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Homoclinic orbits for a singular second order Hamiltonian system

by

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ABSTRACT. — We consider the second order Hamiltonian system:

$$\ddot{q}+V'(q)=0 \tag{HS}$$

where $q=(q_1, \ldots, q_N)\in \mathbf{R}^N (N \geqq 3)$ and $V:\mathbf{R}^N \setminus \{e\} \to \mathbf{R} (e \in \mathbf{R}^N)$ is a potential with a singularity, *i. e.*, $|V(q)| \to \infty$ as $q \to e$. We prove the existence of a homoclinic orbit of (HS) under suitable assumptions. Our main assumptions are the strong force condition of Gordon [8] and the uniqueness of a global maximum of V.

RÉSUMÉ. — On considère le système hamiltonien du second ordre

$$\ddot{q}+V'(q)=0$$

où $q=(q_1, \ldots, q_N)\in \mathbf{R}^N (N \geqq 3)$ et $V:\mathbf{R}^N \setminus \{e\} \to \mathbf{R}$ est un potentiel singulier : $|V(q)| \to \infty$ quand $q \to e$. On montre alors l'existence d'une orbite homocline, sous l'hypothèse dite de « strong farce » (Gordon [8]) et à condition que le maximum de V soit unique.

Mots clés : Homoclinic orbit, singular potential, critical point, minimax argument.

Classification A.M.S. : 58 E 05, 58 F 05.

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0. INTRODUCTION

The purpose of this paper is to study the existence of homoclinic orbits for a singular Hamiltonian system:

$$\ddot{q}+V'(q)=0. \tag{HS}$$

Here $q=(q_1, q_2, \ldots, q_N)$, $N \geqq 3$, $V:\mathbf{R}^N \setminus S \to \mathbf{R}$ and V is singular on S, *i. e.*, $|V(q)| \to \infty$ as $q \to S$. We assume that S is a single point, *i. e.*, $S=\{e\}$, $e \neq 0$. (Slight modifications of our method permit us to treat more general compact sets S. *See* Remark 3.3). We also assume $V:\mathbf{R}^N \setminus \{e\} \to \mathbf{R}$ has a unique global maximum $0 \in \mathbf{R}^N$, and we consider the existence of a homoclinic orbit which begins and ends at 0, *i. e.*, a solution of (HS) which satisfies

$$q(t),\ \dot{q}(t) \to 0 \quad \text{as } t \to \pm\infty.$$

More precisely, our assumptions on V are as follows:

(V1) There is an $e \in \mathbf{R}^N$, $e \neq 0$ and $V \in C^2(\mathbf{R}^N \setminus \{e\}, \mathbf{R})$;

(V2) $V(q) \leqq 0$ for all $q \in \mathbf{R}^N \setminus \{e\}$ and $V(q)=0$ if and only if $q=0$, and $\limsup_{|q| \to \infty} V(q) \equiv \bar{V} < 0$;

(V3) There is a constant $\delta \in \left(0, \frac{1}{2}|e|\right)$ such that $V(q)+\frac{1}{2}(V'(q), q) \leqq 0$ for all $q \in B_\delta(0)$, where $B_\delta(0)=\{x \in \mathbf{R}^N;\ |x|<\delta\}$;

(V4) $-V(q) \to \infty$ as $q \to e$;

(V5) There is a neighbourhood W of e in $\mathbf{R}^N$ and a function $U \in C^1(W \setminus \{e\}, \mathbf{R})$ such that $U(q) \to \infty$ as $q \to e$ and $-V(q) \geqq |U'(q)|^2$ for $q \in W \setminus \{e\}$.

Now we can state our main result.

THEOREM 0.1. – *If* V *satisfies* (V1)-(V5), *then* (HS) *possesses at least one* (*nontrivial*) *homoclinic orbit which begins and ends at* 0.

Remark 0.2. – The assumption (V5) is the so-called *strong force* condition (*cf.* Gordon [8]) and it will be used to verify the Palais-Smale compactness condition for the functional corresponding to the approximate problem (HS : T) (*see* Proposition 1.1). For example, (V5) is satisfied when $V(q)=-|q-e|^{-\alpha}$ $(\alpha \geqq 2)$ in a neighbourhood of e. The assumption (V3) is a kind of concavity condition for $V(q)$ near 0. In particular (V3) holds for small $\delta>0$ when $V''(0)$ is negative definite.

This work is largely motivated by the work of Rabinowitz [12] and the works [1-7]. [12] studied via a variational method the existence and the multiplicity of *heteroclinic* orbits joining global maxima of $V(q)$ for a periodic Hamiltonian system. On the other hand [1-7] studied the existence of time periodic solutions of prescribed period for the second order singular Hamiltonian system (HS).

The proof of Theorem 0.1 will be given in the following sections. We consider the approximate problem:

$$\left.\begin{aligned} \ddot{q}+V'(q)=0, \quad \text{in } (0,T), \\ q(0)=q(T)=0. \end{aligned}\right\} \qquad \text{(HS:T)}$$

Solutions of this approximate problem will be obtained as critical points of the functional $I_T(q)$ (*see* Section 1). We show the existence of critical points of $I_T(q)$ via a minimax argument, which is essentially due to Bahri and Rabinowitz [4] (*see* also Lyusternik and Fet [10] *cf.* Klingenberg [9]). We also get some estimates, which are uniform with respect to $T \geqq 1$, for minimax values and corresponding critical points $q(t; T)$. These uniform estimates permit us to let $T \to \infty$; for a suitable sequence $(\tau_k)_{k=1}^{\infty}$ and a subsequence $T_k \to \infty$, we see $q(t+\tau_k; T_k)$ converges weakly to a homoclinic orbit of (HS) as $k \to \infty$.

1. APPROXIMATE PROBLEM

In this section, we solve the approximate problem (HS:T) via a minimax argument. Let $H_0^1(0,T; \mathbf{R}^N)$ denote the usual Sobolev space on (0, T) with values in $\mathbf{R}^N$ under the norm $\|q\| = \left(\int_0^T |\dot{q}|^2\, dt\right)^{1/2}$. Let

$$\Lambda_T = \{ q \in H_0^1(0,T; \mathbf{R}^N);\ q(t) \neq e \text{ for all } t \in [0,T] \}.$$

Clearly Λ_T is an open subset of $H_0^1(0,T; \mathbf{R}^N)$. Consider

$$I_T(q) = \frac{1}{2}\int_0^T |\dot{q}|^2\, dt - \int_0^T V(q)\, dt \in C^1(\Lambda_T, \mathbf{R}).$$

Then there is an one-to-one correspondence between critical points of $I_T(q)$ and classical solutions of (HS:T).

To obtain a critical point of $I_T(q)$, we use a minimax argument. To do so, $I_T(q)$ must satisfy the Palais-Smale compactness condition (P.S.) on Λ_T.

(P.S.): if $(q_m)_{m=1}^{\infty} \subset \Lambda_T$ is a sequence such that $I_T(q_m)$ is bounded and $I'_T(q_m) \to 0$, then (q_m) possesses a subsequence converging to some $q \in \Lambda_T$.

PROPOSITION 1.1. – *If* $V(q)$ *satisfies* (V1), (V2), (V4), (V5), *then* $I_T(q)$ satisfies (P.S.).

Proof. – Let $(q_m) \subset \Lambda_T$ be a sequence such that $I_T(q_m)$ is bounded and $I'_T(q_m) \to 0$. Then by (V2) and the definition of $I_T(q)$, (q_m) is bounded in $H_0^1(0,T; \mathbf{R}^N)$. Hence we can extract a subsequence of (q_m) – still we denote by (q_m) – such that q_m converges to $q \in \overline{\Lambda_T}$ weakly in $H_0^1(0,T; \mathbf{R}^N)$. On the other hand, by Greco ([6], Lemma 2.1), if $q \in \partial\Lambda_T$, then $-\int_0^T V(q_m)\,dt \to \infty$, *i. e.*, $I_T(q_m) \to \infty$. Hence $q \in \Lambda_T$. Therefore the form of $I'_T(q)$ shows $q_m \to q$ strongly in $H_0^1(0, T; \mathbf{R}^N)$. ■

Now we introduce a minimax procedure for $I_T(q)$. Let

$$D^{N-2} = \{ x \in \mathbf{R}^{N-2};\ |x| \leqq 1 \},$$
$$\Gamma_T = \{ \gamma \in C(D^{N-2}, \Lambda_T);\ \gamma(x)(t) = 0 \text{ for all } x \in \partial D^{N-2} \text{ and } t \in [0, T] \}.$$

For $\gamma \in \Gamma_T$ we observe $\gamma(x)(t) = 0$ for all

$$(x, t) \in (\partial D^{N-2} \times [0, T]) \cup (D^{N-2} \times \{0, T\}) \equiv \partial(D^{N-2} \times [0, T]).$$

Since $D^{N-2} \times [0, T]/\partial(D^{N-2} \times [0, T]) \simeq S^{N-1}$, we can associate for each $\gamma \in \Gamma_T$ a map $\tilde{\gamma}: S^{N-1} \to S^{N-1}$ defined by

$$\tilde{\gamma}(x, t) = \frac{\gamma(x)(t) - e}{|\gamma(x)(t) - e|}.$$

We denote by $\deg \tilde{\gamma}$ the Brouwer degree of a map $\tilde{\gamma}: S^{N-1} \to S^{N-1}$. Let

$$\Gamma_T^* = \{ \gamma \in \Gamma_T;\ \deg \tilde{\gamma} \neq 0 \}.$$

It is clear that $\Gamma_T^* \neq \varnothing$. We define a minimax value of $I_T(q)$ by

$$c(T) = \inf_{\gamma \in \Gamma_T^*} \sup_{x \in D^{N-2}} I_T(\gamma(x)).$$

Then we have

PROPOSITION 1.2. — *$c(T) > 0$ is a critical value of $I_T(q)$.*

Proof. — We will see later that $c(T) > 0$ in Proposition 1.4. Here we assume it and prove that $c(T)$ is a critical value of $I_T(q)$. Since $I_T(q)$ satisfies (P.S.), we have the following deformation theorem (*cf.* Rabinowitz [11]).

LEMMA 1.3. — *Suppose that c is not a critical value of $I_T(q)$. Then for all $\bar{\varepsilon} > 0$ there are an $\varepsilon \in (0, \bar{\varepsilon})$ and $\eta \in C([0, 1] \times \Lambda_T, \Lambda_T)$ such that*

1° $\eta(1, q) = q$ *if* $I_T(q) \notin (c - \bar{\varepsilon}, c + \bar{\varepsilon})$;

2° $I_T(\eta(\tau, q)) \leqq I_T(q)$ *for all* $(\tau, q) \in [0, 1] \times \Lambda_T$;

3° $\eta(1, [I_T \leqq c + \varepsilon]) \subset [I_T \leqq c - \varepsilon]$, *where we use the notation:*

$$[I_T \leqq b] = \{ q \in \Lambda_T;\ I_T(q) \leqq b \}. \quad ■$$

Arguing indirectly, we suppose $c(T) > 0$ is not a critical value. Applying Lemma 1.3 to $c = c(T) > 0$ and $\bar{\varepsilon} = c/2$, we have a deformation flow $\eta(\tau, q)$ with the properties 1°-3°. Moreover, we have

$$\eta(1, \Gamma_T^*) \subset \Gamma_T^*. \tag{1.1}$$

In fact, since $\eta(1, 0) = 0$ (by 1°), we have $\eta(1, \gamma(x)) = 0$ for $x \in \partial D^{N-2}$. On the other hand, we have by 2°

$$I_T(\eta(\tau, \gamma(x))) \leqq I_T(\gamma(x)) < \infty \text{ for all } \gamma \in \Gamma_T^* \text{ and } \tau \in [0, 1].$$

Hence $\eta(\tau, \gamma(x))(t) \neq e$ for all $(x, t) \in D^{N-2} \times [0, T]$ and $\tau \in [0, 1]$. Thus we have

$$\deg \widetilde{\eta(1, \gamma)} = \deg \widetilde{\eta(0, \gamma)} = \deg \tilde{\gamma} \neq 0.$$

Therefore $\eta(1, \gamma) \in \Gamma_T^*$ for $\gamma \in \Gamma_T^*$, that is, we have (1.1).

Choose $\gamma \in \Gamma_T^*$ such that $\max_{x \in D^{N-2}} I_T(\gamma(x)) \leqq c + \varepsilon$ and consider $\eta(1, \gamma(x)) \in \Gamma_T^*$. Then by 3°, we have

$$\max_{x \in D^{N-2}} I_T(\eta(1, \gamma(x))) \leqq c - \varepsilon.$$

This contradicts with $c = c(T)$. Therefore $c(T) > 0$ is a critical value of $I_T(q)$. ■

PROPOSITION 1.4. – *There is a constant $c_0 > 0$ which is independent of $T \geqq 1$ such that*

$$0 < c_0 \leqq c(T) \leqq c(1) \quad \textit{for all } T \geqq 1.$$

Proof. – For any given $\gamma \in \Gamma_1^*$, we define $\gamma^T \in \Gamma_T (T \geqq 1)$ by

$$\gamma^T(x)(t) = \begin{cases} \gamma(x)(t), & \text{for } (x, t) \in D^{N-2} \times [0, 1]; \\ 0, & \text{for } (x, t) \in D^{N-2} \times (1, T]. \end{cases}$$

Then we can easily see the following

1° $\deg \tilde{\gamma}^T = \deg \tilde{\gamma} \neq 0$, that is, $\gamma^T \in \Gamma_T^*$ for all $\gamma \in \Gamma_1^*$;

2° $I_T(\gamma^T(x)) = I_1(\gamma(x))$ for all $x \in D^{N-2}$ and $\gamma \in \Gamma_1^*$.

Therefore we get

$$\begin{aligned} c(T) &= \inf_{\gamma \in \Gamma_T^*} \max_{x \in D^{N-2}} I_T(\gamma(x)) \\ &\leqq \inf_{\gamma \in \Gamma_1^*} \max_{x \in D^{N-2}} I_T(\gamma^T(x)) \\ &= \inf_{\gamma \in \Gamma_1^*} \max_{x \in D^{N-2}} I_1(\gamma(x)) \\ &= c(1). \end{aligned} \tag{1.2}$$

Next we prove the existence of a constant $c_0 > 0$ such that $c(T) \geqq c_0$ for all $T \geqq 1$. For any given $\gamma \in \Gamma_T^*$, we have

$$\{\gamma(x)(t); (x, t) \in D^{N-2} \times [0, T]\} \cap (\mathbf{R}^N \setminus B_{2\delta}(0)) \neq \varnothing.$$

Otherwise, we can easily see that $\deg \tilde{\gamma} = 0$. Hence there is $(x_0, t_0) \in D^{N-2} \times [0, T]$ such that

$$\gamma(x_0)(t_0) \notin B_{2\delta}(0).$$

Since $\gamma(x_0)(0) = 0$, there is an $s_0 \in (0, t_0)$ such that

$$\gamma(x_0)(s_0) \in \partial B_\delta(0), \qquad \gamma(x_0)(t) \notin B_\delta(0) \quad \text{for all } t \in (s_0, t_0).$$

By the Schwarz inequality, we have for $q(t)=\gamma(x_0)(t)$

$$
\begin{aligned}
I_T(\gamma(x_0)) &\geqq \int_{s_0}^{t_0} \frac{1}{2}|\dot{q}|^2\,dt + \int_{s_0}^{t_0} -V(q)\,dt \\
&\geqq \frac{1}{2(t_0-s_0)}\left(\int_{s_0}^{t_0}|\dot{q}|\,dt\right)^2 + m_\delta(t_0-s_0) \\
&\geqq \frac{1}{2(t_0-s_0)}|q(t_0)-q(s_0)|^2 + m_\delta(t_0-s_0) \\
&\geqq (2m_\delta)^{1/2}|q(t_0)-q(s_0)| \\
&\geqq (2m_\delta)^{1/2}\delta \equiv c_0,
\end{aligned}
$$

where

$$m_\delta = \min_{x\in\mathbf{R}^N\setminus B_\delta(0)} -V(x) > 0.$$

Thus we have

$$\max_{x\in D^{N-2}} I_T(\gamma(x)) \geqq c_0 \quad \text{for all } \gamma\in\Gamma_T^*,$$

i. e.,

$$c(T)\geqq c_0. \tag{1.4}$$

By (1.2) and (1.4) we obtain the desired result. ■

From Proposition 1.2 and 1.4, we deduce the following.

PROPOSITION 1.5. – *For* $T\geqq 1$, *the problem* (HS:T) *has a solution* $q(t;T)$ *such that*

$$0<c_0\leqq I_T(q(\,.\,;T))\leqq c_1<\infty \qquad \textit{for} \quad T\geqq 1, \tag{1.5}$$

where c_0, $c_1>0$ *are independent of* $T\geqq 1$. ■

2. SOME ESTIMATES FOR SOLUTIONS $q(t;T)$

Clearly from the definition of $I_T(q)$ and Proposition 1.5, we have

LEMMA 2.1. – *There is a constant* $C>0$ *which is independent of* $T\geqq 1$ *such that*

$$\|\dot{q}(\,.\,;T)\|_{L^2(0,T)},\ \int_0^T -V(q(t;T))\,dt\leqq C \quad \textit{for all}\ T\geqq 1. \quad ■$$

In what follows, we denote by C, C′, . . ., various constants which are independent of $T\geqq 1$.

PROPOSITION 2.2. — $\| q(\,.\,;\mathrm{T}) \|_{\mathrm{L}^\infty(0,\mathrm{T})} \leqq \mathrm{C}$ *for all* $\mathrm{T} \geqq 1$.

Proof. — Suppose that $q(t;\mathrm{T}) \notin \overline{\mathrm{B}_\delta(0)}$ for some $t \in (0, \mathrm{T})$. We can find an interval $(s, t) \subset (0, t)$ such that $q(s;\mathrm{T}) \in \partial \mathrm{B}_\delta(0)$ and $q(\tau;\mathrm{T}) \notin \mathrm{B}_\delta(0)$ for all $\tau \in (s, t)$. Then

$$\begin{aligned}\int_s^t |\dot{q}(\tau;\mathrm{T})|\,d\tau &\leqq (t-s)^{1/2}\left(\int_s^t |\dot{q}(\tau;\mathrm{T})|^2\,d\tau\right)^{1/2}\\ &\leqq (t-s)^{1/2}\,\|\dot{q}(\,.\,;\mathrm{T})\|_{\mathrm{L}^2(0,\mathrm{T})}\\ &\leqq \mathrm{C}\,(t-s)^{1/2}.\end{aligned}$$

On the other hand,

$$t-s \leqq \frac{1}{m_\delta}\int_s^t -\mathrm{V}(q(\tau;\mathrm{T}))\,d\tau \leqq \frac{1}{m_\delta}\int_0^\mathrm{T} -\mathrm{V}(q(\tau;\mathrm{T}))\,d\tau \leqq \mathrm{C}'$$

where $m_\delta > 0$ is a constant defined in (1.3).

Combining the above two inequalities, we get

$$\int_s^t |\dot{q}(\tau;\mathrm{T})|\,d\tau \leqq \mathrm{C}''.$$

Thus we have

$$|q(t;\mathrm{T})| \leqq |q(s;\mathrm{T})| + \int_s^t |\dot{q}(\tau;\mathrm{T})|\,d\tau \leqq \delta + \mathrm{C}''.$$

Therefore we conclude

$$\| q(\,.\,;\mathrm{T}) \|_{\mathrm{L}^\infty(0,\mathrm{T})} \leqq \delta + \mathrm{C}''. \quad \blacksquare$$

Since $q(t;\mathrm{T})$ is a classical solution of (HS:T), we can see

$$\mathrm{E_T} \equiv \frac{1}{2}|\dot{q}(t;\mathrm{T})|^2 + \mathrm{V}(q(t;\mathrm{T})) \tag{2.1}$$

is constant in time $t \in [0, \mathrm{T}]$. Moreover we have

LEMMA 2.3. — $E_T \to 0$ *as* $\mathrm{T} \to \infty$. *In particular,*

$$|\dot{q}(0;\mathrm{T})| = |\dot{q}(\mathrm{T};\mathrm{T})| \to 0 \qquad \textit{as } \mathrm{T} \to \infty. \tag{2.2}$$

Proof. — Integrate (2.1) over (0, T), we have by Lemma 2.1

$$\mathrm{TE_T} = \frac{1}{2}\|\dot{q}(\,.\,;\mathrm{T})\|^2_{\mathrm{L}^2(0,\mathrm{T})} - \int_0^\mathrm{T} -\mathrm{V}(q(t;\mathrm{T}))\,dt \leqq \mathrm{C} \text{ for all } \mathrm{T} \geqq 1.$$

Hence we get $\mathrm{E_T} \to 0$ as $\mathrm{T} \to \infty$. Since $q(0;\mathrm{T}) = q(\mathrm{T};\mathrm{T}) = 0$, we have

$$\mathrm{E_T} = \frac{1}{2}|\dot{q}(0;\mathrm{T})|^2 = \frac{1}{2}|\dot{q}(\mathrm{T};\mathrm{T})|^2 \to 0 \quad \text{as } \mathrm{T} \to \infty. \quad \blacksquare$$

The following proposition gives us an L^∞-bound from below for $q(t; T)$. This is an only place that the condition (V3) plays an role.

PROPOSITION 2.4. – $\| q(\,.\,; T) \|_{L^\infty(0,T)} \geqq \delta$ *for all* $T \geqq 1$.

Proof. – Using (2.1), we get

$$\begin{aligned} \frac{1}{2}\frac{d^2}{dt^2}|q(t;T)|^2 &= \frac{d}{dt}(\dot q(t;T), q(t;T)) \\ &= |\dot q(t;T)|^2 - (V'(q), q) \\ &= -2V(q(t;T)) - (V'(q(t;T)), q(t;T)) + 2E_T. \end{aligned}$$

We observe $E_T = \frac{1}{2}|\dot q(0;T)|^2 > 0$. Otherwise $\dot q(0;T)=0$ and then we have $q(t;T)\equiv 0$ by the uniqueness of the solution of the initial value problem:

$$\ddot q + V'(q) = 0, \quad q(0) = \dot q(0) = 0.$$

But this contradicts with $I_T(q(\,.\,;T)) = c(T) > 0$. Therefore by (V3), we have

$$\frac{d^2}{dt^2}|q(t;T)|^2 > 0 \quad \text{whenever } q(t;T) \in B_\delta(0).$$

Suppose that $|q(t;T)|^2$ takes its maximum at $t_0 \in (0, T)$. From the above inequality we deduce $q(t_0;T) \notin B_\delta(0)$. Thus we have

$$\| q(\,.\,;T) \|_{L^\infty(0,T)} \geqq \delta \quad \text{for all } T \geqq 1. \quad \blacksquare$$

By the above Proposition 2.4, we can find two numbers $0 < \tau_T^1 \leqq \tau_T^2 < T$ such that $q(\tau_T^1; T)$, $q(\tau_T^2; T) \in \partial B_\delta(0)$ and $q(t;T) \in B_\delta(0)$ for all $t \in [0, \tau_T^1] \cup [\tau_T^2, T]$.

Then we have

LEMMA 2.5. – $\tau_T^1,\ T - \tau_T^2 \to \infty$ as $T \to \infty$.

Proof. – Let $q_a(t)$ be a solution of the following initial value problem:

$$\ddot q + V'(q) = 0, \qquad q(0) = 0, \qquad \dot q(0) = a.$$

By the continuous dependence of $q_a(t)$ on the initial data a, for any $l > 0$ there is an $\varepsilon > 0$ such that

$$q_a(t) \in B_\delta(0) \qquad \text{for} \quad t \in [0, l] \quad \text{and} \quad |a| \leqq \varepsilon.$$

Thus by (2.2), for any $l > 0$ we can find $T_l \geqq 1$ such that

$$q(t;T) \in B_\delta(0) \qquad \text{for} \quad t \in [0, l] \quad \text{and} \quad T \geqq T_l.$$

i. e., $\tau_T^1 \geqq l$ for $T \geqq T_l$. Therefore we have $\tau_T^1 \to \infty$ as $T \to \infty$. Similarly we have $T - \tau_T^2 \to \infty$ as $T \to \infty$. $\blacksquare$

3. LIMIT PROCESS AND PROOF OF THEOREM 0.1

In this section, we construct a homoclinic orbit of (HS) as a limit of $q(t; \mathrm{T})$ as $\mathrm{T} \to \infty$ and we complete the proof of Theorem 0.1. An argument similar to the following is used by Rabinowitz [12] to show the existence of heteroclinic orbits joining global maxima via a variational argument.

For each $\mathrm{T} \geqq 1$, we define $\tilde{q}(t; \mathrm{T}) \in \mathrm{H}^1(\mathbf{R}, \mathbf{R}^\mathrm{N})$ by

$$\tilde{q}(t;\mathrm{T}) = \begin{cases} q(t+\tau_\mathrm{T}^1;\, T), & \text{if } t \in [-\tau_\mathrm{T}^1, T-\tau_\mathrm{T}^1]; \\ 0, & \text{otherwise.} \end{cases}$$

Then it clearly follows from Lemma 2.1 and Proposition 2.2 that

1° $\tilde{q}(t; \mathrm{T})$ is a solution of (HS) in $(-\tau_\mathrm{T}^1, \mathrm{T}-\tau_\mathrm{T}^1)$;

2° $\tilde{q}(0; \mathrm{T}) \in \partial \mathrm{B}_\delta(0)$ for all $\mathrm{T} \geqq 1$;

3° $\|\dot{\tilde{q}}(.; \mathrm{T})\|_{\mathrm{L}^2(\mathbf{R}, \mathbf{R}^\mathrm{N})}$, $\|\tilde{q}(.; \mathrm{T})\|_{\mathrm{L}^\infty(\mathbf{R}, \mathbf{R}^\mathrm{N})}$, $\int_{-\infty}^{\infty} -\mathrm{V}(\tilde{q}(t; \mathrm{T}))\,dt$ are uniformly bounded in $\mathrm{T} \geqq 1$.

By 3°, we can extract a subsequence $\mathrm{T}_k \to \infty$ such that $\tilde{q}(t; \mathrm{T}_k)$ converges to some $y(t) \in \mathrm{C}(\mathbf{R}, \mathbf{R}^\mathrm{N}) \cap \mathrm{L}^\infty(\mathbf{R}, \mathbf{R}^\mathrm{N})$ with $\dot{y}(t) \in \mathrm{L}^2(\mathbf{R}, \mathbf{R}^\mathrm{N})$ in the following sense:

$$\tilde{q}(t;\mathrm{T}_k) \to y(t) \quad \text{in } \mathrm{L}^\infty_{\mathrm{loc}}(\mathbf{R}, \mathbf{R}^\mathrm{N}), \tag{3.1}$$

$$\dot{\tilde{q}}(t;\mathrm{T}_k) \rightharpoonup \dot{y}(t) \quad \text{weakly in } \mathrm{L}^2(\mathbf{R}, \mathbf{R}^\mathrm{N}). \tag{3.2}$$

Moreover we have

$$\int_{-\infty}^{\infty} -\mathrm{V}(y(t))\,dt \leqq \limsup_{\mathrm{T} \to \infty} \int_{-\infty}^{\infty} -\mathrm{V}(\tilde{q}(t;\mathrm{T}))\,dt \leqq \mathrm{C} < \infty. \tag{3.3}$$

Similarly as in [6], Lemma 2.1, we also have

$$y(t) \neq e \quad \text{for all } t \in \mathbf{R}. \tag{3.4}$$

PROPOSITION 3.1. – *$y(t)$ a nontrivial solution of* (HS) *on* $\mathbf{R}$.

Proof. – Noting (3.4), it is suffices to prove for any $\varphi \in \mathrm{C}_0^\infty(\mathbf{R}, \mathbf{R}^\mathrm{N})$

$$\int_{-\infty}^{\infty} [\dot{y}(t)\,\dot{\varphi}(t) - \mathrm{V}'(y(t))\,\varphi(t)]\,dt = 0. \tag{3.5}$$

By Lemma 2.5, we can choose $k_0 \in \mathbf{N}$ such that $\operatorname{supp}\varphi \subset (-\tau_{\mathrm{T}_k}^1, \mathrm{T}_k - \tau_{\mathrm{T}_k}^1)$ for all $k \geqq k_0$. By the property 1° of $\tilde{q}(t; \mathrm{T})$, we have for $k \geqq k_0$

$$\int_{-\infty}^{\infty} [\dot{\tilde{q}}(t;\mathrm{T}_k)\,\dot{\varphi}(t) - \mathrm{V}'(\tilde{q}(t;\mathrm{T}_k))\,\varphi(t)]\,dt = 0.$$

By (3.1) and (3.2), we can pass to the limit $k \to \infty$ and we get (3.5). Nontriviality of $y(t)$ clearly follows from the fact $y(0) \in \partial \mathrm{B}_\delta(0)$, that is a consequence of the property 2° of $\tilde{q}(t; \mathrm{T})$ and (3.1). ■

As a last step of the proof of Theorem 0.1, we prove

PROPOSITION 3.2. – *$y(t)$, $\dot{y}(t)\to 0$ as $t\to\pm\infty$.*

Proof. – First we prove $y(t)\to 0$ as $t\to\infty$. Arguing indirectly, we assume $y(t)\not\to 0$. Then for some sequence $t_k\to\infty$ and for some $\varepsilon>0$, we have

$$y(t_k)\notin B_\varepsilon(0) \qquad \text{for all} \quad k\in\mathbf{N}.$$

On the other hand, by (V2) and (3.3),

$$\operatorname{meas}\{t\in\mathbf{R}\,;y(t)\notin B_{\varepsilon/2}(0)\}<\infty.$$

Hence there is a sequence $\bar{t}_k\to\infty$ such that $y(\bar{t}_k)\in B_{\varepsilon/2}(0)$. Thus the curve $y(t)$ must intersect $\partial B_\varepsilon(0)$ and $\partial B_{\varepsilon/2}(0)$ infinitely often as $t\to\infty$. But this contradicts with $\dot{y}(t)\in L^2(\mathbf{R},\mathbf{R}^N)$ and (3.3). In fact, suppose $(a, b)\subset\mathbf{R}$ is an interval such that

$$\begin{gathered} y(a)\in\partial B_{\varepsilon/2}(0), \qquad y(b)\in\partial B_\varepsilon(0), \\ y(t)\in B_\varepsilon(0)\setminus B_{\varepsilon/2} \quad \text{in } (a,b). \end{gathered} \tag{3.6}$$

Then we have

$$m_{\varepsilon/2}(b-a)\leqq\int_a^b -V(y(t))\,dt \qquad \text{where} \quad m_{\varepsilon/2}=\min_{x\in\mathbf{R}^N\setminus B_{\varepsilon/2}(0)} -V(x)>0.$$

By Schwarz inequality, we have

$$\begin{aligned} \frac{\varepsilon}{2}\leqq|y(b)-y(a)|&\leqq\int_a^b|\dot{y}(t)|\,dt \\ &\leqq(b-a)^{1/2}\left(\int_a^b|\dot{y}(t)|^2\,dt\right)^{1/2} \\ &\leqq\frac{1}{2}\int_a^b|\dot{y}(t)|^2\,dt+\frac{b-a}{2} \\ &\leqq\frac{1}{2}\int_a^b|\dot{y}(t)|^2\,dt+\frac{1}{2m_{\varepsilon/2}}\int_a^b -V(y(t))\,dt. \end{aligned}$$

If $y(t)$ intersects $\partial B_\varepsilon(0)$ and $\partial B_{\varepsilon/2}(0)$ infinitely often as $t\to\infty$, we can find infinitely many disjoint interval (a_i, b_i) with the property (3.6). Thus we find

$$\int_{-\infty}^{\infty}|\dot{y}(t)|^2\,dt+\frac{1}{2m_{\varepsilon/2}}\int_{-\infty}^{\infty} -V(y(t))\,dt=\infty.$$

This contradicts with $\dot{y}(t)\in L^2(\mathbf{R},\mathbf{R}^N)$ and (3.3). Thus we obtain $y(t)\to 0$ as $t\to\infty$.

Since $\tilde{q}(t;T_k)$ satisfies the equation (HS), $\ddot{\tilde{q}}(t;T_k)=-V'(\tilde{q}(t;T_k))$ is bounded on each compact interval by 3° and (3.4). Thus $\tilde{q}(t;T_k)$ converges in $W^{1,\infty}_{loc}(\mathbf{R}, \mathbf{R}^N)$ to $y(t)$. Hence

$$\frac{1}{2}|\dot{y}(t)|^2 + V(y(t)) = \lim_{k \to \infty} \left[\frac{1}{2}|\dot{\tilde{q}}(t; T_k)|^2 + V(\tilde{q}(t; T_k)) \right]$$
$$= . \lim_{k \to \infty} E_{T_k} = 0 \quad \text{for all } t \in \mathbf{R}$$

Since $y(t) \to 0$ as $t \to \infty$, we have

$$|\dot{y}(t)|^2 = -2V(y(t)) \to 0 \quad \text{as } t \to \infty.$$

In a similar way, we get $y(t)$, $\dot{y}(t) \to 0$ as $t \to -\infty$. ■

Proof of Theorem 0.1. – Obviously by Propositions 3.1 and 3.2, $y(t)$ is a homoclinic orbit for (HS). ■

Remark 3.3. – Slight modifications of our argument permit us to treat more general compact sets S. More precisely, we assume

(V0) $S \subset \mathbf{R}^N$ is a compact subset such that $0 \notin S$ and 0 belongs to an unbounded component of $\mathbf{R}^N \setminus S$.

We also assume the following instead of (V1)-(V5).

(V1') $V \in C^2(\mathbf{R}^N \setminus S, \mathbf{R})$;

(V2') $V(q) \leqq 0$ for all $q \in \mathbf{R}^N \setminus S$ and $V(q) = 0$ if and only if $q = 0$, and $\limsup_{|q| \to \infty} V(q) \equiv \bar{V} < 0$;

(V3') There is a constant $\delta \in \left(0, \frac{1}{2}\text{dist}(0, S)\right)$ such that

$$V(q) + \frac{1}{2}(V'(q), q) \leqq 0 \quad \text{for all } q \in B_\delta(0);$$

(V4') $-V(q) \to \infty$ as $q \to S$;

(V5') There is a neighbourhood W of S in $\mathbf{R}^N$ and a function $U \in C^1(W \setminus S, \mathbf{R})$ such that $U(q) \to \infty$ as $q \to S$ and $-V(q) \geqq |U'(q)|^2$ for $q \in W \setminus S$.

Then we have the following theorem.

THEOREM 3.4. – *If* V *satisfies* (V0) *and* (V1')-(V5'), *then* (HS) *possesses at least one* (*nontrivial*) *homoclinic orbit which begins and ends at* 0. ■

Remark 3.5. – After completing this work, the author learned from Professor Rabinowitz that Benci and Giannoni [13] and Coti-Zelati and Ekeland [14] have also recently obtained results on homoclinic orbits of Hamiltonian systems.

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