, \quad (2.11)$$

hence  $z_\varepsilon^- \in L^\infty(\Omega)$  and, since  $z_\varepsilon \in W_0^{1,p}(\Omega)$ , clearly  $v_\varepsilon \in W_0^{1,p}(\Omega)$ . Therefore  $v_\varepsilon$  is an admissible test function for (2.2). We deduce

$$\int_{\Omega} a(x, u_\varepsilon, Du_\varepsilon) Dz_\varepsilon^- \varphi'_\lambda(z_\varepsilon^-) dx + \int_{\Omega} g_\varepsilon(x, u_\varepsilon, Du_\varepsilon) \varphi_\lambda(z_\varepsilon^-) dx = \langle h, \varphi_\lambda(z_\varepsilon^-) \rangle. \quad (2.12)$$

Define

$$E_\varepsilon = \{x \in \Omega : u_\varepsilon^+(x) \leq u_k^+(x)\}, \quad F_\varepsilon = \{x \in \Omega : 0 \leq u_\varepsilon(x) \leq u_k^+(x)\}.$$

We have

$$\int_{\Omega} g_{\varepsilon}(x, u_{\varepsilon}, Du_{\varepsilon}) \varphi_{\lambda}(z_{\varepsilon}^{-}) dx = \int_{E_{\varepsilon}} g_{\varepsilon}(x, u_{\varepsilon}, Du_{\varepsilon}) \varphi_{\lambda}(z_{\varepsilon}^{-}) dx. \quad (2.13)$$

Note that  $\varphi_{\lambda}(z_{\varepsilon}^{-}) \geq 0$  and that  $g_{\varepsilon}(x, u_{\varepsilon}, Du_{\varepsilon}) \leq 0$  whenever  $u_{\varepsilon} \leq 0$ . Using then (1.2) and (1.1) we have

$$\begin{aligned} \int_{E_{\varepsilon}} g_{\varepsilon}(x, u_{\varepsilon}, Du_{\varepsilon}) \varphi_{\lambda}(z_{\varepsilon}^{-}) dx &\leq \int_{F_{\varepsilon}} g_{\varepsilon}(x, u_{\varepsilon}, Du_{\varepsilon}) \varphi_{\lambda}(z_{\varepsilon}^{-}) dx \\ &\leq \int_{F_{\varepsilon}} b(|u_{\varepsilon}|) [|Du_{\varepsilon}|^p + c(x)] \varphi_{\lambda}(z_{\varepsilon}^{-}) dx \\ &\leq b(k) \int_{F_{\varepsilon}} [|Du_{\varepsilon}|^p + c(x)] \varphi_{\lambda}(z_{\varepsilon}^{-}) dx \\ &\leq \frac{b(k)}{\alpha} \int_{F_{\varepsilon}} a(x, u_{\varepsilon}, Du_{\varepsilon}) Du_{\varepsilon} \varphi_{\lambda}(z_{\varepsilon}^{-}) dx + b(k) \int_{\Omega} c(x) \varphi_{\lambda}(z_{\varepsilon}^{-}) dx. \end{aligned} \quad (2.14)$$

Since  $u_{\varepsilon} \leq 0$  implies  $z_{\varepsilon}^{-} = u_k^{+}$ , we obtain

$$\begin{aligned} \int_{\Omega} -a(x, u_{\varepsilon}, Du_{\varepsilon}^{+}) Dz_{\varepsilon}^{-} \varphi'_{\lambda}(z_{\varepsilon}^{-}) dx \\ = \int_{\Omega} -a(x, u_{\varepsilon}, Du_{\varepsilon}) Dz_{\varepsilon}^{-} \varphi'_{\lambda}(z_{\varepsilon}^{-}) dx \\ + \int_{\Omega} [a(x, u_{\varepsilon}, Du_{\varepsilon}) - a(x, u_{\varepsilon}, Du_{\varepsilon}^{+})] Du_k^{+} \varphi'_{\lambda}(u_k^{+}) dx. \end{aligned}$$

Moreover, using (2.12), (2.13) and (2.14), we obtain

$$\begin{aligned} \int_{\Omega} -[a(x, u_{\varepsilon}, Du_{\varepsilon}^{+}) - a(x, u_{\varepsilon}, Du_k^{+})] D(u_{\varepsilon}^{+} - u_k^{+})^{-} \varphi'_{\lambda}(z_{\varepsilon}^{-}) dx \\ = \int_{\Omega} [a(x, u_{\varepsilon}, Du_{\varepsilon}) - a(x, u_{\varepsilon}, Du_{\varepsilon}^{+})] Du_k^{+} \varphi'_{\lambda}(u_k^{+}) dx + \langle -h, \varphi_{\lambda}(z_{\varepsilon}^{-}) \rangle \\ + \int_{\Omega} g_{\varepsilon}(x, u_{\varepsilon}, Du_{\varepsilon}) \varphi_{\lambda}(z_{\varepsilon}^{-}) dx + \int_{\Omega} a(x, u_{\varepsilon}, Du_k^{+}) Dz_{\varepsilon}^{-} \varphi'_{\lambda}(z_{\varepsilon}^{-}) dx \\ \leq \int_{\Omega} [a(x, u_{\varepsilon}, Du_{\varepsilon}) - a(x, u_{\varepsilon}, Du_{\varepsilon}^{+})] Du_k^{+} \varphi'_{\lambda}(u_k^{+}) dx + \langle -h, \varphi_{\lambda}(z_{\varepsilon}^{-}) \rangle + \end{aligned}$$

$$\begin{aligned}
 & + \int_{\Omega} a(x, u_{\varepsilon}, Du_k^+) Dz_{\varepsilon}^- \varphi'_{\lambda}(z_{\varepsilon}^-) dx \\
 & + \frac{b(k)}{\alpha} \int_{F_{\varepsilon}} a(x, u_{\varepsilon}, Du_{\varepsilon}) Du_{\varepsilon} \varphi_{\lambda}(z_{\varepsilon}^-) dx \\
 & + b(k) \int_{\Omega} c(x) \varphi_{\lambda}(z_{\varepsilon}^-) dx \\
 & = \int_{\Omega} [a(x, u_{\varepsilon}, Du_{\varepsilon}) - a(x, u_{\varepsilon}, Du_{\varepsilon}^+)] Du_k^+ \varphi'_{\lambda}(u_k^+) dx + \langle -h, \varphi_{\lambda}(z_{\varepsilon}^-) \rangle \\
 & + \int_{\Omega} a(x, u_{\varepsilon}, Du_k^+) Dz_{\varepsilon}^- \varphi'_{\lambda}(z_{\varepsilon}^-) dx \\
 & + \frac{b(k)}{\alpha} \int_{\Omega} [a(x, u_{\varepsilon}, Du_{\varepsilon}^+) - a(x, u_{\varepsilon}, Du_k^+)] D(u_{\varepsilon}^+ - u_k^+) \varphi_{\lambda}(z_{\varepsilon}^-) dx \\
 & + \frac{b(k)}{\alpha} \int_{\Omega} a(x, u_{\varepsilon}, Du_{\varepsilon}^+) Du_k^+ \varphi_{\lambda}(z_{\varepsilon}^-) dx \\
 & + \frac{b(k)}{\alpha} \int_{\Omega} a(x, u_{\varepsilon}, Du_k^+) D(u_{\varepsilon}^+ - u_k^+) \varphi_{\lambda}(z_{\varepsilon}^-) dx \\
 & \qquad \qquad \qquad + b(k) \int_{\Omega} c(x) \varphi_{\lambda}(z_{\varepsilon}^-) dx. \tag{2.15}
 \end{aligned}$$

Now we choose  $\lambda = \frac{b(k)^2}{4\alpha^2}$ . So  $\varphi'_{\lambda} - \frac{b(k)}{\alpha} \varphi_{\lambda} \geq \frac{1}{2}$  and we deduce

$$\begin{aligned}
 & - \frac{1}{2} \int_{\Omega} [a(x, u_{\varepsilon}, Du_{\varepsilon}^+) - a(x, u_{\varepsilon}, Du_k^+)] D(u_{\varepsilon}^+ - u_k^+) dx \\
 & \leq \int_{\Omega} [a(x, u_{\varepsilon}, Du_{\varepsilon}) - a(x, u_{\varepsilon}, Du_{\varepsilon}^+)] Du_k^+ \varphi'_{\lambda}(u_k^+) dx + \langle -h, \varphi_{\lambda}(z_{\varepsilon}^-) \rangle \\
 & + \int_{\Omega} a(x, u_{\varepsilon}, Du_k^+) Dz_{\varepsilon}^- \varphi'_{\lambda}(z_{\varepsilon}^-) dx \\
 & + \frac{b(k)}{\alpha} \int_{\Omega} a(x, u_{\varepsilon}, Du_{\varepsilon}^+) Du_k^+ \varphi_{\lambda}(z_{\varepsilon}^-) dx + \dots
 \end{aligned}$$

$$\begin{aligned}
 & + \frac{b(k)}{\alpha} \int_{\Omega} a(x, u_{\varepsilon}, Du_k^+) D(u_{\varepsilon}^+ - u_k^+) \varphi_{\lambda}(z_{\varepsilon}^-) dx \\
 & + b(k) \int_{\Omega} c(x) \varphi_{\lambda}(z_{\varepsilon}^-) dx. \quad (2.16)
 \end{aligned}$$

Extracting a subsequence such that

$$\begin{aligned}
 a(x, u_{\varepsilon}, Du_{\varepsilon}) & \rightharpoonup \gamma \quad \text{in } (L^{p'}(\Omega))^N \text{ weakly} \\
 a(x, u_{\varepsilon}, Du_{\varepsilon}^+) & \rightharpoonup \sigma \quad \text{in } (L^{p'}(\Omega))^N \text{ weakly}
 \end{aligned}$$

(which is still possible), and using Lebesgue's dominated convergence Theorem, it is easy to pass to the limit in  $\varepsilon$  (for  $k$  fixed) in the right hand side of (2.16). The limit is

$$\begin{aligned}
 & \int_{\Omega} [\gamma(x) - \sigma(x)] Du_k^+ \varphi'_{\lambda}(u_k^+) dx + \langle -h, \varphi_{\lambda}((u^+ - u_k^+)^-) \rangle \\
 & + \int_{\Omega} a(x, u, Du_k^+) D(u^+ - u_k^+)^- \varphi'_{\lambda}((u^+ - u_k^+)^-) dx \\
 & + \frac{b(k)}{\alpha} \int_{\Omega} \sigma(x) Du_k^+ \varphi_{\lambda}((u^+ - u_k^+)^-) dx \\
 & + \frac{b(k)}{\alpha} \int_{\Omega} a(x, u, Du_k^+) D(u^+ - u_k^+) \varphi_{\lambda}((u^+ - u_k^+)^-) dx \\
 & + b(k) \int_{\Omega} c(x) \varphi_{\lambda}((u^+ - u_k^+)^-) dx = \int_{\Omega} [\gamma(x) - \sigma(x)] Du_k^+ \varphi'_{\lambda}(u_k^+) dx,
 \end{aligned}$$

since  $(u^+ - u_k^+)^- = 0$  and  $\varphi_{\lambda}(0) = 0$ ; moreover

$$[a(x, u_{\varepsilon}, Du_{\varepsilon}) - a(x, u_{\varepsilon}, Du_{\varepsilon}^+)](u_{\varepsilon})_k^+ = 0 \quad \text{a. e.}$$

which implies  $[\gamma(x) - \sigma(x)] u_k^+ = 0$ ; so the last term is zero too. Thus passing to the limit in  $\varepsilon$  for  $k$  fixed in (2.16) gives

$$\overline{\lim}_{\varepsilon \rightarrow 0} - \int_{\Omega} [a(x, u_{\varepsilon}, Du_{\varepsilon}^+) - a(x, u_{\varepsilon}, Du_{\varepsilon}^+)] D(u_{\varepsilon}^+ - u_k^+)^- dx \leq 0. \quad (2.17)$$

*Third step: conclusion*

From (2.10) and (2.17) we deduce

$$\begin{aligned}
 & \overline{\lim}_{\varepsilon \rightarrow 0} \int_{\Omega} [a(x, u_{\varepsilon}, Du_{\varepsilon}^+) - a(x, u_{\varepsilon}, Du^+)] D(u_{\varepsilon}^+ - u^+) dx \\
 & \leq \overline{\lim}_{\varepsilon \rightarrow 0} \int_{\Omega} [a(x, u_{\varepsilon}, Du_{\varepsilon}^+) - a(x, u_{\varepsilon}, Du_k^+)] D(u_{\varepsilon}^+ - u_k^+) dx \\
 & + \overline{\lim}_{\varepsilon \rightarrow 0} \int_{\Omega} [a(x, u_{\varepsilon}, Du_{\varepsilon}^+) - a(x, u_{\varepsilon}, Du_k^+)] D(u_{\varepsilon}^+ - u_k^+) dx \\
 & + \overline{\lim}_{\varepsilon \rightarrow 0} \int_{\Omega} [a(x, u_{\varepsilon}, Du_{\varepsilon}^+) - a(x, u_{\varepsilon}, Du^+)] D(u_k^+ - u^+) dx \\
 & + \overline{\lim}_{\varepsilon \rightarrow 0} \int_{\Omega} [a(x, u_{\varepsilon}, Du_k^+) - a(x, u_{\varepsilon}, Du^+)] D(u_{\varepsilon}^+ - u^+) dx \\
 & \leq R_k + \int_{\Omega} [\sigma(x) - a(x, u, Du_k^+)] D(u_k^+ - u^+) dx. \quad (2.18)
 \end{aligned}$$

Letting  $k$  tend to  $+\infty$ , the right hand side tends to zero. By a variation of a result of Leray-Lions (for the proof see e. g. [12], [10]), this implies

$$u_{\varepsilon}^+ \rightarrow u^+ \quad \text{in } W_0^{1,p}(\Omega) \text{ strongly.} \quad (2.19)$$

**2.3. Convergence of the negative part of  $u_{\varepsilon}$**

Similarly to the preceding Section, we want to prove now that

$$u_{\varepsilon}^- \rightarrow u^- \quad \text{in } W_0^{1,p}(\Omega) \text{ strongly.} \quad (2.20)$$

Define

$$u_k^- = u^- \wedge k, y_{\varepsilon}^- = u_{\varepsilon}^- - u_k^-.$$

Multiplying (2.2) by  $y_{\varepsilon}^+$ , we get

$$\int_{\Omega} a(x, u_{\varepsilon}, Du_{\varepsilon}) Dy_{\varepsilon}^+ dx + \int_{\Omega} g_{\varepsilon}(x, u_{\varepsilon}, Du_{\varepsilon}) y_{\varepsilon}^+ dx = \langle h, y_{\varepsilon}^+ \rangle.$$

But

$$g_{\varepsilon}(x, u_{\varepsilon}, Du_{\varepsilon}) y_{\varepsilon}^+ \leq 0 \quad \text{a. e.}$$

therefore we obtain as in (2.10) that for  $k$  fixed:

$$\overline{\lim}_{\varepsilon \rightarrow 0} \int_{\Omega} -[a(x, u_{\varepsilon}, -Du_{\varepsilon}^{-}) - a(x, u_{\varepsilon}, -Du_k^{-})] D(u_{\varepsilon}^{-} - u_k^{-})^+ dx \leq Q_k, \tag{2.21}$$

where  $Q_k \rightarrow 0$ , if  $k \rightarrow +\infty$ .

The next step is to study the behaviour of  $(u_{\varepsilon}^{-} - u_k^{-})^{-}$ . Considering again as test function

$$v_{\varepsilon} = \varphi_{\lambda}(y_{\varepsilon}^{-}),$$

we deduce as in (2.17) that

$$\overline{\lim}_{\varepsilon \rightarrow 0} \int_{\Omega} [a(x, u_{\varepsilon}, -Du_{\varepsilon}^{-}) - a(x, u_{\varepsilon}, -Du_k^{-})] D(u_{\varepsilon}^{-} - u_k^{-})^{-} dx \leq 0. \tag{2.22}$$

Combining (2.21) and (2.22) we deduce, as in (2.18), that

$$u_{\varepsilon}^{-} \rightarrow u^{-} \text{ in } W_0^{1,p}(\Omega) \text{ strongly.}$$

### 2.4. Convergence

From (2.19) and (2.20) we deduce that for a subsequence

$$u_{\varepsilon} \rightarrow u \text{ in } W_0^{1,p}(\Omega) \text{ and a. e.} \tag{2.23}$$

$$Du_{\varepsilon} \rightarrow Du \text{ a. e.} \tag{2.24}$$

Since  $g$  is continuous in the two last arguments we have

$$\left. \begin{aligned} g_{\varepsilon}(x, u_{\varepsilon}, Du_{\varepsilon}) &\rightarrow g(x, u, Du) \text{ a. e.} \\ g_{\varepsilon}(x, u_{\varepsilon}, Du_{\varepsilon}) u_{\varepsilon} &\rightarrow g(x, u, Du) u \text{ a. e.} \end{aligned} \right\} \tag{2.25}$$

From (2.2) and (2.3) we infer that

$$0 \leq \int_{\Omega} g_{\varepsilon}(x, u_{\varepsilon}, Du_{\varepsilon}) u_{\varepsilon} dx \leq c_2. \tag{2.26}$$

For any measurable subset  $E$  of  $\Omega$  and any  $m > 0$ , we have

$$\int_E |g_\varepsilon(x, u_\varepsilon, Du_\varepsilon)| dx = \int_{E \cap X_m^\varepsilon} |g_\varepsilon(x, u_\varepsilon, Du_\varepsilon)| dx + \int_{E \cap Y_m^\varepsilon} |g_\varepsilon(x, u_\varepsilon, Du_\varepsilon)| dx,$$

where

$$X_m^\varepsilon = \{x \in \Omega : |u_\varepsilon(x)| \leq m\}$$

$$Y_m^\varepsilon = \{x \in \Omega : |u_\varepsilon(x)| > m\}.$$

So

$$\begin{aligned} \int_E |g_\varepsilon(x, u_\varepsilon, Du_\varepsilon)| dx &\leq \int_{E \cap X_m^\varepsilon} |g_\varepsilon(x, u_\varepsilon, Du_\varepsilon)| dx + \frac{1}{m} \int_\Omega g_\varepsilon(x, u_\varepsilon, Du_\varepsilon) u_\varepsilon dx \\ &\leq b(m) \int_E (|Du_\varepsilon|^p + c(x)) dx + c_2 \frac{1}{m}. \end{aligned} \quad (2.27)$$

Since the sequence  $Du_\varepsilon$  strongly converges in  $(L^p(\Omega))^N$ , (2.27) implies the equi-integrability of  $g_\varepsilon(x, u_\varepsilon, Du_\varepsilon)$ . Now (2.25) and Vitali's Theorem yield:

$$g_\varepsilon(x, u_\varepsilon, Du_\varepsilon) \rightarrow g(x, u, Du) \quad \text{strongly in } L^1(\Omega). \quad (2.28)$$

Because of (2.23) and (2.28) it is easy to pass to the limit in

$$\langle A(u_\varepsilon), v \rangle + \int_\Omega g_\varepsilon(x, u_\varepsilon, Du_\varepsilon) v dx = \langle h, v \rangle$$

to obtain

$$\left. \begin{aligned} \langle A(u), v \rangle + \int_\Omega g(x, u, Du) v dx = \langle h, v \rangle \\ \text{for any } v \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega). \end{aligned} \right\} \quad (2.29)$$

Moreover since  $g_\varepsilon(x, u_\varepsilon, Du_\varepsilon) u_\varepsilon \geq 0$  a. e. it follows from (2.25), (2.26) and Fatou's Lemma that

$$g(x, u, Du) u \in L^1(\Omega). \quad (2.30)$$

Thus Theorem 1.1 is proved.

Finally let us note that

$$\langle A(u), u \rangle + \int_{\Omega} g(x, u, Du) u \, dx = \langle h, u \rangle; \quad (2.31)$$

indeed put  $v = u_k$  in (2.29) where  $u_k$  is the truncation of  $u$ . We have

$$\langle A(u) - h, u_k \rangle \rightarrow \langle Au - h, u \rangle$$

and

$$g(x, u, Du) u_k \rightarrow g(x, u, Du) u \quad \text{in } L^1(\Omega),$$

by Lebesgue's dominated convergence Theorem, since

$$|g(x, u, Du) u_k| \leq |g(x, u, Du)| \cdot |u| \in L^1(\Omega),$$

by (2.30), and

$$g(x, u, Du) u_k \rightarrow g(x, u, Du) u \quad \text{a. e.}$$

### 3. VARIATIONAL INEQUALITIES

In this Section we extend our main result (Theorem 1.1) to variational inequalities. Let  $\psi$  be a measurable function with values in  $\bar{\mathbb{R}}$  such that  $\psi^+ \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ ; note that this implies:

$$\{v \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega) : v \geq \psi \text{ a. e.}\} \neq \emptyset. \quad (3.1)$$

We consider the problem

$$\left. \begin{aligned} \langle A(u), v-u \rangle + \int_{\Omega} g(x, u, Du)(v-u) \, dx &\geq \langle h, v-u \rangle \\ \forall v \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega), \quad v &\geq \psi \quad \text{a. e.} \\ u \in W_0^{1,p}(\Omega), \quad u &\geq \psi \quad \text{a. e.} \\ g(x, u, Du) \in L^1(\Omega), \quad u g(x, u, Du) &\in L^1(\Omega) \end{aligned} \right\} \quad (3.2)$$

We have the following result

**THEOREM 3.1.** — *Under the assumptions (1.1), (1.2), (1.3) and (3.1), there exists a solution of (3.2).*

*Proof.* — We follow the developments made in Section 2. We just emphasize the necessary changes. We start with the approximate problem

$$\begin{aligned} \langle A(u_\varepsilon), v - u_\varepsilon \rangle + \int_{\Omega} g_\varepsilon(x, u_\varepsilon, Du_\varepsilon)(v - u_\varepsilon) dx &\geq \langle h, v - u_\varepsilon \rangle \\ \forall v \in W_0^{1,p}(\Omega), \quad v &\geq \psi \quad \text{a. e.} \\ u \in W_0^{1,p}(\Omega), \quad u &\geq \psi \quad \text{a. e.} \end{aligned} \tag{3.3}$$

using the same approximation  $g_\varepsilon$  of  $g$  as in the case of the equation [see (2.1)]. For the existence of a solution of (3.3), see [18].

Using  $v = \psi^+$  as test function in (3.3), we easily deduce that  $u_\varepsilon$  remains in a bounded set of  $W_0^{1,p}(\Omega)$  and that  $\int_{\Omega} g_\varepsilon(x, u_\varepsilon, Du_\varepsilon) u_\varepsilon dx$  remains bounded. Actually this is the only point in the present proof where the hypothesis  $\psi^+ \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ ; is used; with some technicalities it is possible to obtain the same estimates just using (3.1) as hypothesis.

Pick now a subsequence such that

$$u_\varepsilon \rightharpoonup u \quad \text{in } W_0^{1,p}(\Omega) \text{ weakly and a. e.}$$

Note that  $u \geq \psi$  a. e.

We first study the convergence of the positive part  $u_\varepsilon^+$ . Define again

$$z_\varepsilon = u_\varepsilon^+ - u_k^+$$

and consider the test function

$$v = u_\varepsilon - (u_\varepsilon^+ - u_k^+)^+.$$

Since  $k$  will tend to  $+\infty$ , we may without loss of generality assume that

$$k \geq \psi \quad \text{a. e.}$$

which is possible since  $\psi$  is bounded above. By this choice of  $k$ , the above test function is admissible. We recover immediately the inequality (2.10).

We then prove (2.17). We consider for that the test function

$$v_\varepsilon = u_\varepsilon + \phi_\lambda(z_\varepsilon^-)$$

which is clearly admissible. Using this test function we recover (2.15) with the first  $=$  replaced by  $\leq$  and (2.17) follows.

Therefore (2.19) is proved as in the case of the equation.

We turn now to the convergence of the negative part. As in Section 2.3 we define

$$y_\varepsilon = u_\varepsilon^- - u_k^-.$$

We begin proving (2.21). We can use

$$v_\varepsilon = u_\varepsilon + y_\varepsilon^+$$

as a test function in (3.3) because it is clearly admissible. We easily deduce

$$\int_{\Omega} a(x, u_\varepsilon, Du_\varepsilon) Dy_\varepsilon^+ dx \geq \langle h, y_\varepsilon^+ \rangle$$

from which (2.21) follows as in the case of the equation.

The final step is to recover (2.22). Our test function will now be

$$v_\varepsilon = u_\varepsilon - \delta_\varepsilon \phi_\lambda(y_\varepsilon^-)$$

where  $\delta_\varepsilon$  is a positive constant such that  $\delta_\varepsilon E_\varepsilon(x) \leq 1$ , where  $E_\varepsilon = \exp \lambda (y_\varepsilon^-)^2$ ; such a constant exists since  $0 \leq y_\varepsilon^- \leq k$ . Let us check that this test function is admissible. Consider a point where  $\psi \geq 0$ , then  $u_\varepsilon \geq 0$ ,  $u \geq 0$ ,  $y_\varepsilon = 0$ , and  $v_\varepsilon = u_\varepsilon$ . Consider next a point where  $\psi < 0$ ,  $u \geq 0$ , then  $y_\varepsilon = u_\varepsilon^-$ ,  $y_\varepsilon^- = 0$ , and again  $v_\varepsilon = u_\varepsilon$ . Assume from now on  $\psi < 0$ ,  $u < 0$ ; note that  $u^- \leq -\psi$ ,  $u_k^- \leq -\psi$ . Suppose that  $u_\varepsilon \geq 0$ , then  $y_\varepsilon = -u_k^-$ ,  $y_\varepsilon^- = u_k^-$ ,  $v_\varepsilon = u_\varepsilon - \delta_\varepsilon E_\varepsilon u_k^-$ . But  $\delta_\varepsilon E_\varepsilon u_k^- \leq u_k^- \leq -\psi$ , thus  $v_\varepsilon \geq -\delta_\varepsilon E_\varepsilon u_k^- \geq \psi$ . Suppose finally  $u_\varepsilon < 0$  and  $u_\varepsilon^- < u_k^-$  (otherwise  $y_\varepsilon^- = 0$ ), then

$$v_\varepsilon = -u_\varepsilon^- - \delta_\varepsilon E_\varepsilon (u_k^- - u_\varepsilon^-) = -(1 - \delta_\varepsilon E_\varepsilon) u_\varepsilon^- - \delta_\varepsilon E_\varepsilon u_k^-.$$

Noting that  $u_\varepsilon^- \leq -\psi$ ,  $u_k^- \leq -\psi$  we deduce again  $v_\varepsilon \geq \psi$ .

With this choice, we derive (2.22) and (2.20) follows as in the case of the equation.

From the above arguments we can assert that

$$\left. \begin{aligned} u_\varepsilon &\rightarrow u \text{ in } W_0^{1,p}(\Omega) \text{ and a. e.} \\ Du_\varepsilon &\rightarrow Du \text{ a. e.} \end{aligned} \right\} \tag{3.4}$$

By the same argument as in Section 2.4, we deduce again that

$$g_\varepsilon(x, u_\varepsilon, Du_\varepsilon) \rightarrow g(x, u, Du) \quad \text{in } L^1(\Omega),$$

and by Fatou's Lemma that  $g(x, u, Du)u \in L^1(\Omega)$  with

$$\int_{\Omega} g(x, u, Du)u \, dx \leq \underline{\lim} \int_{\Omega} g_\varepsilon(x, u_\varepsilon, Du_\varepsilon)u_\varepsilon \, dx.$$

By passing to the limit in (3.3) with  $v \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ , one recovers immediately (3.2), which completes the proof of Theorem 3.1.

#### ACKNOWLEDGEMENTS

Parts of this work were done at the times the first and the third authors were visiting the University of Roma I and the second author was "Professeur associé" at the University of Paris-Sud-Orsay. The research of the second author was partially supported by M.P.I. (40%) and G.N.A.F.A.-C.N.R.

#### REFERENCES

- [1] H. AMANN and M. G. CRANDALL, On Some Existence Theorems for Semi Linear Elliptic Equations, *Indiana Univ. Math. J.*, Vol. 27, 1978, pp. 779-790.
- [2] A. BENSOUSSAN and J. FREHSE, Nonlinear Elliptic Systems in Stochastic Game Theory, *J. Reine Ang. Math.*, Vol. 350, 1984, pp. 23-67.
- [3] A. BENSOUSSAN, J. FREHSE and U. MOSCO, A Stochastic Impulse Control Problem with Quadratic Growth Hamiltonian and the Corresponding Quasi Variational Inequality, *J. Reine Ang. Math.*, Vol. 331, 1982, pp. 124-145.
- [4] L. BOCCARDO and D. GIACHETTI, Strongly Non Linear Unilateral Problems, *Appl. Mat. Opt.*, Vol. 9, 1983, pp. 291-301.
- [5] L. BOCCARDO and D. GIACHETTI, Alcune osservazioni sulla regolarità delle soluzioni di problemi fortemente non lineari e applicazioni, *Ricerche di Matematica*, Vol. 34, 1985, pp. 309-323.
- [6] L. BOCCARDO, D. GIACHETTI and F. MURAT, On a Generalization of a Theorem of Brezis-Browder, 1984, unpublished.
- [7] L. BOCCARDO, F. MURAT and J. P. PUEL, Existence de solutions non bornées pour certaines équations quasi-linéaires, *Portugaliae Math.*, Vol. 41, 1982, pp. 507-534.
- [8] L. BOCCARDO, F. MURAT and J. P. PUEL, Existence de solutions faibles pour des équations elliptiques quasi-linéaires à croissance quadratique, *Nonlinear Partial Differ-*

- ential Equations and Their Applications, Collège de France Seminar, Vol. IV, J. L. LIONS and H. BREZIS Eds., Research Notes in Mathematics, No. 84, Pitman, London, 1983, pp. 19-73.*
- [9] L. BOCCARDO, F. MURAT and J. P. PUEL, Résultats d'existence pour certains problèmes elliptiques quasi-linéaires, *Ann. Sc. Norm. Sup. Pisa*, Vol. 11, 1984, pp. 213-235.
- [10] L. BOCCARDO, F. MURAT and J. P. PUEL, Existence of Bounded Solutions for Non Linear Elliptic Unilateral Problems, *Annali di Mat. Pura Appl.*, (to appear).
- [11] H. BREZIS and F. E. BROWDER, Some Properties of Higher Order Sobolev Spaces, *J. Math. Pures Appl.*, Vol. 61, 1982, pp. 245-259.
- [12] F. E. BROWDER, Existence Theorems for Non Linear Partial Differential Equations, *Proceedings of Symposia in Pure Mathematics*, Vol. 16, S. S. CHERN and S. SMALE Eds., A.M.S., Providence, 1970, pp. 1-60.
- [13] T. DEL VECCHIO, Strongly Non Linear Problems with Gradient Dependent Lower Order Non Linearity, *Nonlinear Anal. T.M.A.*, Vol. 11, 1987, pp. 5-15.
- [14] J. FREHSE, A Refinement of Rellich's Theorem, *Rendiconti di Matematica*, (to appear).
- [15] P. HESS, Variational Inequalities for Strongly Non Linear Elliptic Operators, *J. Math. Pures Appl.*, Vol. 52, 1973, pp. 285-298.
- [16] R. LANDES, *Solvability of Perturbated Elliptic Equations with Critical Growth Exponent for the Gradient*, Preprint.
- [17] J. LERAY and J. L. LIONS, Quelques résultats de Visik sur les problèmes elliptiques non linéaires par la méthode de Minty-Browder, *Bull. Soc. Math. France*, Vol. 93, 1965, pp. 97-107.
- [18] J. L. LIONS, *Quelques méthodes de résolution des problèmes aux limites non linéaires*, Dunod, Paris, 1969.
- [19] J. M. RAKOTOSON and R. TEMAM, Relative Rearrangement in Quasilinear Variational Inequalities, *Indiana Univ. Math. J.*, 36, 1987, pp. 757-810.
- [20] J. R. L. WEBB, Boundary Value Problems for Strongly Non Linear Elliptic Equations, *J. London Math. Soc.*, Vol. 21, 1980, pp. 123-132.

(Manuscrit reçu le 4 août 1987.)