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# MOREAU'S DECOMPOSITION THEOREM REVISITED

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## ABSTRACT

Given two convex functions  $g$  and  $h$  on a Hilbert space, verifying  $g + h = \frac{1}{2} \|\cdot\|^2$ , we show there necessarily exists a lower-semicontinuous convex function  $F$  such that  $g = F \square \frac{1}{2} \|\cdot\|^2$  and  $h = F^* \square \frac{1}{2} \|\cdot\|^2$ . An explicit formulation of  $F$  is given as a deconvolution of a convex function by another one. The approach taken here as well as the way of factorizing  $g$  and  $h$  shed a new light on what is known as Moreau's theorem in the literature on Convex Analysis.

1 - INTRODUCTION

The starting point of our study was the following question, which takes root in the regularization processes studied in [9]: Let  $(H, \langle \cdot, \cdot \rangle)$  be a Hilbert space, let  $f$  be a function on  $H$  and  $\alpha > 0$  such that

(1.1) both  $\frac{\alpha}{2} \|\cdot\|^2 - f$  and  $\frac{\alpha}{2} \|\cdot\|^2 + f$  are convex functions on  $H$  (Here  $\|\cdot\|$  denotes the norm on  $H$  associated with the inner product  $\langle \cdot, \cdot \rangle$ ).

How to show that  $f$  is Gâteaux-differentiable on  $H$  with

(1.2)  $\|f'(x) - f'(y)\| \leq \alpha \|x-y\|$  for all  $x, y$  in  $H$ ?

The question of differentiability of  $f$  offers no difficulty since it readily comes from (1.1) that both  $g := \frac{\alpha}{2} \|\cdot\|^2 - f$  and  $h := \frac{\alpha}{2} \|\cdot\|^2 + f$  are finite convex functions on  $H$ , so that the directional derivative  $f'(x, \cdot)$  of  $f$  exists and satisfies:

(1.3)  $f'(x, \cdot) = \alpha \langle x, \cdot \rangle - g'(x, \cdot) = h'(x, \cdot) - \alpha \langle x, \cdot \rangle$  for all  $x \in H$ , whence  $f'(x, \cdot)$  is linear and continuous (since convex and concave) for all  $x \in H$ . The problem now is to prove that  $f'$  is Lipschitz on  $H$ , with Lipschitz constant  $\alpha$ . It is clear, in view of (1.1), that  $\alpha$  is the best Lipschitz constant one can expect on  $f'$ . Even if the problem can be reduced (by an argument of projection) to the same problem in a 2-dimensional context (cf.[6]), it is not simpler for all that. So, the question should be broached in a different way.

When reading (1.1), our first reaction is to observe that  $f$  is necessarily a d.c. function (i.e., a difference of convex functions):

(1.4)  $f = \frac{\alpha}{2} \|\cdot\|^2 - g$  or  $f = h - \frac{\alpha}{2} \|\cdot\|^2$ .

D.C. functions enjoy differentiability properties similar to

those of convex functions, but to keep control of their derivatives is hopeless in general ([3, §II.2]). Things are however made easier since one of the functions involved in the decomposition of  $f$  is merely  $\frac{\alpha}{2} \|\cdot\|^2$ . Referring back to (1.4), we see we are in the presence of two convex functions  $g$  and  $h$  such that

$$(1.5) \quad g + h = \alpha \|\cdot\|^2.$$

We thus reformulate the question posed at the beginning in the following way : Let  $g$  and  $h$  be convex functions on  $H$  and  $\alpha > 0$  such that

$$(1.6) \quad g + h = \alpha \|\cdot\|^2$$

Show that both  $g$  and  $h$  are Gâteaux-differentiable on  $H$  with

$$(1.7) \quad \langle g'(x) - g'(y), h'(x) - h'(y) \rangle \geq 0 \text{ for all } x, y \text{ in } H.$$

Let us prove that the two formulations are equivalent.

Suppose we have answered the question in its second formulation and wish to answer it in its first one. Then, posing  $g = \frac{\alpha}{2} \|\cdot\|^2 - f$  and  $h = \frac{\alpha}{2} \|\cdot\|^2 + f$ , we get that  $f$  is differentiable and

$$(1.8) \quad \begin{aligned} &\langle g'(x) - g'(y), h'(x) - h'(y) \rangle \\ &= \alpha^2 \|x-y\|^2 - \|f'(x) - f'(y)\|^2 \geq 0 \text{ for all } x, y \in H, \end{aligned}$$

which is (1.2) precisely.

Conversely, suppose we have answered the question in its original formulation and wish to answer it in its second one.

Posing  $f = \frac{\alpha}{2} \|\cdot\|^2 - g = h - \frac{\alpha}{2} \|\cdot\|^2$ , we indeed have a function

$f$  such that both  $\frac{\alpha}{2} \|\cdot\|^2 + f$  and  $\frac{\alpha}{2} \|\cdot\|^2 - f$  are convex functions on  $H$ . Then, the differentiability of  $f$  induces that of  $g$  and  $h$ , and, in view of (1.8), the inequality (1.2) induces (1.7).

Starting from convex functions  $g$  and  $h$  such that  $g + h = \alpha \|x\|^2$ , we actually can prove more about  $g$  and  $h$ , namely that  $g$  and  $h$  can be *factorized* in the following form :  $g = 2\alpha(F \square \frac{1}{2} \|\cdot\|^2)$  and  $h = 2\alpha(F^* \square \frac{1}{2} \|\cdot\|^2)$  for some lower-semicontinuous convex function  $F$ . As a result,  $g$  and  $h$  will appear as Moreau-Yosida regularized versions of  $F$  and  $F^*$  respectively, so that all the announced properties on  $g$  and  $h$  follow.

## 2 - MOREAU'S DECOMPOSITION THEOREM REVISITED

2.1 - Let  $\Gamma_0(H)$  denote the set of convex functions  $F$  from  $H$  into  $(-\infty, +\infty]$  which are lower-semicontinuous and not identically equal to  $+\infty$ . What is known as Moreau's theorem in the context of Convex Analysis asserts the following : *for any*  $F \in \Gamma_0(H)$

$$(2.1) \quad F \square \frac{1}{2} \|\cdot\|^2 + F^* \square \frac{1}{2} \|\cdot\|^2 = \frac{1}{2} \|\cdot\|^2. \quad ([9])$$

By choosing  $F$  as the indicator function of a closed convex cone  $K$  of  $H$ ,  $F^*$  is the indicator function of the polar cone  $K^\circ$  to  $K$ ,  $F \square \|\cdot\|^2$  is the square of the distance function to  $K$ , so that (2.1) reads as a kind of Pythagore's theorem :

$$(2.2) \quad d_K^2 + d_{K^\circ}^2 = \|\cdot\|^2. \quad ([7,9])$$

Such a decomposition has proved useful in all areas involving a Hilbertian structure (Euclidean spaces of matrices in Statistics, Sobolev spaces in Nonlinear Analysis [7,11], etc).

Our goal now is to prove a sort of *converse* to Moreau's theorem : starting with convex functions  $g$  and  $h$  such that  $g + h = \frac{1}{2} \|\cdot\|^2$ , we want to factorize  $g$  and  $h$  in the form  $F \square \frac{1}{2} \|\cdot\|^2$  and  $F^* \square \frac{1}{2} \|\cdot\|^2$  respectively, by providing also an *explicit* formulation for  $F$ .

THEOREM (of factorization)

Let  $g$  and  $h$  be convex functions on  $H$  such that  $g + h = \frac{1}{2} \|\cdot\|^2$ .  
There then exists  $F \in \Gamma_0(H)$  such that

$$(2.3) \quad g = F \square \frac{1}{2} \|\cdot\|^2 \quad \text{and} \quad h = F^* \square \frac{1}{2} \|\cdot\|^2.$$

Moreover

$$(2.4) \quad g'(x) \in \partial F(h'(x)) \quad \text{and} \quad h'(x) \in \partial F^*(g'(x)) \quad \text{for all} \\ x \in H.$$

Before going into the details of the proof, we need to recall some facts about an operation on convex functions which has been recently introduced ([4]), and which bears the name of *deconvolution of a function by another one*.

Given  $\varphi$  and  $\psi$  in  $\Gamma_0(H)$ , the deconvolution of  $\varphi$  by  $\psi$  is the function denoted  $\varphi \square \psi$  and defined as:

$$\forall x \in H; (\varphi \square \psi)(x) = \sup_{\psi(u) < +\infty} \{\varphi(x+u) - \psi(u)\}.$$

The two main properties to be noticed are :  $\varphi \square \psi \in \Gamma_0(H)$  (or possibly identically equal to  $+\infty$ ) and  $(\varphi \square \psi)^* = (\varphi^* - \psi^*)^{**}$  (see [5] and the references therein).

*Proof of Theorem 1*

We set  $F = g \square \frac{1}{2} \|\cdot\|^2$ , that is :

$$\forall x \in H, F(x) = \sup_{u \in H} \left\{ g(x+u) - \frac{1}{2} \|u\|^2 \right\}.$$

Since  $g + h = \frac{1}{2} \|\cdot\|^2$ , we also have :

$$\forall x \in H, F(x) = \sup_{w \in H} \left\{ g(w) - \frac{1}{2} \|x-w\|^2 \right\}$$

$$\begin{aligned}
&= \sup_{v \in H} \left\{ \frac{1}{2} \|v\|^2 - h(v) - \frac{1}{2} \|x-v\|^2 \right\} \\
&= \sup_{v \in H} \left\{ \langle x, v \rangle - h(v) - \frac{1}{2} \|x\|^2 \right\} \\
&= h^*(x) - \frac{1}{2} \|x\|^2.
\end{aligned}$$

Whence

$$(2.5) \quad F = g \equiv \frac{1}{2} \|\cdot\|^2 = h^* - \frac{1}{2} \|\cdot\|^2 \quad (\in \Gamma_0(H)).$$

By inverting the role of  $g$  and  $h$ , we get in a same way :

$$(2.6) \quad h \equiv \frac{1}{2} \|\cdot\|^2 = g^* - \frac{1}{2} \|\cdot\|^2 \quad (\in \Gamma_0(H)).$$

But the formula giving the conjugate function of  $g \equiv \frac{1}{2} \|\cdot\|^2$  (as aforesaid) yields that

$$\left( g \equiv \frac{1}{2} \|\cdot\|^2 \right)^* = \left( g^* - \frac{1}{2} \|\cdot\|^2 \right)^{**} = g^* - \frac{1}{2} \|\cdot\|^2.$$

Thus, the function defined in (2.6) is nothing else than  $F^*$ . Consequently, the usual calculus rules on conjugate functions, applied to

$$\begin{aligned}
h^* = F + \frac{1}{2} \|\cdot\|^2 \quad \text{and} \quad g^* = F^* + \frac{1}{2} \|\cdot\|^2, \quad \text{induce that} \\
g = F \square \frac{1}{2} \|\cdot\| \quad \text{and} \quad h = F^* \square \frac{1}{2} \|\cdot\|^2.
\end{aligned}$$

Now, calculus rules on subdifferentials, applied to

$$h^* = F + \frac{1}{2} \|\cdot\|^2 \text{ for example, yield that}$$

$$\partial h^*(h'(x)) = \partial F(h'(x)) + \{h'(x)\} \text{ for all } x \in H.$$

But  $x \in \partial h^*(h'(x))$  for all  $x \in H$ , whence

$g'(x) \in \partial F(h'(x))$  for all  $x \in H$ .

*Remark 1* The factorization of  $g$  and  $h$  in the form  $F \square \frac{1}{2} \|\cdot\|^2$  and  $F^* \square \frac{1}{2} \|\cdot\|^2$  respectively, with  $F \in \Gamma_0(H)$ , is unique :

indeed, if  $\phi \in \Gamma_0(H)$  verifies  $\phi \square \frac{1}{2} \|\cdot\|^2 = g$  and

$\phi^* \square \frac{1}{2} \|\cdot\|^2 = h$ , we get that

$$(2.7) \quad \phi = h^* - \frac{1}{2} \|\cdot\|^2 = \left( g^* - \frac{1}{2} \|\cdot\|^2 \right)^*,$$

that is  $\phi = g \square \frac{1}{2} \|\cdot\|^2$ .

*Remark 2.* The dual formulation of the theorem of factorization is as follows : If  $k, \ell \in \Gamma_0(H)$  satisfy

$k \square \ell = \frac{1}{2} \|\cdot\|^2$ , there then exists an unique  $K \in \Gamma_0(H)$  such that

$$k = K + \frac{1}{2} \|\cdot\|^2 \text{ and } \ell = K^* + \frac{1}{2} \|\cdot\|^2.$$

*Example.* Let  $S$  be a nonempty closed convex set of  $H$ . We have that

$$\underbrace{\frac{1}{2} d_S^2}_g + \underbrace{\frac{1}{2} (\|\cdot\|^2 - d_S^2)}_h = \frac{1}{2} \|\cdot\|^2.$$

It is known that  $h = \frac{1}{2} (\|\cdot\|^2 - d_S^2)$  is convex ([1]) (\*). Then the only solution  $F$  yielded by the factorization theorem is  $F = \phi_S$  (the indicator function of  $S$ ). Note incidentally the pairing result :

$$(2.8) \quad \frac{1}{2} (\|\cdot\|^2 - d_S^2) = \phi_S^* \square \frac{1}{2} \|\cdot\|^2,$$

which also can be obtained from direct calculations or as an example of Moreau's theorem (cf. (2.1)).



## 2.2. Applications

2.2.1. As a first application of the factorization theorem, we look back at the question posed in the Introduction and which motivated our study.

Consider two convex functions  $g$  and  $h$  on  $H$ ,  $\alpha > 0$ , such that  $g + h = \alpha \|\cdot\|^2$ . According to the factorization theorem, there exists a unique  $F \in \Gamma_0(H)$  such that :

$$g/2\alpha = F \square \frac{1}{2} \|\cdot\|^2 \text{ and } h/2\alpha = F^* \square \frac{1}{2} \|\cdot\|^2, \\ g'(x) \in \partial F(h'(x)) \text{ for all } x \in H.$$

Due to the monotonicity property of  $\partial F$ , the second relation above induces that

$$\langle g'(x) - g'(y), h'(x) - h'(y) \rangle \geq 0 \text{ for all } x \in H,$$

which is the relation (1.7) required.

2.2.2. A second application of the factorization theorem is the following result.

*COROLLARY 2. Let  $f : H \rightarrow \mathbb{R}$  be a Gâteaux-differentiable function and  $\alpha > 0$ . Then the next statements are equivalent :*

$$(2.9) \quad |\langle f'(x) - f'(y), x-y \rangle| \leq \alpha \|x-y\|^2 \text{ for all } x, y \in H ;$$

$$(2.10) \quad \|f'(x) - f'(y)\| \leq \alpha \|x-y\| \text{ for all } x, y \in H.$$

Although it was known for  $C^2$  - functions, this equivalence is rather surprising ; clearly, (2.9) which involves  $f$  on line segments is easier to check.

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(\*) Actually,  $h$  is convex whatever  $S$  be. But to ensure the convexity of  $g$  also, we need the convexity of  $S$ .

To prove that (2.9) implies (2.10), it suffices to observe that both  $\frac{\alpha}{2} \|\cdot\|^2 - f$  and  $\frac{\alpha}{2} \|\cdot\|^2 + f$  are convex functions on  $H$ ; (2.10) then follows from the equivalence properties stated in the Introduction.

Corollary 2 answers a question the first author alluded to in [3, p. 48 bottom] concerning the comparison between (globally)  $C^{1,1}$  functions  $f$  and those satisfying an inequality like (2.9).

2.3. A third application of the factorization theorem is a characterization of the so-called  $\alpha$ -strongly convex functions. We recall that, given  $\alpha > 0$ ,  $f \in \Gamma_0(H)$  is said to be  $\alpha$ -strongly convex (or strongly convex with modulus  $\alpha$ ) if

$$f(tx + (1-t)x') \leq t f(x) + (1-t) f(x') - \frac{\alpha}{2} t(1-t) \|x-x'\|^2$$

for all  $x, x'$  in  $H$  and  $t \in ]0,1[$ . In other words, that means that  $f - \frac{\alpha}{2} \|\cdot\|^2$  is still a convex function ( $\in \Gamma_0(H)$ ). The next characterization of  $\alpha$ -strongly convex functions has also been observed by Volle ([10]) who, furthermore, introduced a new conjugacy mapping for such functions by substituting the "coupling functional"

$$(x, y) \mapsto \frac{\alpha}{2} \|x-y\|^2 \text{ for the usual bilinear functional}$$

$$(x, y) \mapsto \langle x, y \rangle.$$

COROLLARY 3. Let  $f \in \Gamma_0(H)$ . The following are equivalent :

(2.11)  $f$  is  $\alpha$ -strongly convex ;

(2.12)  $\frac{1}{2\alpha} \|\cdot\|^2 - f^* \in \Gamma_0(H)$  ;

(2.13) There exists  $\varphi \in \Gamma_0(H)$  such that  $f \square \varphi = \frac{\alpha}{2} \|\cdot\|^2$ .

Condition (2.12) actually says more than what is stated : since  $f^*$  is itself in  $\Gamma_0(H)$ , condition (2.12) implies that

$f^*$  is finite on  $H$ ; in fact we will see in the course of the proof that  $f^*$  is a  $C^{1,1}$  function (\*).

Likewise, a consequence of (2.13) is that  $\varphi^* = \frac{1}{2\alpha} \|\cdot\|^2 - f^*$ , whence the exhibited function  $\varphi$  is  $\alpha$ -strongly convex; indeed,

$$(2.14) \quad \varphi = \left( \frac{1}{2\alpha} \|\cdot\|^2 - f^* \right)^* = \frac{\alpha}{2} \|\cdot\|^2 \equiv f,$$

$$(2.15) \quad \psi = \left( \frac{1}{2\alpha} \|\cdot\|^2 - \varphi^* \right)^* = \frac{\alpha}{2} \|\cdot\|^2 \equiv \varphi.$$

*Proof.* (2.12)  $\Rightarrow$  (2.11). Let  $g$  denote the convex function  $\frac{1}{2\alpha} \|\cdot\|^2 - f^*$ . Since  $\alpha g + \alpha f^* = \frac{1}{2} \|\cdot\|^2$ , the theorem of

factorization yields that there exists  $F \in \Gamma_0(H)$  such that

$\alpha f^* = F \square \frac{1}{2} \|\cdot\|^2$ . Consequently,  $f$  assigns

$\frac{1}{\alpha} F^*(\alpha x) + \frac{\alpha}{2} \|x\|^2$  to  $x \in H$ , so that  $f - \frac{\alpha}{2} \|\cdot\|^2$  is still a convex function. We thus have proved  $f$  is  $\alpha$ -strongly convex.

(2.11)  $\Rightarrow$  (2.13). Let  $\chi$  denote the convex function  $\frac{f}{\alpha} - \frac{1}{2} \|\cdot\|^2$ ; we set  $\varphi = \alpha \chi^* + \frac{\alpha}{2} \|\cdot\|^2$ . Starting from the

relation  $\frac{f}{\alpha} = \chi + \frac{1}{2} \|\cdot\|^2$ ,

we get successively

(\*) The equivalence of (2.11) and (2.12) appears also as a by-product of more general results on the duality relations between uniformly convex functions and uniformly smooth convex functions ([2]).

$$\begin{aligned} \left(\frac{f}{\alpha}\right)^* &= x^* \square \frac{1}{2} \|\cdot\|^2 \\ (2.16) \qquad &= \frac{1}{2} \|\cdot\|^2 - \left(x \square \frac{1}{2} \|\cdot\|^2\right) \quad \text{by Moreau's} \\ &\text{theorem.} \end{aligned}$$

Let us calculate  $g = \left(\frac{f}{\alpha}\right) \square \left(\frac{\varphi}{\alpha}\right)$ . Since  $g^* = \left(\frac{f}{\alpha}\right)^* + \left(\frac{\varphi}{\alpha}\right)^*$ , we infer from the definition of  $\varphi$  and (2.16) :

$$g^* = \frac{1}{2} \|\cdot\|^2 - \left(x \square \frac{1}{2} \|\cdot\|^2\right) + x \square \frac{1}{2} \|\cdot\|^2 = \frac{1}{2} \|\cdot\|^2.$$

Whence  $g = \frac{1}{2} \|\cdot\|^2$  and (2.13) is secured.

(2.13)  $\Rightarrow$  (2.12) From  $f \square \varphi = \frac{\alpha}{2} \|\cdot\|^2$  we derive

$$f^* + \varphi^* = \frac{1}{2\alpha} \|\cdot\|^2, \text{ so that } \frac{1}{2\alpha} \|\cdot\|^2 - f^* = \varphi^* \in \Gamma_0(H). \quad \blacksquare$$

### 3 - COMPARISON WITH MOREAU'S APPROACH

In his seminal 1965 paper ([8]), Moreau extensively studied the functions of the form  $F \square \frac{1}{2} \|\cdot\|^2$ ,  $F \in \Gamma_0(H)$ , and defined the so-called *proximal mapping*  $\text{prox}_F$  which assigns to  $x \in H$  the unique point where the infimum of  $u \mapsto F(u) + \frac{1}{2} \|x - u\|^2$  is achieved. Among other properties, he proved that  $\text{prox}_F$  is a Lipschitz mapping (with Lipschitz constant 1) and that  $\text{prox}_F$  is actually a gradient mapping (i.e., there is a differentiable function  $\phi$ , called primitive function of  $\text{prox}_F$ , such that  $\phi'(x) = \text{prox}_F(x)$  for all  $x \in H$ ).

In a much less read section ([8, §9]), Moreau introduced a binary relation between convex functions by defining what he meant by "a convex function  $g$  less convex than a convex function  $f$ ". More interesting is the characterization of such a

relationship when  $f$  is  $\frac{1}{2} \|\cdot\|^2$  precisely, which now allows us to make connections with our approach.

According to Moreau ([8, définition 9.b]), a convex function  $g$  is less convex than a convex function  $f$  (or  $f$  is more convex than  $g$ ) if there exists a convex function  $h$  such that  $f = g + h$ . He then proved the equivalence of the following properties ([8, Proposition 9.b and Proposition 10.b]) :

(3.1)  $g \in \Gamma_o(H)$  is less convex than  $\frac{1}{2} \|\cdot\|^2$  ;

(3.2) The conjugate function of  $g \in \Gamma_o(H)$  is more convex than  $\frac{1}{2} \|\cdot\|^2$  ;

(3.3)  $g$  is the primitive function of a proximal mapping ;

(3.4)  $g \in \Gamma_o(H)$  is differentiable and  $g'$  is Lipschitz on  $H$  with a Lipschitz constant 1.

(3.1) expresses the existence of a convex function  $h$  such that  $g + h = \frac{1}{2} \|\cdot\|^2$ , which is precisely the situation we have considered here. According to (3.4), such a  $g$  is differentiable and  $\|g'(x) - g'(y)\| \leq \|x-y\|$  for all  $x, y \in H$  ; the property we were looking for from the beginning is stronger, namely :  $\|g'(x) - g'(y) - \frac{x-y}{2}\| \leq \frac{1}{2} \|x-y\|$  (cf. Introduction).

Moreover, the factorization of  $g$  (and  $h$ ) does not appear explicitly and a characterization like (3.3) uses heavily the properties of the proximal mapping.

Our approach, based on the deconvolution operation, allowed us to get at an explicit formulation of  $F$  in the factorization theorem (Theorem 1), thereby shedding a new light on Moreau's theorem.

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