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ABSTRACT. — We first show that no oscillation can *appear* in a 2×2 hyperbolic system whose two eigenvalues are linearly degenerate. For such a system, oscillations can only propagate. The method uses both compensated compactness ideas and the characteristics of the system, which is partly formal. We then make some heuristic remarks on the extension to the 3×3 gas dynamics system, and we link this question to the "separation of the wave cone and the constitutive manifold" (Ron DiPerna).

Key words : Conservation laws, compensated compactness, young measures.

RÉSUMÉ. — On montre d'abord qu'aucune oscillation ne peut *apparaître* dans un système 2×2 dont les deux valeurs propres sont linéairement dégénérées. Pour un tel système, les oscillations peuvent seulement se propager. La méthode utilise à la fois des idées de compacité par compensation et les caractéristiques du système, ce qui est partiellement formel. On fait ensuite quelques remarques heuristiques sur le système 3×3 de la dynamique des gaz, et on relie cette question à la « séparation du cône d'ondes et de la variété constitutive » (Ron DiPerna).

This paper is dedicated to the memory of Ron DiPerna.

1. INTRODUCTION

The results and comments we present here have been strongly inspired by the ideas of Ron DiPerna. We want to study the Young measure (*see* e.g. Young [16], Tartar [13]) associated to a sequence of approximate solutions to a general 2×2 nonlinear strictly hyperbolic system of conservation laws:

$$u_t + f(u)_x = 0 \qquad (1.1)$$

A typical problem is to prescribe the initial data:

$$u(x, 0) = u_0(x) \qquad (1.2)$$

and to study the convergence of a (sub) sequence of approximate solutions to (1.1), (1.2) to an admissible weak solution of the Cauchy Problem (1.1), (1.2).

Another motivation is to study homogenization problems: e. g. we include oscillations (numerical oscillations, eddies...) in the initial data:

$$u_t^\varepsilon + f(u^\varepsilon)_x = 0 \qquad (1.3)$$
$$u^\varepsilon(x, 0) = u_0(x, x/\varepsilon), \qquad \varepsilon \to 0+ \qquad (1.4)$$

and we want to study the weak limit of u^ε as $\varepsilon \to 0+$. Observe that the wave-length — but not the amplitude — of such oscillations is thus vanishing as ε goes to 0.

Throughout the paper, we will assume that the system (1.1) is strictly hyperbolic and 2×2 (except in Sections 4 and 5). In the genuinely nonlinear (GNL) case — resp. in the linearly degenerate (LD) case — an eigenvalue λ is strictly increasing (or decreasing) — resp. is constant — across a corresponding simple wave. Between these two extreme situations, there are of course many intermediate cases, corresponding for instance to a change of convexity in some constitutive relation.

If the system is GNL, it is well known, since the pioneering paper of Ron DiPerna [4], that, even with oscillating initial data (1.4), no oscillation can persist for any positive time. Roughly speaking, the time-life of oscillations of wave-length ε is of the order of ε.

In contrast, in the LD case, such oscillations can persist. Typically, if we choose:

$$u^\varepsilon(x, 0) = \begin{cases} a & \text{for} \quad k\varepsilon < x < (k+\theta)\varepsilon \\ & \qquad\quad 0 < \theta < 1 \\ b & \text{for} \quad (k+\theta)\varepsilon < x < (k+1)\varepsilon \end{cases} \qquad (1.5)$$

with suitable constant states a and b, these oscillations will just *propagate* along contact discontinuities connecting a and b. So in this case initial oscillations can persist, precisely because there is no entropy condition, no dissymmetry between a and b, no dissipation of energy along a contact

discontinuity. However, in this example, oscillations are not *created:* they were already present in the initial data. In this case, the goal is precisely to show that no oscillation can develop if it was not present in the initial data.

In other words, in the LD case, the goal is to investigate the *dynamics* of the Young measure. This is an important problem since the classical reproach to this theory is precisely it's static feature: the Young measure (Y.M.) at some given point (x, t) only describes the values at the *same* point of weak limits of subsequences $g(u^\varepsilon)$, without any information on the situation at other points (x, t). This is the great weakness of this tool and there are current investigations (L. Tartar [14], independently P. Gerard [7]) to develop a more powerful tool.

However, in this problem we know two things:

(*a*) the Y. M. at any point (x, t) has a tensor-product structure. This is a result of compensated compactness: *see* further;

(*b*) initially this Y.M. is a delta-function for all x under consideration, *i.e.* there is no oscillation in the initial data.

As we will show in Section 3, this implies that no oscillation will develop later on. In other words, this is a result on the dynamics of the Young measure in the LD case, in the spirit of the notion of measure-valued solution introduced by Ron DiPerna [5]. The method we use here has been developped a few years ago and has been presented in a few Conferences. It is partially heuristic, since it uses a (technically difficult) Green formula along generalized characteristics. Very recently, Chen Gui-Qiang [2] has given a rigorous treatment of this result, using a Lagrange-Euler type change of coordinates which avoids this technical problem.

The sequel of the paper is organized as follows. In Section 4, we make a few (very!) heuristic remarks on the possibility of extending this approach to the 3×3 system of the full Gas Dynamics Equations. We conclude it is very unlikely that oscillations could appear.

In Section 5, we turn to the other extreme situation: the GNL case. This is another beautiful paper of Ron DiPerna [6], on the geometric separation (in the phase portrait) of what he called the "constitutive manifold" and the "wave cone", for general systems of conservation laws. The purpose of this Section is just to summarize the main ideas of this paper, whose importance has been completely underestimated. The most striking result is that a GNL hyperbolic system is some weak version of an elliptic system, (at least if we add suitable Entropy conditions). Indeed, this elliptic feature "explains" why oscillations cannot propagate for such a system. Of course, in general, realistic systems have both GNL and LD eigenvalues. So there is of course a difficult coupling betwen these two types of behaviour.

2. BASIC FACTS ABOUT COMPENSATED COMPACTNESS

We denote by $\rightharpoonup$ the weak convergence in any L^p space ($1 \leqq p < \infty$) or the weak-star convergence in L^∞ as $\varepsilon \to 0$. In either case, a weakly convergent sequence of (possibly) oscillating functions (u^ε) satisfies:

$$\forall f \in C_0^\infty, \quad \int f(u^\varepsilon - u) \to 0 \qquad (2.1)$$

and the weak limit u is some averaged values of u^ε.

We first recall that for any sequence of functions (u^ε), uniformly bounded in L^∞, the associated Young measure has the following property: there exists a subsequence, still denoted by (u^ε), such that, for any continuous function g, we have:

$$g(u^\varepsilon) \rightharpoonup g^* : (x, t) \to \langle v_{x,t}, g(u) \rangle = \int g(u) \, dv_{x,t}(u) \qquad (2.2)$$

see for instance Tartar [13]. For a L^p version of this result, *see* Ball [1]. We just mention here that the sequence (u^ε) is strongly convergent, *i.e.* does not oscillate, if and only if $v_{x,t}$ is a delta-function.

We now give two basic examples of compensated compactness.

Example 1. — Let $Q = \{(x, t)\}$ and let (u^ε), (v^ε) be two weakly convergent sequences in $L^2(Q)$: $u^\varepsilon \rightharpoonup u$, $v^\varepsilon \rightharpoonup v$. Assume that

$$\left.\begin{array}{l} \partial_t u^\varepsilon \in K1 \\[4pt] \partial_x v^\varepsilon \in K2 \end{array}\right\} \qquad (2.3)$$

where K1 and K2 are two strongly compact subsets of the negative Sobolev space $H_{loc}^{-1}(Q)$. In such a case, we will briefly say that $\partial_t u^\varepsilon$ and $\partial_x v^\varepsilon$ are "nice".

Then

$$u^\varepsilon . v^\varepsilon \rightharpoonup u . v \quad \text{in the distribution sense}$$

and under the assumptions (2.3) the quadratic function $Q(u, v) = uv$ is only nonlinear function of u^ε and v^ε which behaves nicely as $\varepsilon \to 0$.

Example 2 (Div-curl lemma in $\mathbb{R}^2$). — Let (u^ε, v^ε, y^ε, z^ε) $\rightharpoonup (u, v, y, z)$ in L^2 weak such that

$$\left.\begin{array}{l} \partial_t u^\varepsilon + \partial_x v^\varepsilon \in K1 \\[4pt] \partial_t y^\varepsilon + \partial_x z^\varepsilon \in K2 \end{array}\right\} \qquad (2.4)$$

then

$$u^\varepsilon z^\varepsilon - v^\varepsilon y^\varepsilon \rightharpoonup uz - vy \quad \text{in the distribution sense}$$

and again, under the assumption (24), $Q(u, v, y, z) = uz - vy$ is the only non linear function of u^ε, v^ε, y^ε, z^ε which behaves nicely as $\varepsilon \to 0$.

We will come back to these examples later. We now recall how this type of information has been applied to 2×2 systems of conservation laws by L. Tartar and Ron DiPerna. We first consider the additional conservation laws of the system (1.1), *i.e.* the entropy-flux (E-F) pairs ($\varphi(u)$, $\psi(u)$). For smooth solutions of (1.1), they satisfy:

$$\varphi(u)_t + \psi(u)_x = 0 \qquad (2.5)$$

while for discontinuous solutions the classical Lax admissibility criterion requires:

$$\varphi(u)_t + \psi(u)_x \leqq 0 \qquad (2.6)$$

for any convex entropy φ of the system. Then we consider a sequence (u^ε) of approximate solutions of (1.1). For instance (u^ε) is the solution of the nonlinear parabolic Cauchy Problem:

$$u_t^\varepsilon + f(u^\varepsilon)_x = \varepsilon (D u_x^\varepsilon)_x, \qquad \varepsilon \to 0+ \qquad (2.7)$$
$$u^\varepsilon(x, 0) = u_0(x) \qquad (2.8)$$

where D is some diffusion matrix. We assume two *a priori* estimates:

$$\| u^\varepsilon \|_{L^\infty (Q)} \leqq c \qquad (2.9)$$

(uniform L^∞ bound) and

$$\| \varepsilon^{1/2} u_x^\varepsilon \|_{L^2 (Q)} \leqq c \qquad (2.10)$$

(energy estimate). The latter is quite easy, while the former is in general extremely difficult. Then, using Murat's Lemma [8], it is easy to obtain, for any E-F pairs (φ, ψ), (η, q):

$$\left. \begin{array}{l} \varphi(u^\varepsilon)_t + \psi(u^\varepsilon)_x \quad \text{is ``nice''} \\ \eta(u^\varepsilon)_t + q(u^\varepsilon)_x \quad \text{is ``nice''.} \end{array} \right\} \qquad (2.11)$$

So we can apply the Div-Curl Lemma to these four sequences of functions φ^ε, ψ^ε, η^ε, q^ε, where $\varphi^\varepsilon = \varphi(u^\varepsilon) \rightharpoonup \varphi^*$ (and similar notations for the other terms). Therefore we obtain

$$\varphi^\varepsilon q^\varepsilon - \psi^\varepsilon \eta^\varepsilon \rightharpoonup \varphi^* q^* - \psi^* \eta^*. \qquad (2.12)$$

In other words, the associated Young measure satisfies, for almost all points (x, t):

$$\langle \nu_{x,t}, \varphi q - \psi \eta \rangle = \langle \nu_{x,t}, \varphi \rangle \cdot \langle \ , q \rangle - \langle \nu_{x,t}, \psi \rangle \cdot \langle \nu_{x,t}, \eta \rangle. \qquad (2.13)$$

The problem was thus reduced by L. Tartar to the following *Static Identification Problem:*

For any given point (x, t), find a probability measure $\nu_{x,t}$ satisfying (2.13) for all E-F pairs (φ, ψ) and (η, q).

In the 2×2 GNL case, this problem has been solved by Ron DiPerna [4], using a large family of entropies constructed by P. Lax. The only probability measure solution is a delta-function. So, even if there were oscillations at time $t = 0$, they cannot persist for any positive time.

In contrast with this static approach, the concept of measure-valued solution, also introduced by Ron [5], is dynamical. Taking the weak limit in (2.11), we trivially obtain [compare to 2.6)]:

$$\partial_t \langle v_{x,t}, \varphi(u) \rangle + \partial_x \langle v_{x,t}, \psi(u) \rangle \leqq 0 \qquad (2.14)$$

for any convex entropy φ. Of course, any component u_j (or $-u_j$) of u is a convex entropy of the system, so (2.14) implies (1.1).

So, in the LD case, if we start with given (non oscillating) initial data u_0, the problem is to combine the static information (2.13), the dynamical information (2.14) and the initial condition:

$$\forall x, \quad v_{x,0} = \delta_{u_0(x)} \qquad (2.15)$$

to show that no oscillation will develop. This is the purpose of the next Section.

3. NON APPEARANCE OF OSCILLATIONS IN THE 2×2 LD CASE: A FORMAL RESULT

We first write any 2×2 system — for smooth solutions — in non conservative form:

$$\left. \begin{array}{l} \partial_t w(u) + \lambda_2(u) \partial_x w(u) = 0 \\ \partial_t z(u) + \lambda_1(u) \partial_x z(u) = 0. \end{array} \right\} \qquad (3.1)$$

$\lambda_1(u)$ and $\lambda_2(u)$ are the real and distinct eigenvalues of the Jacobian matrix $f'(u)$, $w(u)$ and $z(u)$ are respectively the Riemann invariants in the sense of Lax associated to λ_1 and λ_2. The following prototype of LD system has been introduced by D. Serre (see [11]):

$$\left. \begin{array}{l} \partial_t w + z \partial_x w = 0 \\ \partial_t z + w \partial_x z = 0 \end{array} \right\} \qquad (3.2)$$

The result we are going to state would be the same if only one eigenvalue was LD but we only show it for the particular case (3.2).

For this system, $\lambda_2 = \lambda_2(z) = z$, $\lambda_2 = \lambda_1(w) = w$, i.e. the two eigenvalues are LD, and we just have made a particular choice of the Riemann Invariants. This system has no physical interpretation, but e.g. it would correspond to the elasticity system in the linear case, or to the isentropic gas dynamics equations in the (unrealistic) case where the pressure $p(\rho) = -1/\rho$. We can write this system in conservative form, e.g.

$$\left. \begin{array}{l} \partial_t u + \partial_x v = 0 \qquad (u > 0) \\ \partial_t v + \partial_x (v(v+1)/u) = 0 \end{array} \right\} \qquad (3.3)$$

where $u = 1/(z - w) > 0$ and $v = w/(z - w)$. In fact, u and v just two particular entropies of (3.2). Due to the LD feature of this system, there are enough

bounded invariant regions to guarantee that u will remain strictly positive and bounded if it is the case for u_0.

We now consider a sequence of approximate solutions

$$(u^\varepsilon, v^\varepsilon) = (1/(z^\varepsilon - w^\varepsilon), w^\varepsilon/(z^\varepsilon - w^\varepsilon))$$

of (3.2). For instance, assume that $(u^\varepsilon, v^\varepsilon)$ is the solution of the "viscous" problem:

$$\left.\begin{array}{c} \partial_t u^\varepsilon + \partial_x v^\varepsilon = \varepsilon\, \partial_x^2 u^\varepsilon \\ \partial_t v^\varepsilon + \partial_x (v^\varepsilon (v^\varepsilon + 1)/u^\varepsilon) = \varepsilon\, \partial_x^2 v^\varepsilon \end{array}\right\} \qquad (3.3_\varepsilon)$$

with initial data in L^∞, bounded independently of ε:

$$(u^\varepsilon, v^\varepsilon)(x, 0) = (u_0^\varepsilon(x), v_0^\varepsilon(x)), \qquad u_0(x) \geqq c > 0 \qquad (3.4)$$

Let $\nu_{x,t}$ be the measure- valued solution $\nu_{x,t}$ associated to the sequence $(w^\varepsilon, z^\varepsilon)$.

As a particular example, we obtain the following formal result:

THEOREM 1. — *Let $(w^\varepsilon, z^\varepsilon)$ be the solutions of (3.3), (3.4). Assume there is no oscillation in w at time $t = 0$. To simplify, assume that*

$$\forall\, \varepsilon > 0, \quad \forall\, x, \qquad w^\varepsilon(x, 0) = w_0(x) \qquad (3.5)$$

where w_0 is a piecewise-smooth function. Moreover, assume we can justify the formal steps 4 and 6 below. Then:

(i) *No oscillation in w will develop later on: the measure $\mu_{1, x, t}$ defined below in (3.6) remains a delta-function.*

(ii) *Of course, here λ_1 and λ_2 play symmetric roles, so the same result holds if we replace w by the other Riemann Invariant z.*

Remark. — The same conclusion is true for some systems with only one LD eigenvalue. In this case, the result strongly depends on the conservative form of the system, and not only on the nonconservative form (3.1).

Proof. — First observe that the other Riemann Invariant can be either oscillating or non oscillating. In other words, the information on w and z are *uncoupled*. The proof consists of several steps.

Step 1. — We first show that for almost all (x, t) the Young measure $\nu_{x,t}$ has a precise structure, namely a *tensor-product structure:*

$$\nu_{x,\, t} = (\rho(w, z))^{-1} \cdot \mu_{1, x, t} \otimes \mu_{2, x, t} \qquad (3.6)$$

where $\mu_{1, x, t}$ and $\mu_{2, x, t}$ are non negative measures which operate respectively on w and z and ρ is a suitable weight-function.

This is a result of "compensated compactness with varying directions": in fact, w^ε and z^ε satisfy:

$$\left.\begin{array}{c} \partial_t w^\varepsilon + \lambda_2^\varepsilon\, \partial_x w^\varepsilon \quad \text{is "nice"} \\ \partial_t z^\varepsilon + \lambda_1^\varepsilon\, \partial_x z^\varepsilon \quad \text{is "nice"} \end{array}\right\} \qquad (3.7)$$

which is of course an approximation of (3.1). Therefore it is natural to expect that, roughly speaking, "we can pass to the limit on the product of the Riemann Invariants": if $w^{\varepsilon} \rightharpoonup w^{*}$ and $z^{\varepsilon} \rightharpoonup z^{*}$ then

$$w^{\varepsilon} \cdot z^{\varepsilon} \rightharpoonup w^{*} \cdot z^{*} \qquad (3.8)$$

Since the same result would be true if we replace w and z respectively by arbitrary functions $g(w)$ and $h(z)$, such a result would imply

$$\forall g, \quad \forall h, \quad \langle \nu_{x,t}, g(w) h(z) \rangle = \langle \nu_{x,t}, g(w) \rangle \cdot \langle \nu_{x,t}, h(z) \rangle \qquad (3.9)$$

[Since the left hand side is nothing else than the weak limit of $g(w^{\varepsilon}) h(z^{\varepsilon})$]. We can rewrite (3.9) under the form:

$$\nu_{x,t} = \mu_{1,x,t} \otimes \mu_{2,x,t} \qquad (3.10)$$

If the coefficients λ_1^{ε} and λ_2^{ε} were smooth, this result would be a trivial extension of the Example 1 in Section 2, but here these coefficients can be wildly oscillating as ε goes to 0.

The relation (3.10) has been conjectured in M. Rascle ([9], [10]) for the elasticity system. This sort of idea was also mentioned in L. Tartar [13] and in R. DiPerna [4].

In fact, the relation (3.10) is *false* for a general 2×2 system: D. Serre [11] has constructed a counter-exemple which is precisely based on system (3.2): for this system (3.2), there exists oscillating solutions $(w^{\varepsilon}, z^{\varepsilon})$ for which (3.10) is false.

However, even for these solutions, this relation becomes true if we add a suitable weight-function $\rho(w, z)$, whose expression is precisely given. So the correct general formula, conjectured by D. Serre [11], is (3.6).

For system (3.2), this weight-function in (3.6) is given by

$$\rho(w, z) = 1/(\lambda_2 - \lambda_1) = 1/(z - w) \qquad (3.11)$$

and in some sense it describes how the characteristics are twisted in the interaction of waves of the two families. In the general case, to prove (3.6) is a very difficult problem, but if (at least) one of the eigenvalues is LD, then the proof is much easier. It is given in D. Serre [11].

Therefore the measure-valued solutions of system (3.2) satisfy (3.6), (3.11).

Step 2. — The system (3.2) has two large families of entropies, respectively associated to the two characteristic fields, defined for *arbitrary* (smooth) functions g and h:

$$\varphi = \rho g(w), \qquad \psi = \lambda_2(z) \varphi = \rho z g(w) \qquad (3.12)$$
$$\varphi = \rho h(z), \qquad \psi = \lambda_1(w) \varphi = \rho w h(z) \qquad (3.13)$$

with the *same* function $\rho = (z - w)^{-1}$ as in (3.6), (3.11). The proof is quite easy: We multiply the equations in (3.2) respectively by $\partial_w \varphi$ and $\partial_z \varphi$

and we add. Necessarily φ and ψ satisfy:

$$\partial_w \psi = \lambda_2 \, \partial_w \varphi = z \, \partial_w \varphi; \qquad \partial_z \psi = \lambda_1 \, \partial_z \varphi = w \, \partial_z \varphi \qquad (3.14)$$

which is a first order linear hyperbolic system, with non constant coefficients. Since λ_2 (resp. λ_1) only depends of z (resp. w), it is very easy to solve this system. So we just let the reader check that functions φ and ψ given in (3.12), (3.13) satisfy (3.14) for *any* functions g and h. Actually, this is the deep reason for which (3.6) holds.

Step 3. − Moreover, as a function of the conservative variables u and v in (3.3), the entropy φ in (3.12) [or (3.13)] is convex with respect to u and v if and only if the function g (or h) is convex *with respect to w* (or z). Indeed, we have in (3.12)

$$\varphi = \rho \, g(w) = (z-w)^{-1} \, g(w) = u g(v/u); \qquad u > 0$$

so that the eigenvalues of the Hessian matrix of φ with respect to u and v are 0 and $((u^2 + v^2)/u^3) \cdot g''(v/u)$. Therefore they are nonnegative if and only if g is convex.

Step 4. − Since we only want to study the oscillations of w, we only use the family of entropies (3.12). They satisfy the very nice relation:
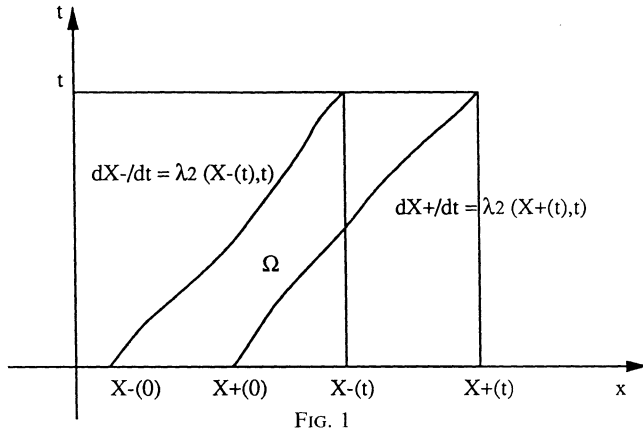
$$\psi = \lambda_2 \, \varphi \qquad (3.15)$$

which is never true in the GNL case, but which is always true in the linear case. We come now to the *formal* part of the proof. Let us first define the (generalized) characteristics of the system:

$$dX/dt = \lambda_2 ((w, z) (X(t), t)) = z (X(t), t) \qquad (3.16)$$

For BV solutions, even in the GNL case, they would be well defined, *see* for instance C. M. Dafermos [3], as solutions in the sense of Filippov of ordinary differential equations with discontinuous right-hand side. Of course, in such a case, there is no unique solution, but for instance there is a unique maximal (or minimal) solution, which is enough for our purpose. We also remark that generalized characteristics are Lipschitz curves. Of course, here we dont't want to consider only BV solutions. However, the definition of these characteristics is easier here in a LD case (since there is no shock). Anyway, this step is formal.

Let us integrate (2.6) with respect to x and t, in the domain Ω, bounded in the (x, t) plane by the lines $\{t=0\}$, $\{t=T\}$ and two characteristics, say $\{(X_+(t), t)\}$ and $\{(X_-(t), t))\}$, starting respectively from arbitrary points $X_+(0)$ and $X_-(0)$: see Figure 1. Then we apply the Divergence Theorem. Due to (3.15) and (3.16), there is no integral over the characteristics. We obtain, for any convex entropy φ:

$$\int_{X_-(t)}^{X_+(t)} \varphi(x, t) \, dx - \int_{X_-(0)}^{X_+(0)} \varphi(x, t) \, dx \leqq 0 \qquad (3.17)$$

FIG. 1

Step 5. — We rewrite the definition of a measure-valued solution: for any convex entropy φ, $v_{x,\,t}$ satisfies:

$$\partial_t \langle v_{x,\,t},\, \varphi(u,v) \rangle + \partial_x \langle v_{x,\,t},\, \psi(u,v) \rangle \leqq 0 \qquad (2.14)$$

Let us choose the entropies (3.12), with a convex function g. Using (3.6) and (3.11), we obtain:

$$\langle v_{x,\,t},\, \varphi(u,v) \rangle = \langle (\rho(w,z))^{-1} \cdot \mu_{1,\,x,\,t} \otimes \mu_{2,\,x,\,t},\, \rho g(w) \rangle$$
$$= \langle \mu_{1,\,x,\,t},\, g(w) \rangle \cdot \langle \mu_{2,\,x,\,t},\, 1 \rangle \quad (3.18)$$

Let us normalize $\mu_{2,\,x,\,t}$ so that:

$$\langle \mu_{2,\,x,\,t},\, 1 \rangle = 1 \qquad (3.19)$$

Similarly

$$\langle v_{x,\,t},\, \psi(u,v) \rangle = \langle (\rho(w,z))^{-1} \cdot \mu_{1,\,x,\,t} \otimes \mu_{2,\,x,\,t},\, \rho z g(w) \rangle$$
$$= \langle \mu_{1,\,x,\,t},\, g(w) \rangle \cdot \langle \mu_{2,\,x,\,t},\, z \rangle.$$

Let us define

$$\bar{\lambda}_2(x,t) = \langle \mu_{2,\,x,\,t},\, z \rangle = \langle v_{x,\,t},\, \rho \lambda_2(z) \rangle / \langle v_{x,\,t},\, \rho \rangle. \qquad (3.20)$$

This is the *group velocity* of the possible oscillations. Observe that this expression is *different* from the weak limit of λ_2^ε, which is of course:

$$\lambda_2^*(x,t) = \langle v_{x,\,t},\, \lambda_2(z) \rangle \qquad (3.21)$$

(*see* D. Serre [11]). Using all these equations, (2.14) becomes: *for any convex function g*,

$$\partial_t \langle \mu_{1,\,x,\,t},\, g(w) \rangle + \partial_x (\bar{\lambda}_2(x,t) \langle \mu_{1,\,x,\,t},\, g(w) \rangle) \leqq 0 \qquad (3.22)$$

Step 6. — We use the same formal method as in Step 4 we formally define the "homogenized characteristics": we just replace in (3.16) λ_2

by $\bar{\lambda}_2$

$$dX/dt = \bar{\lambda}_2(w, z)(X(t), t) = \langle \mu_{2, X(t), t}, z \rangle \qquad (3.23)$$

and we integrate with respect to x and t on the corresponding domain Ω. So we rewrite inequality (3.17) under the form: *for any convex function g*.

$$\int_{X_-(t)}^{X_+(t)} \langle \mu_{1, x, t}, g(w) \rangle dx - \int_{X_-(0)}^{X_+(0)} \langle \mu_{1, x, 0}, g(w) \rangle dx \leqq 0. \quad (3.24)$$

In other words, knowing the precise (static) structure (3.6) of the measurevalued solution $v_{x, t}$, we can completely *uncouple* the information on each Riemann Invariant: z only plays a role through $X_\pm(t)$.

Step 7. $-$ We can now conclude this formal proof. Under the simplifying assumptions of Theorem 1, there is initially no oscillation in w, *i.e.*

$$\forall x, \quad \mu_{1, x, 0} = \delta_{w_0(x)} \qquad (3.25)$$

and we want to show that

$$\forall x, \quad \forall t, \quad \mu_{1, x, t} = \delta_{w^*}(x, t) \qquad (3.26)$$

So we assume w_0 is piecewise smooth. Let us choose two adjacent sequences (x_-^n) and (x_+^n) converging to the same arbitrary point x_0, and let us consider the generalized characteristics $X_\pm^n(.)$ starting from $x_\pm^n$. If there are several solutions, we pick up any of them, for instance $-$ since this step is formal $-$ the minimal one for X_+ and the maximal one for X_-. Let us define

$$\left. \begin{array}{l} a_n = \inf \text{ ess}\{w_0(x), x_-^n \leqq x \leqq x_+^n\} \\ b_n = \sup \text{ ess}\{w_0(x), x_-^n \leqq x \leqq x_+^n\} \end{array} \right\} \qquad (3.27)$$

Now we can choose a_n and b_n such that

$$b_n - a_n \to 0 \quad \text{as } n \to +\infty. \qquad (3.28)$$

For each n, we choose the function $g = g_n$ in (3.24) such that

$$g_n \geqq 0, \quad g_n \text{ is convex, } \quad \text{and } g_n = 0 \text{ on } [a_n, b_n]. \qquad (3.29)$$

So the first integral in (3.24) must be equal to zero. Therefore

$$\langle \mu_{1, x, t}, g_n(w) \rangle = 0 \quad \text{a.e. on } [X_-^n(t), X_+^n(t)]. \qquad (3.30)$$

In other words, *as $n \to \infty$*,

$$\sup \text{ ess}\{\langle \mu_{1, x, t}, g_n(w) \rangle, X_-^n(t) \leqq x \leqq X_+^n(t)\}$$
$$- \inf \text{ ess}\{\langle \mu_{1, x, t}, g_n(w) \rangle, X_-^n(t) \leqq x \leqq X_+^n(t)]\} \to 0 \quad (3.31)$$

This precisely means that oscillations in w cannot develop if they were not present in the initial data. So the (formal) proof is complete.

Of course, in system (3.2), the two Riemann Invariants w and z play symmetric roles. So the same result would be true if we replaced w by z.

To conclude this Section, we make two remarks:

(i) We have only used the information on w. Again, the structure (3.6) enables to almost completely uncouple the evolution of $\mu_{1, \ x, \ t}$ and $\mu_{2, \ x, \ t}$. The method also works if only one eigenvalue is LD, at least if we properly choose the conservative form of system (3.1).

(ii) Our proof is of course formal, since we use (generalized) characteristics in a context where we don't know if they exist. However, the result would be rigorous if for instance λ_2 was constant. On the other hand, for smooth solutions, (3.12) implies:

$$\partial_t (\rho g (w)) + \partial_x (\lambda_2 (z) \rho z g (w)) = 0 \qquad (3.32)$$

So there exists a function $y (x, t)$ such that

$$\partial_t y = -\lambda_2 (z) \rho z g (w), \qquad \partial_x y = \rho z g (w) \qquad (3.33)$$

This idea comes from the Euler-Lagrange change of coordinates and has been used in D. Wagner[15] and in unpublished results of D. Serre. It is used by Chen Gui-Qiang in a recent preprint[2] to avoid these technical difficulties with characteristics.

4. SOME HEURISTIC REMARKS ON THE 3×3 CASE

Let us consider the system of Gas dynamics Equations (GDE):

$$\begin{aligned}
\partial_t \rho + \partial_x (\rho u) &= 0 \\
\partial_t (\rho u) + \partial x (\rho u^2 + p) &= 0 \\
\partial_t (\rho E) + \partial_x (u (\rho E + p)) &= 0
\end{aligned} \qquad (4.1)$$

with the classical notations. For this system, the second eigenvalue λ_2 is LD, associated to a strict Riemann Invariant (*i.e.* a function whose gradient is a left eigenvector of the Jacobian matrix of the system): the specific entropy S. The entropy production inequality is:

$$\partial_t (\rho S) + \partial_x (u \rho S) = \alpha \geqq 0 \qquad (4.2)$$

More generally, for any non decreasing function g,

$$\partial_t (\rho g (S)) + \partial_x (u \rho g (S)) = \alpha (g) \geqq 0 \qquad (4.3)$$

where $\alpha (g)$ is a bounded measure, depending on g. Compare with (3.32): the structure is the same. However, in Section 3, the entropy production measure was in fact 0, due to the LD feature of the full system (although we dud not use this information). Here, any attempt to imitate step 7 in Section 3 fails, since the function g would have to be non decreasing, non-positive, identically equal to zero on some interval $[a, b]$. So we cannot control the evolution of the width of this interval with respect to the time.

This difficulty reflects the following possibility. Consider this system in Lagrangian mass coordinates:

$$\partial_t v - \partial_x u = 0$$
$$\partial_t u + \partial_x p = 0 \qquad (4.4)$$
$$\partial_t E + \partial_x (up) = 0$$

The entropy production is now another nonnegative measure:

$$\partial_t S = \alpha \geqq 0 \qquad (4.5)$$

Let us consider the case where

$$S(x, 0) = 0; \qquad \alpha = 1 + \sin(1/x)$$

Then the solution is:

$$S(x, t) = t(1 + \sin(1/x)) \qquad (4.6)$$

In this case, oscillations do appear, which were not present in the initial data. The reason is that the entropy production measure α is singular along the line $x = 0$, which is precisely a characteristic curve of the second family.

In fact, this case is completely unrealistic, since entropy is only produced along shock waves of either the first or the third family. Due to the Lax Entropy Criterion, these shock curves are *transverse* to the characteristics of the second family, as long as the system is strictly hyperbolic, *i. e.* away from the vacuum state.

However, a more realistic case is the following. Let us consider a sequence of initial data for system (4.4). Clearly, if there are initial oscillations – of wave-length ε – in the GNL fields, the interaction of waves will produce oscillations in the LD field, roughly speaking before a time of order ε. The entropy production measure α^ε is:

$$\alpha^\varepsilon = \sum_j \beta_j^\varepsilon(t) \, \delta(x - X_j^\varepsilon(t)), \qquad (4.7)$$

$$\delta : \text{delta-function,} \qquad X_j^\varepsilon(.) : \text{shock curves}$$

where the β_j^ε are related to the strength of the shocks. Therefore, oscillations do appear between time 0 and time $0(\varepsilon)$, although α^ε is supported by non-vertical shock curves: the reason is that the β_j^ε oscillate each time the waves interact. The extreme (but unrealistic) case would be the one where, say a 1-shock interacting with a 3 rarefaction would produce a 1-rarefaction, a contact, and a 3-shock. In such a case, depicted in Figure 2, the β_j^ε would oscillate between zero and "large" positive values. In realistic cases, the strength of the shocks would have much smaller amplitude oscillations (so that interactions would not transform shocks into rarefactions and vice-versa) but nevertheless oscillations would be created.

So, for system (4.4), oscillations of wave-length ε in the GNL fields can produce oscillations in the LD field, before time $0(\varepsilon)$, and then
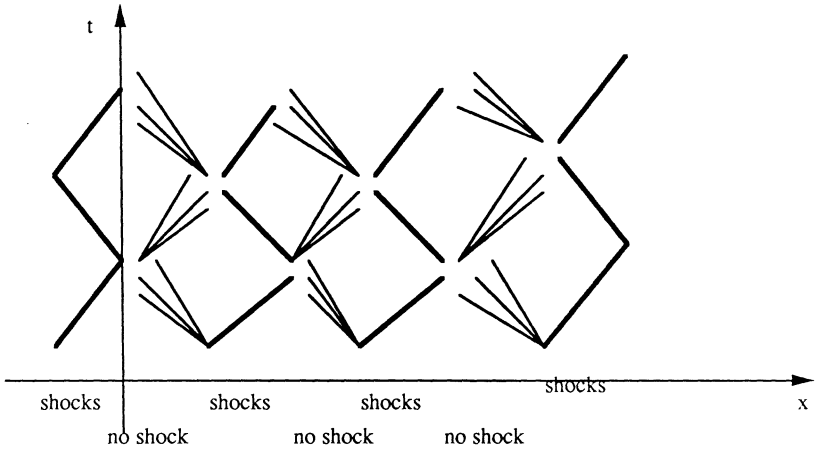
FIG. 2. — 3×3 system (4.4): extreme (but unrealistic) picture in which oscillations in the GNL fields would produce oscillations in the LD field. The (vertical) contact discontinuities are not represented.

cancel. Therefore, for larger time, there would be no other production of oscillations in the LD field, we only expect a propagation of the ones which have been created during this initial boundary layer.

Now, if there is no initial oscillation in any field (GNL as well as LD), this initial boundary layer should not exist, so that no oscillation would be created, even in the LD field.

5. THE GNL CASE: GEOMETRIC SEPARATION OF THE WAVE CONE AND THE CONSTITUTIVE MANIFOLD

We now recall some very nice ideas developed in R. DiPerna [6]. First, examples 1 and 2 in Section 2 are particular examples of the following general result of F. Murat, L. Tartar, *see* e. g. L. Tartar [13]. Assume that a sequence (u^ε) is weakly convergent in L^2 and satisfies

$$\sum_{\substack{1 \leq j \leq N \\ 1 \leq k \leq n}} a_{ijk} \partial_k u_j^\varepsilon = 0 \quad \text{or ``nice''}, \qquad 1 \leq i \leq q \qquad (5.1)$$

where $\partial_k = \partial/\partial x_k$, $t = x_0$. Then the quadratic functions $Q(u)$ such that $Q(u^\varepsilon) \rightharpoonup Q(u)$ (as $\varepsilon \to 0$) are exactly the ones which vanish on the "wave cone":

$$\Lambda = \Big\{ u \in \mathbb{R}^N / \exists \, \xi \in \mathbb{R}^n, \, \xi \neq 0 / \sum_{\substack{1 \leq j \leq N \\ 1 \leq k \leq n}} a_{ijk} \partial_k u_j = 0 \Big\}$$

In the particular case of Example 2 in Section 2 (Div-Curl Lemma), we had

$$\Lambda = \{ u \in \mathbb{R}^4 / u_1 . u_4 - u_2 . u_3 = 0 \}. \tag{5.3}$$

As the simplest example, let us consider the Laplace Equation in two coordinates (x, t), and let us write it as an elliptic first order system:

$$\left.\begin{array}{l} \partial_t u + \partial_x v = 0 \\ \partial_t v - \partial_x u = 0 \end{array}\right\} \tag{5.4}$$

Now let us consider a weakly convergent sequence of approximate solutions $(u^\varepsilon, v^\varepsilon)$ to this system, such that

$$\left.\begin{array}{l} \partial_t u^\varepsilon + \partial_x v^\varepsilon \quad \text{"nice"} \\ \partial_t v^\varepsilon - \partial_x u^\varepsilon \quad \text{"nice"} \end{array}\right\} \tag{5.5}$$

So we can apply the Div-Curl Lemma, to obtain

$$(u^\varepsilon)^2 + (v^\varepsilon)^2 \rightharpoonup u^2 + v^2 \quad \text{in the distribution sense.} \tag{5.6}$$

Since this is a strictly convex function of u, v, we easily deduce the strong convergence of sequences (u^ε) and (v^ε). Of course, nobody was really anxious about this problem (!), but the geometric view is the following: in (5.5), we have applied the Div-Curl Lemma to a very particular sequence, namely to a sequence of functions $(u^\varepsilon_1, u^\varepsilon_2, u^\varepsilon_3, u^\varepsilon_4)$ with values in the "constitutive manifold":

$$M = \{ u \in \mathbb{R}^4 / u_2 = u_3, u_1 = -u_4 \} \tag{5.7}$$

Now the geometric view of the elliptic nature of system (5.4) is quite simple: the set Λ and the set $(M - M) = \{ u - v / (u, v) \in M \times M \}$ are transverse, see Figure 3, case (a). In particular,

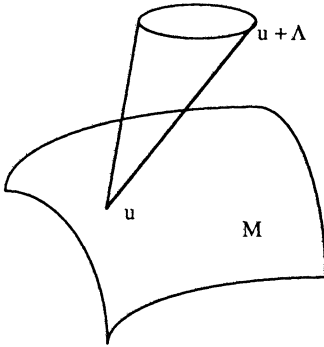$$(M - M) \cap \Lambda = \{ 0 \} \tag{5.8}$$

In fact, this condition is necessary to avoid oscillations, *see* L. Tartar [13]: if $(u_+ - u_-) \in \Lambda$ for some pair $(u_-, u_+) \in M \times M$, then for any step-function u of one variable, taking the only values u_- and u_+, the sequence (u^ε) defined by:

$$u^\varepsilon(x) = u((\xi . x)/\varepsilon), \quad \xi \text{ related to as in (5.2)}, \quad \varepsilon \to 0 \tag{5.9}$$
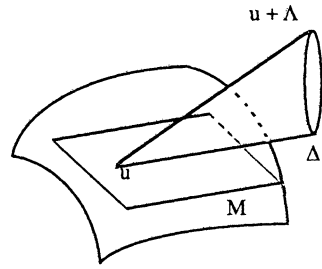
is a sequence of (faster and faster) oscillatin solutions to a general system (5.1).

As we have seen, condition (5.8) is satisfied for an elliptic system such as (5.4). Now the beautiful idea of Ron DiPerna is the following: consider the nonlinear elasticity system:
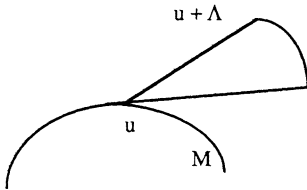
$$\begin{array}{ll} u_t - \sigma(v)_x = 0, & \sigma'(v) > 0 \\ v_t - u_x = 0, & \sigma''(v) > 0 \\ \varphi(u, v)_t + \psi(u, v)_x = 0 \\ \eta(u, v)_t + q(u, v)_x = 0 \end{array} \tag{5.10}$$
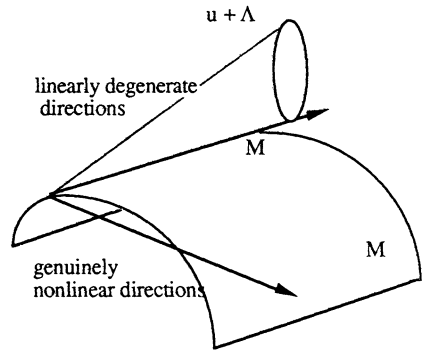
Case (a): elliptic case

Case (b): hyperbolic GNL
(genuinely nonlinear) case.
Δ is a characteristic
direction.

Case (b): hyperbolic GNL
case. Transverse view

Case (c): hyperbolic case,
with GNL and LD
eigenvalues

Fig. 3

In fact, we have added the two "natural" entropy-flux pairs, corresponding
to the conservation of energy

$$\varphi(u, v) = u^2/2 + \Sigma(v); \qquad \psi(u, v) = -u\sigma(v); \qquad \Sigma'(v) = \sigma(v)$$

and to the dual relation (when we exchange stress and strain, space and
time)

$$\eta(u, v) = -uv; \qquad q(u, v) = u^2/2 + \Sigma^*(v)$$

(Σ^* is the Legendre transform of Σ). Of course, system (5.10) is clearly strictly hyperbolic and GNL. Therefore the last two equations are only satisfied for smooth solutions.

It is now clear that (5.8) is satisfied for system (5.10): if not, a pair $(U_-, U_+) \in M \times M$ whose difference lies in the wave-cone Λ would satisfy the Rankine-Hugoniot relations

$$[-\sigma(v)] = s[u]; \qquad [-u] = s[v]$$

and

$$[\psi(u, v)] = s[\varphi(u, v)]; \qquad [q(u, v)] = s[\eta(u, v)]$$

where $s = -\xi_0/\xi_1$ and ξ is associated to $[U] = (U_+ - U_-)$ as in (5.2). Obviously, this is impossible, since the system is *genuinely nonlinear*. The corresponding geometric situation is depicted in Figure 3, case(*b*). The wave cone Λ contains directions which are rangent to the constitutive manifold M: the characteristic directions. However, as Ron showed in [6], Λ is separated – at least locally – from the manifold M itself.

This is a weak version of "ellipticity" and that "explains" why GNL systems – at least in the 2×2 case – don't admit oscillating solutions, whatever the initial data are.

On the contrary, this is no longer the case for a *linearly degenerate*, in particular for a linear system. Naturally, realistic systems, e.g. system (4.1) or (4.4) of gas dynamics, have both GNL and LD eigenvalues. We represent the corresponding situation in Figure 3, case(*c*).

6. CONCLUSION

In conclusion, to understand "real" hyperbolic systems, it is certainly necessary to mix techniques and ideas from linear hyperbolic systems, in order to study LD eigenvalues and (weak versions of) elliptic techniques, in order to study GNL eigenvalues. In this direction, despite it's static feature, the theory of Young measures and "classical" compensated compactness has given some important results, both in the genuinely nonlinear and in the linearly degenerate case. It has certainly been the most powerful tool of the eighties. Recently, more sophisticated techniques have been developed by L. Tartar[14] and independently by P. Gérard[7]. It is still too early to know if these new tools will be powerful enough to go further, and which other ingredients will be necessary.

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