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Integral representation of nonconvex functionals defined on measures

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ABSTRACT. — We show that every functional which is defined on the space $\mathcal{M}(\Omega; \mathbf{R}^n)$ of all $\mathbf{R}^n$ valued measures with bounded variation on Ω, and which is local and sequentially weakly* lower semicontinuous, can be represented in a suitable integral form.

RÉSUMÉ. — *Représentation intégrale de fonctionnelles nonconvexes définies sur des mesures.* — Une formule de représentation intégrale est établie pour toute fonctionnelle définie sur l'espace $\mathcal{M}(\Omega; \mathbf{R}^n)$ des mesures boréliennes sur Ω à valeurs dans $\mathbf{R}^n$, qui est locale et séquentiellement semicontinue inférieurement pour la topologie $*$faible.

1. INTRODUCTION

In a recent paper (*see* Bouchitté & Buttazzo [3]) we introduced a class of functionals defined on the space $\mathcal{M}(\Omega; \mathbf{R}^n)$ of all $\mathbf{R}^n$-valued measures with bounded variation on Ω; more precisely, we considered functionals $F: \mathcal{M}(\Omega; \mathbf{R}^n) \to [0, +\infty]$ of the form

$$F(\lambda) = \int_\Omega f\left(x, \frac{d\lambda}{d\mu}\right) d\mu + \int_{\Omega \setminus A_\lambda} \varphi(x, \lambda^s) + \int_{A_\lambda} g(x, \lambda(x))\, d\sharp \qquad (1.1)$$

where μ is a given finite positive non-atomic measure on Ω, f, φ, g are suitable Borel functions on $\Omega \times \mathbf{R}^n$, $\lambda = (d\lambda/d\mu)\,\mu + \lambda^s$ is the usual Lebesgue-Nikodym decomposition of λ with respect to μ, A_λ is the set of all atoms of λ, and $\sharp$ is the counting measure. Under suitable conditions on the integrands f, φ, g, we proved that the functional F in (1.1) is sequentially lower semicontinuous with respect to the weak* convergence in $\mathcal{M}(\Omega; \mathbf{R}^n)$. We stress the fact that, due to the presence of the "atomic term" $\int_{A_\lambda} g(x, \lambda(x))\, d\sharp$, in general the functional F in (1.1) is not convex. Moreover, when $f^\infty = \varphi = g$ (where $f^\infty(x, s)$ denotes the recession function of $f(x, s)$ with respect to s), the functional above reduces to the well-known form

$$F(\lambda) = \int_\Omega f\left(x, \frac{d\lambda}{d\mu}\right) d\mu + \int_\Omega f^\infty(x, \lambda^s) \qquad (1.2)$$

studied by Goffman & Serrin [11], and more recently by Valadier [16], Demengel & Temam [9], Bouchitté [2], De Giorgi & Ambrosio & Buttazzo [8]. In these last two papers it is also proved that (1.2) is the most general form for a mapping $F: \mathcal{M}(\Omega; \mathbf{R}^n) \to [0, +\infty]$ which is convex, sequentially weakly* l.s.c., and additive in the sense that

$$F(\lambda + \nu) = F(\lambda) + F(\nu) \qquad \text{whenever} \quad \lambda \perp \nu. \qquad (1.3)$$

Then, the following question naturally arises: given a sequentially weakly* l.s.c. and additive (in the sense of (1.3)) mapping $F: \mathcal{M}(\Omega; \mathbf{R}^n) \to [0, +\infty]$, not necessarily convex, can we find three Borel functions f, φ, g such that the integral representation formula (1.1) holds? In the present paper we answer affirmatively this question (*see* Theorem 2.3) and we list some conditions on f, φ, g which are necessary for the lower semicontinuity of F. Moreover, we show that under a very mild additional hypothesis, these necessary conditions are actually sufficient (*see* Theorem 2.6): this extends our previous semicontinuity results obtained in [3]. Finally, the uniqueness of the representation of F in the form (1.1) is discussed (*see* Proposition 2.9).

The integral representation result of Theorem 2.3 will be used in a forthcoming paper to characterize the lower semicontinuous envelope of a given nonconvex functional like (1.1).

2. NOTATIONS AND STATEMENT OF THE RESULTS

In this section we fix the notation we shall use in the following. In all the paper $(\Omega, \mathscr{B})$ will denote a measurable space, where Ω is a separable locally compact metric space with distance d and $\mathscr{B}$ is the σ-algebra of all Borel subsets of Ω. For every vector-valued measure $\lambda: \mathscr{B} \to \mathbf{R}^n$ and every $B \in \mathscr{B}$ the variation of λ on B is defined by

$$|\lambda|(B) = \sup \left\{ \sum_{h=1}^{\infty} |\lambda(B_h)| : \bigcup_{h=1}^{\infty} B_h \subset B,\ B_h \text{ pairwise disjoint} \right\}.$$

In this way, the set function $B \to |\lambda|(B)$ turns out to be a positive measure, which will be denoted by $|\lambda|$.

In the following we shall consider the spaces:

$C_0(\Omega; \mathbf{R}^n)$: the space of all continuous functions $u: \Omega \to \mathbf{R}^n$ "vanishing on the boundary", that is such that for every $\varepsilon > 0$ there exists a compact set $K_\varepsilon \subset \Omega$ with $|u(x)| < \varepsilon$ for all $x \in \Omega \setminus K_\varepsilon$;

$\mathscr{M}(\Omega; \mathbf{R}^n)$: the space of all vector-valued measures $\lambda: \mathscr{B} \to \mathbf{R}^n$ with finite variation on Ω. It is well-known that $\mathscr{M}(\Omega; \mathbf{R}^n)$ can be identified with the dual space of $C_0(\Omega; \mathbf{R}^n)$ by the duality

$$\langle \lambda, u \rangle_\Omega = \int_\Omega u \, d\lambda \,(u \in C_0(\Omega; \mathbf{R}^n),\ \lambda \in \mathscr{M}(\Omega; \mathbf{R}^n)),$$

and for every $\lambda \in \mathscr{M}(\Omega; \mathbf{R}^n)$ we have

$$|\lambda|(\Omega) = \sup \{ |\langle \lambda, u \rangle_\Omega| : u \in C_0(\Omega; \mathbf{R}^n),\ \|u\|_{C_0(\Omega; \mathbf{R}^n)} \leqq 1 \}.$$

The space $\mathscr{M}(\Omega; \mathbf{R}^n)$ will be endowed with the weak* topology deriving from the duality between $\mathscr{M}(\Omega; \mathbf{R}^n)$ and $C_0(\Omega; \mathbf{R}^n)$; in particular, a sequence $\{\lambda_n\}$ in $\mathscr{M}(\Omega; \mathbf{R}^n)$ will be said to w^*-converge to $\lambda \in \mathscr{M}(\Omega; \mathbf{R}^n)$ (and this will be indicated by $\lambda_h \to \lambda$) if and only if

$$\langle \lambda_h, u \rangle_\Omega \to \langle \lambda, u \rangle_\Omega \qquad \text{for every } u \in C_0(\Omega; \mathbf{R}^n).$$

In the sequel, we denote by $\mu: \mathscr{B} \to [0, +\infty[$ a positive, finite measure.

DEFINITION 2.1. – *We say that* $\lambda \in \mathscr{M}(\Omega; \mathbf{R}^n)$ *is*

(i) *absolutely continuous with respect to* μ *(and we write* $\lambda \ll \mu$*) if*

$$|\lambda|(B) = 0 \textit{ whenever } B \in \mathscr{B} \textit{ and } \mu(B) = 0;$$

(ii) *singular with respect to* μ *(and we write* $\lambda \perp \mu$*) if*

$$|\lambda|(\Omega \setminus B) = 0 \textit{ for a suitable } B \in \mathscr{B} \textit{ with } \mu(B) = 0.$$

In the following, given $u \in L^1(\Omega; \mathbf{R}^n; \mu)$ we shall denote by $u\mu$ (or simply by u when no confusion is possible) the measure of $\mathscr{M}(\Omega; \mathbf{R}^n)$ defined by

$$(u\mu)(B) = \int_B u \, d\mu \qquad \text{for every } B \in \mathscr{B}.$$

moreover, if $u: \Omega \to \mathbf{R}$ is a bounded Borel function and $\lambda \in \mathcal{M}(\Omega; \mathbf{R}^n)$, we denote by $u\lambda$ the measure of $\mathcal{M}(\Omega; \mathbf{R}^n)$

$$(u\lambda)(\mathrm{B}) = \int_{\mathrm{B}} u\, d\lambda \qquad \text{for every} \quad \mathrm{B} \in \mathscr{B}.$$

It is well-known that every measure $\lambda \in \mathcal{M}(\Omega; \mathbf{R}^n)$ which is absolutely continuous with respect to μ is representable in the form $\lambda = u\mu$ for a suitable $u \in \mathrm{L}^1(\Omega; \mathbf{R}^n; \mu)$; moreover, by the Legesgue-Nikodym decomposition theorem, for every $\lambda \in \mathcal{M}(\Omega; \mathbf{R}^n)$ there exists a unique function $u \in \mathrm{L}^1(\Omega; \mathbf{R}^n; \mu)$ (often indicated by $d\lambda/d\mu$) and a unique measure $\lambda^s \in \mathcal{M}(\Omega; \mathbf{R}^n)$ such that

$$\begin{cases} \text{(i)} & \lambda = u\mu + \lambda^s; \\ \text{(ii)} & \lambda^s \text{ is singular with respect to } \mu. \end{cases}$$

If $\varphi: \Omega \times \mathbf{R}^n \to [0, \infty]$ is a Borel function, with $\varphi(x, .)$ positively 1-homogeneous for every $x \in \Omega$, it is well-known (see for instance Goffman & Serrin [11]) that for every $\mathrm{B} \in \mathscr{B}$ and every $\lambda \in \mathcal{M}(\Omega; \mathbf{R}^n)$ the quantity

$$\int_{\mathrm{B}} \varphi\left(x, \frac{d\lambda}{d\vartheta}\right) d\vartheta$$

does not depend on ϑ, when ϑ varies over all positive measures such that $|\lambda| \ll \vartheta$. Therefore, in the following, we shall denote the quantity above simply by

$$\int_{\mathrm{B}} \varphi(x, \lambda).$$

For every subset B of Ω we denote by 1_{B} the function

$$1_{\mathrm{B}}(x) = \begin{cases} 1 & \text{if} \quad x \in \mathrm{B} \\ 0 & \text{if} \quad x \in \Omega \setminus \mathrm{B}; \end{cases}$$

moreover, for every $x \in \Omega$ we indicate by δ_x the measure of $\mathcal{M}(\Omega; \mathbf{R})$

$$\delta_x(\mathrm{B}) = \begin{cases} 1 & \text{if} \quad x \in \mathrm{B} \\ 0 & \text{if} \quad x \in \Omega \setminus \mathrm{B}. \end{cases}$$

Besides to $\mathcal{M}(\Omega; \mathbf{R}^n)$ the following spaces of measures will be considered: $\mathcal{M}^0(\Omega; \mathbf{R}^n)$ the space of all non-atomic measures of $\mathcal{M}(\Omega; \mathbf{R}^n)$; $\mathcal{M}^{\#}(\Omega; \mathbf{R}^n)$ the space of all "purely atomic" measures of $\mathcal{M}(\Omega; \mathbf{R}^n)$, that is the measures of the form

$$\lambda = \sum_{i=1}^{\infty} a_i \delta_{x_i} \; (x_i \in \Omega,\ a_i \in \mathbf{R}^n).$$

It is easy to see that for every $\lambda \in \mathcal{M}(\Omega; \mathbf{R}^n)$ there exist two unique measures $\lambda^0 \in \mathcal{M}^0(\Omega; \mathbf{R}^n)$ and $\lambda^\# \in \mathcal{M}^\#(\Omega; \mathbf{R}^n)$ (respectively called non-atomic part and atomic part of λ) such that $\lambda = \lambda^0 + \lambda^\#$. In other words we have

$$\mathcal{M}(\Omega; \mathbf{R}^n) = \mathcal{M}^0(\Omega; \mathbf{R}^n) \oplus \mathcal{M}^\#(\Omega; \mathbf{R}^n).$$

For every $\lambda \in \mathcal{M}(\Omega; \mathbf{R}^n)$ we simply write $\lambda(x)$ instead of $\lambda(\{x\})$, and we denote by A_λ the set of all atomes of λ, that is

$$A_\lambda = \{x \in \Omega : \lambda(x) \neq 0\}.$$

For every proper function $f: \mathbf{R}^n \to]-\infty, +\infty]$ we define as usual the conjugate function f^* by

$$f^*(s) = \sup\{sw - f(w) : w \in \mathbf{R}^n\} \qquad (s \in \mathbf{R}^n),$$

and the recession function f^∞ by

$$f^\infty(s) = \sup\{f(s+t) - f(t) : t \in \mathbf{R}^n, f(t) < +\infty\} \qquad (s \in \mathbf{R}^n).$$

When f is convex and l.s.c. it is well-known that f^* is convex l.s.c. and proper, and we have $f^{**} = f$; moreover, in this case, for the recession function f^∞ the following formula holds (see for instance Rockafellar [15]):

$$f^\infty(s) = \lim_{t \to +\infty} \frac{f(s_0 + ts)}{t}$$

where s_0 is any point such that $f(s_0) < +\infty$. It can be shown that the definition above does not depend on s_0, and that the function f^∞ turns out to be convex, l.s.c., and positively 1-homogeneous on $\mathbf{R}^n$.

In the sequel we deal with Borel functions $f: \Omega \times \mathbf{R}^n \to [0, +\infty]$ such that $f(x, .)$ is convex l.s.c. and proper for μ-a.e. $x \in \Omega$. For this kind of functions we define for every $(x, s) \in \Omega \times \mathbf{R}^n$

$$\varphi_{f,\mu}(x, s) = \sup\left\{u(x)s : u \in C_0(\Omega; \mathbf{R}^n), \int_\Omega f^*(x, u)\,d\mu < +\infty\right\}.$$

The function $\varphi_{f,\mu}(x, s)$ is l.s.c. in (x, s), convex and positively 1-homogeneous in s, and we have (*see* for instance Bouchitté & Valadier [4], Proposition 7)

$$\varphi_{f,\mu}(x, .) \leqq f^\infty(x, .) \text{ for } \mu\text{-a.e. } x \in \Omega;$$
$$\varphi_{f,\mu}(x, .) \geqq f^\infty(x, .) \text{ for every } x \in \Omega$$
if the multimapping $x \to \operatorname{epi} f^*(x, .)$ is l.s.c. on Ω.

In particular we have $\varphi_{f,\mu} = f^\infty$ when f does not depend on x.

For every function $g: \mathbf{R}^n \to [0, +\infty]$ with $g(0) = 0$ we define

$$g^0(s) = \limsup_{t \to 0^+} \frac{g(ts)}{t} \qquad (s \in \mathbf{R}^n);$$

moreover, g will be called *subadditive* if

$$g(s_1+s_2) \leqq g(s_1)+g(s_2) \qquad \text{for every} \quad s_1, s_2 \in \mathbf{R}^n.$$

We remark that g is subadditive if and only if $g^\infty \leqq g$, hence $g^\infty = g$ for every subadditive function g with $g(0)=0$. Finally, if $g: \Omega \times \mathbf{R}^n \to [0, +\infty]$ is a function, we denote by $\hat{g}: \Omega \times \mathbf{R}^n \to [0, +\infty]$ the function

$$\hat{g}(x, s) = \liminf_{\substack{(y, t) \to (x, s) \\ y \neq x}} g(y, t).$$

The following proposition holds (*see* Bouchitté & Buttazzo [3], Proposition 2.2).

Proposition 2.2. – *Let* $g: \mathbf{R}^n \to [0, +\infty]$ *be a subadditive l.s.c. function, with* $g(0)=0$. *Then we have:*

(i) *the function* $g^0: \mathbf{R}^n \to [0, +\infty]$ *is convex, l.s.c., and positively* 1-*homogeneous;*

(ii) $g^0(s) = \sup_{t>0} \dfrac{g(ts)}{t} = \lim_{t \to 0^+} \dfrac{g(ts)}{t}$ *for every* $s \in \mathbf{R}^n$.

The following theorem is our main integral representation result for functionals F: $\mathcal{M}(\Omega; \mathbf{R}^n) \to [0, +\infty]$.

Theorem 2.3. – *Let* F: $\mathcal{M}(\Omega; \mathbf{R}^n) \to [0, +\infty]$ *be a functional such that*

(i) F *is additive (i.e.* $F(\lambda+\nu) = F(\lambda) + F(\nu)$ *whenever* $\lambda \perp \nu$*);*

(ii) F *is sequentially weakly* l.s.c. on* $\mathcal{M}(\Omega; \mathbf{R}^n)$.

Then there exist a non-atomic positive measure $\mu \in \mathcal{M}(\Omega)$ *and three Borel functions* $f, \varphi, g: \Omega \times \mathbf{R}^n \to [0, +\infty]$ *which satisfy*

(H_1) $f(x, .)$ *is convex and l.s.c. on* $\mathbf{R}^n$, *and* $f(x, 0)=0$ *for* μ-*a.e.* $x \in \Omega$,

(H_2) $f^\infty(x, .) = \varphi_{f,\mu}(x, .)$ *for* μ-*a.e.* $x \in \Omega$,

(H_3) g *and* g^∞ *are l.s.c. on* $\Omega \times \mathbf{R}^n$, *and* $g(x, 0)=0$ *for every* $x \in \Omega$,

(H_4) $g^\infty \leqq \varphi_{f,\mu}$ *and* $g^\infty \leqq \hat{g}$ *on* $\Omega \times \mathbf{R}^n$,

(H_5) $g^0 = \varphi = \varphi_{f,\mu}$ *on* $(\Omega \setminus N) \times \mathbf{R}^n$, *where* N *is a suitable subset of* Ω, *and such that for every* $\lambda \in \mathcal{M}(\Omega; \mathbf{R}^n)$ *the following integral representation formula holds:*

$$F(\lambda) = \int_\Omega f\left(x, \frac{d\lambda}{d\mu}\right) d\mu + \int_{\Omega \setminus A_\lambda} \varphi(x, \lambda^s) + \int_{A_\lambda} g(x, \lambda(x))\, d\#. \qquad (2.1)$$

Remark 2.4. – The function $g(x, .)$ in Theorem 2.3 is not, in general, subadditive for every $x \in \Omega$ (*see* Example 2.7); however, setting

$$D = \{x \in \Omega : g(x, .) \text{ is not subadditive}\}, \qquad (2.2)$$

we may prove that D is at most countable. Indeed, let (x_h, s_h, τ_h) a sequence which is dense in

$$\operatorname{epi} g = \{(x, s, r) \in \Omega \times \mathbf{R}^n \times \mathbf{R} : g(x, s) \leqq r\},$$

and let $D' = \{x_h : h \in \mathbf{N}\}$; it is enough to show that $D \subset D'$. Since by (H_4) it is $g^\infty \leqq \hat{g}$, by (2.2) we have

$$D \subset \{x \in \Omega : g(x, s) < \hat{g}(x, s) \text{ for some } s \in \mathbf{R}^n\}.$$

Hence, if $x \in D$, it is

$$g(x, s) < r < \hat{g}(x, s) \quad \text{for some } s \in \mathbf{R}^n,\ r \in \mathbf{R}, \tag{2.3}$$

so that $(x, s, r) \in \operatorname{epi} g$; then there exists a subsequence $(x_{h_k}, s_{h_k}, \tau_{h_k})$ converging to (x, s, r). Assume by contradiction that $x \notin D'$; then $x_{h_k} \neq x$ for every $k \in \mathbf{N}$, so that

$$\hat{g}(x, s) \leqq \liminf_{k \to +\infty} g(x_{h_k}, s_{h_k}) \leqq \liminf_{k \to +\infty} \tau_{h_k} = r,$$

which contradicts (2.3).

Remark 2.5. — Conditions $(H_1)-(H_5)$ are not sufficient, in general, for the weak* lower semicontinuity of the functional (2.1) (*see* Example 2.8). However, they become sufficient (*see* Theorem 2.6) if the set D introduced in (2.2) has no accumulation points.

The following lower semicontinuity theorem is an extension of a previous result obtained by us (*see* Bouchitté & Buttazzo [3]) under the assumption that $g(x, .)$ is subadditive for every $x \in \Omega$.

THEOREM 2.6. — *Let $\mu \in \mathcal{M}(\Omega)$ be a non-atomic positive measure and let f, φ, g: $\Omega \times \mathbf{R}^n \to [0, +\infty]$ three Borel functions satisfying conditions $(H_1)-(H_5)$ of Theorem 2.3. Assume in addition that the at most countable set D introduced in (2.2) has no accumulation points. Then the functional F defined in (2.1) is sequentially weakly* l.s.c.*

Example 2.7. — Let $\Omega =]-1, 1[$, let μ be the Lebesque measure on Ω, and let f, φ, $g : \Omega \times \mathbf{R}^n \to [0, +\infty]$ be the functions

$$f(x, s) = |s|^2$$
$$\varphi(x, s) = \begin{cases} 0 & \text{if } s = 0 \\ +\infty & \text{if } s \neq 0 \end{cases}$$
$$g(x, s) = \begin{cases} k|s|^2 & \text{if } xs = 0 (k > 0) \\ +\infty & \text{if } xs \neq 0. \end{cases}$$

Then it is easy to check that all conditions of the semicontinuity theorem 2.6 are fulfilled; indeed it is

$$g^\infty = \hat{g} = \varphi_{f,\mu} = \varphi = f^\infty \quad \text{on } \Omega \times \mathbf{R},$$

and the set D introduced in (2.2) reduces to $\{0\}$, that is $g(0, .)$ is not subadditive.

Example 2.8. — Let Ω, μ, f, φ be as in Example 2.7, and let $g:\Omega\times\mathbf{R}\to[0, +\infty]$ be the function

$$g(x, s)=\begin{cases} 0 & \text{if } s=0 \\ k|s|^2 & \text{if } x=1/k \quad \text{with } k\in\mathrm{N} \\ +\infty & \text{otherwise.} \end{cases}$$

Then it is easy to check that all conditions $(\mathrm{H}_1)-(\mathrm{H}_5)$ of Theorem 2.3 are fulfilled; indeed it is

$$g^\infty=\hat{g}=\varphi_{f,\mu}=\varphi=f^\infty \quad \text{on } \Omega\times\mathbf{R}.$$

However, the functional F defined in (2.1) is not weakly* l.s.c. In fact, take $\lambda=\delta_0$ and

$$\lambda_h=\sum_{k\geq h}\frac{h}{k^2}\delta_{1/k}.$$

We have $\lambda_h\to\lambda$ and $\mathrm{F}(\lambda)=+\infty$, whereas

$$\liminf_{h\to+\infty}\mathrm{F}(\lambda_h)=\liminf_{h\to+\infty}\sum_{k\geq h}k\left(\frac{h}{k^2}\right)^2=\frac{1}{2}.$$

Note that the set D introduced in (2.2) is in this case the set $\{1/k:k\in\mathbf{N}\}$ which has 0 as an accumulation point.

We now discuss the uniqueness of the integral representation of F in the form (2.1).

PROPOSITION 2.9. — *Let $\mu\in\mathcal{M}(\Omega)$ be a positive non-atomic measure, let f, φ, $g:\Omega\times\mathbf{R}^n\to[0, +\infty]$ be three Borel functions, let* F *be the functional defined in* (2.1), *and let* A *be the set*

$$\mathrm{A}=\{x\in\Omega:f(x, .)\neq\varphi(x, .)\}.$$

Then, another representation of F *in the form*

$$\int_\Omega f_1\left(x, \frac{d\lambda}{d\mu_1}\right)d\mu_1+\int_{\Omega\setminus\mathrm{A}_\lambda}\varphi_1(x, \lambda^s)+\int_{\mathrm{A}_\lambda}g_1(x, \lambda(x))\,d\# \tag{2.4}$$

holds, where in (2.4) $f_1^\infty(x, .)=\varphi_1(x, .)$ *for* μ_1*-a.e.* $x\in\Omega$ *and* λ^s *is the singular part of* λ *with respect to* μ_1, *if and only if we have*

(i) $g=g_1$ *on* $\Omega\times\mathbf{R}^n$,

(ii) *there exists a countable subset* N *of* Ω *such that* $\varphi=\varphi_1$ *on* $(\Omega\setminus\mathrm{N})\times\mathbf{R}^n$,

(iii) *There exists* $\alpha\in\mathrm{L}^1(\Omega;\mu_1)$ *such that* $1_\mathrm{A}\mu=\alpha\mu_1$,

(iv) *there exists a Borel subset* M *of* Ω *with* $\mu_1(\mathrm{M})=0$ *such that*

$$f_1(x, s)=\begin{cases} \alpha(x)f\left(x, \dfrac{s}{\alpha(x)}\right) & \text{if } \alpha(x)\neq 0 \\ \varphi(x, s) & \text{if } \alpha(x)=0 \end{cases} \quad \text{on } (\Omega\setminus\mathrm{M})\times\mathbf{R}^n.$$

3. PROOF OF THE RESULTS

In this section we prove Theorem 2.3, Theorem 2.6, and Proposition 2.9 stated in Section 2. We shall use some preliminary results which we state separately in the following lemmas.

LEMMA 3.1. – *Let* $F:\mathcal{M}(\Omega;\mathbf{R}^n)\to[0,+\infty]$ *be a functional satisfying conditions* (*i*) *and* (*ii*) *of Theorem* 2.3. *Then the Moreau-Yosida regularizations* F_k *defined by*

$$F_k(\lambda)=\inf\{F(\nu)+k\|\lambda-\nu\|:\nu\in\mathcal{M}(\Omega;\mathbf{R}^n)\}$$

satisfy (*i*) *and* (*ii*) *too. Moreover we have*

$$F=\sup\{F_k:k\in\mathbf{N}\}. \tag{3.1}$$

Proof. – Equality (3.1) is a well-known consequence of the fact that F is l.s.c. with respect to the norm topology (*see* for instance Proposition 1.3.7 of Buttazzo [6]).

Let us prove property (i) for F_k. We first notice that for every $\lambda\in\mathcal{M}(\Omega;\mathbf{R}^n)$ and $B\in\mathcal{B}$ there exists $\eta\in\mathcal{M}(\Omega;\mathbf{R}^n)$ such that

$$F_k(1_B\lambda)=F(1_B\eta)+k\|1_B\eta-1_B\lambda\|.$$

Indeed, there exists $\eta\in\mathcal{M}(\Omega;\mathbf{R}^n)$ such that the infimum in the definition of $F_k(1_B\lambda)$ is achieved, so that

$$F_k(1_B\lambda)=F(\eta)+k\|\eta-1_B\lambda\|\geqq F(1_B\eta)+k\|1_B\eta-1_B\lambda\|\geqq F_k(1_B\lambda).$$

Now, if $B_1, B_2\in\mathcal{B}$ with $B_1\cap B_2=\varnothing$, and $\lambda\in\mathcal{M}(\Omega;\mathbf{R}^n)$, we have for suitable $\eta_1,\eta_2\in\mathcal{M}(\Omega;\mathbf{R}^n)$

$$\begin{aligned}F_k(1_{B_1}\lambda)+F_k(1_{B_2}\lambda)&=F(1_{B_1}\eta_1)+F(1_{B_2}\eta_2)\\&\quad+k\|1_{B_1}\eta_1-1_{B_1}\lambda\|+k\|1_{B_2}\eta_2-1_{B_2}\lambda\|\\&=F(1_{B_1}\eta_1+1_{B_2}\eta_2)+k\|1_{B_1}\eta_1+1_{B_2}\eta_2-1_{B_1\cup B_2}\lambda\|\geqq F_k(1_{B_1\cup B_2}\lambda),\end{aligned}$$

which proves F_k is subadditive. The superadditivity of F_k follows from the fact that F_k is an infimum of additive functionals.

Finally, we prove property (ii) for F_k. Let $\lambda_h\to\lambda$ be such that

$$\liminf_{h\to\infty}F_k(\lambda_h)<+\infty.$$

Then for suitable $\eta_h\in\mathcal{M}(\Omega;\mathbf{R}^n)$ we have

$$F_k(\lambda_h)=F(\eta_h)+k\|\eta_k-\lambda_h\|.$$

Possibly passing to subsequences we may assume η_h is bounded in $\mathcal{M}(\Omega;\mathbf{R}^n)$ and weakly* converges to a measure $\eta\in\mathcal{M}(\Omega;\mathbf{R}^n)$. Therefore, by the weak* lower semicontinuity of the norm in $\mathcal{M}(\Omega;\mathbf{R}^n)$ we get

$$\liminf_{h\to\infty}F_k(\lambda_h)\geqq F(\eta)+k\|\eta-\lambda\|\geqq F_k(\lambda),$$

and the proff of the lemma completely achieved. ■

LEMMA 3.2. – *Let* $F:\mathcal{M}(\Omega;\mathbf{R}^n)\to[0,+\infty]$ *be a functional satisfying conditions* (*i*) *and* (*ii*) *of Theorem* 2.3, *and let* ν *be a positive non-atomic measure of* $\mathcal{M}(\Omega)$. *Then there exist two Borel functions* $j, f:\Omega\times\mathbf{R}^n\to[0,+\infty]$ *such that*

(*a*) $j(x,.)$ *and* $f(x,.)$ *are convex l.s.c. and* $j(x,0)=f(x,0)=0$ *for every* $x\in\Omega$;

(*b*) $f^\infty(x,.)=\varphi_{f,\nu}(x,.)$ *for every* $x\in\Omega$;

(*c*) $F(\lambda)=\int_\Omega j(x, d\lambda/d\nu)\,d\nu$ *for every* $\lambda\ll\nu$;

(*d*) $\overline{F_\nu}(\lambda)=\int_\Omega f(x, d\lambda/d\nu)\,d\nu+\int_\Omega \varphi_{f,\nu}(x,\lambda^s)$ *for every* $\lambda\in\mathcal{M}(\Omega;\mathbf{R}^n)$;

where F_ν *is the functional defined by*

$$F_\nu(\lambda)=\begin{cases}F(\lambda) & \text{if } \lambda\ll\nu\\ +\infty & \text{otherwise}\end{cases}\tag{3.2}$$

and where $\overline{F_\nu}$ *denotes the greatest sequentially weakly* l.s.c. functional which is less than or equal to* F_ν.

Proof. – Let us define $J: L^1(\Omega;\mathbf{R}^n;\nu)\to[0,+\infty]$ by setting

$$J(u)=F(u\nu)\qquad\text{for every}\quad u\in L^1(\Omega;\mathbf{R}^n;\nu);$$

it is clear that

$$\begin{cases}J\text{ is sequentially weakly l.s.c. on } L^1(\Omega;\mathbf{R}^n;\nu)\\ J\text{ is additive on } L^1(\Omega;\mathbf{R}^n;\nu)\text{ (i.e. } J(u+v)\\ \qquad =J(u)+J(v)\text{ whenever } uv=0\ \nu\text{-a.e. on }\Omega).\end{cases}$$

Therefore (*see* for instance Hiai [12], Ioffe [13], Buttazzo & Dal Maso [7], or the recent book of Buttazzo [6]) there exists a Borel function $j:\Omega\times\mathbf{R}^n\to[0,+\infty]$ such that $j(x,.)$ is convex and l.s.c., $j(x,0)=0$ for every $x\in\Omega$, and

$$J(u)=\int_\Omega j(x,u)\,d\nu\qquad\text{for every}\quad u\in L^1(\Omega;\mathbf{R}^n;\nu),$$

so that (*c*) is proved. The functional $\overline{F_\nu}$ is then convex, proper, and sequentially weakly* l.s.c. on $\mathcal{M}(\Omega;\mathbf{R}^n)$. Therefore (*see* for instance Dunford & Schwartz [10], Chapter V) $\overline{F_\nu}$ is topologically weakly* l.s.c. on $\mathcal{M}(\Omega;\mathbf{R}^n)$, and so, by a standard argument of convex analysis,

$$\overline{F_\nu}=F_\nu^{**}.$$

By a theorem of Rockafellar [14], we have

$$F_\nu^*(u)=\int_\Omega j^*(x,u)\,d\nu\qquad\text{for every}\quad u\in C_0(\Omega;\mathbf{R}^n),$$

so that

$$\overline{F_\nu} = F_\nu^{**} = \sup \left\{ \int_\Omega u\, d\lambda - \int_\Omega j^*(x, u)\, d\nu : u \in C_0(\Omega; \mathbf{R}^n) \right\}.$$

By Theorem 4 of Bouchitté & Valadier [4] we obtain

$$\overline{F_\nu}(\lambda) = \int_\Omega f\left(x, \frac{d\lambda}{d\nu}\right) d\nu + \int_\Omega h(x, \lambda^s) \qquad \text{for every} \quad \lambda \in \mathcal{M}(\Omega; \mathbf{R}^n),$$

where $h = \varphi_{j,\nu}$ and

$$f(x, .) = (j^*(x, .) + h^*(x, .)) \qquad \text{for every} \quad x \in \Omega.$$

From Proposition 7 of Bouchitté & Valadier [4] it follows

$$f^\infty(x, .) = h(x, .) \qquad \text{for} \quad \nu\text{-a.e. } x \in \Omega,$$

so that it remains only to prove that $h = \varphi_{f,\nu}$, that is

$$\left. \begin{array}{c} \displaystyle\int_\Omega j^*(x, u)\, d\nu < +\infty \Leftrightarrow \int_\Omega f^*(x, u)\, d\nu < +\infty \\ \text{for every} \quad u \in C_0(\Omega; \mathbf{R}^n). \end{array} \right\} \tag{3.3}$$

Since $f^* = j^* + h^* \geqq j^*$, the implication $\Leftarrow$ in (3.3) is obvious; conversely, if $u \in C_0(\Omega; \mathbf{R}^n)$ with $\int_\Omega j^*(x, u)\, d\nu < +\infty$, by definition of h we have

$$h^*(x, u) = 0 \qquad \text{for every} \quad x \in \Omega,$$

which yields

$$j^*(x, u) = f^*(x, u) \qquad \text{for every} \quad x \in \Omega. \blacksquare$$

COROLLARY 3.3. – *Let* F *be as in Lemma* 3.2. *Then* F *is convex on* $\mathcal{M}^0(\Omega; \mathbf{R}^n)$.

Proof. – Let $\lambda_1, \lambda_2 \in \mathcal{M}^0(\Omega; \mathbf{R}^n)$ and let $\nu = |\lambda_1| + |\lambda_2|$. By Lemma 3.2 we get

$$F(t\lambda_1 + (1-t)\lambda_2) \leqq t F(\lambda_1) + (1-t) F(\lambda_2) \qquad \text{for every} \quad t \in [0, 1]. \quad \blacksquare$$

We are now in a position to prove our results.

Proof of Theorem 2.3. – Possibly passing to Moreau-Yosida regularizations F_k, using Lemma 3.1, and letting $k \to \infty$, it is not restrictive to assume on F that

$$|F(\lambda_1) - F(\lambda_2)| \leqq k \|\lambda_1 - \lambda_2\| \qquad \text{for every} \quad \lambda_1, \lambda_2 \in \mathcal{M}(\Omega; \mathbf{R}^n). \tag{3.4}$$

Let us define on $\mathcal{M}(\Omega; \mathbf{R}^n)$ the functional

$$F^0(\lambda) = \begin{cases} F(\lambda) & \text{if} \quad \lambda \in \mathcal{M}^0(\Omega; \mathbf{R}^n) \\ +\infty & \text{otherwise.} \end{cases}$$

For every $u \in C_0(\Omega; \mathbf{R}^n)$ we have

$$(F^0)^*(u) = \sup\left\{\int_\Omega u\, d\lambda - F^0(\lambda) : \lambda \in \mathcal{M}^0(\Omega; \mathbf{R}^n)\right\},$$

and by the Lindelöff property of $C_0(\Omega; \mathbf{R}^n)$, the supremum above is attained by taking a suitable sequence (λ_h) in $\mathcal{M}^0(\Omega; \mathbf{R}^n)$ (independent of u). Setting

$$\mu = \sum_{h\in N} 2^{-h} \frac{|\lambda_h|}{\|\lambda_h\|}$$

we have that μ is a positive non-atomic measure of $\mathcal{M}(\Omega)$, and

$$\{\lambda_h : h \in \mathbf{N}\} \subset \{\lambda \in \mathcal{M}(\Omega; \mathbf{R}^n) : \lambda \ll \mu\} \subset \mathcal{M}^0(\Omega; \mathbf{R}^n).$$

Therefore, for every $u \in C_0(\Omega; \mathbf{R}^n)$,

$$(F^0)^*(u) = \sup\left\{\int_\Omega u\, d\lambda - F^0(\lambda) : \lambda \ll \mu\right\} = (F_\mu)^*(u),$$

where F_μ is defined as in (3.2). Now, by Lemma 3.2 and Corollary 3.3, F_μ and F^0 are convex; then

$$\overline{F^0} = (F^0)^{**} = (F_\mu)^{**} = \overline{F_\mu}.$$

Hence, by Lemma 3.2 there exists a Borel function $f : \Omega \times \mathbf{R}^n \to [0, +\infty]$ with $f(x, .)$ convex and l.s.c. and $f(x, 0) = 0$ for every $x \in \Omega$, such that

$$\overline{F^0}(\lambda) = \int_\Omega f\left(x, \frac{d\lambda}{d\mu}\right) d\mu + \int_\Omega \varphi_{f,\mu}(x, \lambda^s) \qquad \text{for every} \quad \lambda \in \mathcal{M}(\Omega; \mathbf{R}^n). \tag{3.5}$$

We remark that, since F is sequentially l.s.c., it is $F = \overline{F^0}$ on $\mathcal{M}^0(\Omega; \mathbf{R}^n)$. Let us define for every $(x, s) \in \Omega \times \mathbf{R}^n$

$$\varphi(x, s) = \varphi_{f,\mu}(x, s) \qquad \text{and} \qquad g(x, s) = F(s\,\delta_x).$$

It is easy to verify that condition (3.4) implies

$$\left.\begin{array}{l} |f(x, s_1) - f(x, s_2)| \leqq k\,|s_1 - s_2| \\ |\varphi(x, s_1) - \varphi(x, s_2)| \leqq k\,|s_1 - s_2| \\ |g(x, s_1) - g(x, s_2)| \leqq k\,|s_1 - s_2|. \end{array}\right\} \tag{3.6}$$

By using the additivity and the lower semicontinuity of F, for every integer M and every $\lambda \in \mathcal{M}^\#(\Omega; \mathbf{R}^n)$ of the form

$$\lambda = \sum_{h\in\mathbf{N}} s_h\, \delta_{x_h}$$

we obtain

$$\sum_{h=0}^{M} F(s_h\, \delta_{x_h}) \leqq F(\lambda) \leqq \sum_{h\in\mathbf{N}} F(s_h\, \delta_{x_h}),$$

which yields

$$F(\lambda)=\int_{A_\lambda} g(x, \lambda(x))\, d\sharp.$$

Therefore, for every $\lambda \in \mathcal{M}(\Omega; \mathbf{R}^n)$ we get

$$F(\lambda)=F(\lambda^0)+F(\lambda^\sharp)=\overline{F_0}(\lambda^0)+\int_{A_\lambda} g(x, \lambda(x))\, d\sharp$$
$$=\int_\Omega f\left(x, \frac{d\lambda}{d\mu}\right) d\mu+\int_{\Omega\setminus A_\lambda} \varphi(x, \lambda^s)+\int_{A_\lambda} g(x, \lambda(x))\, d\sharp,$$

that is (2.1).

It remains only to prove properties (H_3), (H_4), (H_5). The lower semicontinuity of g and the fact that $g(x, 0)=0$ for every $x\in\Omega$ follow immediately from the definition of g and from the lower semicontinuity of F; then property (H_3) will be achieved if we prove that g^∞ is l.s.c. Since the recession function F^∞ of F is sequentially weakly* l.s.c. (being supremum of sequentially weakly* l.s.c. mappings), it will be enough to show that

$$F^\infty(s\delta_x)=g^\infty(x, s) \qquad \text{for every} \quad (x, s)\in\Omega\times\mathbf{R}^n.$$

Every measure $\lambda\in\mathcal{M}(\Omega; \mathbf{R}^n)$ can be decomposed in the form $\lambda=t\delta_x+\nu$ with $\nu\perp\delta_x$, therefore, by using the additivity of F, we have

$$\begin{aligned}F^\infty(s\delta_x)&=\sup\{F(s\delta_x+\lambda)-F(\lambda):\lambda\in\mathcal{M}(\Omega; \mathbf{R}^n),\ F(\lambda)<+\infty\}\\&=\sup\{F((s+t)\delta_x)-F(t\delta_x): t\in\mathbf{R}^n,\ F(t\delta_x)<+\infty\}\\&=\sup\{g(x, s+t)-g(x, t): t\in\mathbf{R}^n,\ g(x, t)<+\infty\}=g^\infty(x, s).\end{aligned}$$

In order to prove (H_4) let us fix $x\in\Omega$ and $s\in\mathbf{R}^n$. By (3.5) we get $\varphi_{f,\mu}(x, s)=\overline{F^0}(s\delta_x)$, hence, since $F\leqq F^0$, we have

$$g(x, s)=F(s\delta_x)\leqq\overline{F^0}(s\delta_x)=\varphi_{f,\mu}(x, s).$$

By the subadditivity of $\varphi_{f,\mu}(x, .)$ this implies

$$g^\infty(x, s)\leqq\varphi_{f,\mu}(x, s).$$

In order to prove the inequality $g^\infty\leqq\hat{g}$, let $t\in\mathbf{R}^n$ and let $x_h\in\Omega\setminus\{x\}$, $s_h\in\mathbf{R}^n$ be such that $x_h\to x$, $s_h\to s$, and

$$\hat{g}(x, s)=\lim_{h\to+\infty} g(x_h, s_h).$$

Taking $\lambda_h=t\delta_z+s_h\delta_{x_h}$ we obtain $\lambda_h\to(t+s)\delta_x$ and, by the lower semicontinuity and additivity of F,

$$\begin{aligned}g(x, s+t)=F((s+t)\delta_x)&\leqq\liminf_{h\to+\infty} F(\lambda_h)\\&=\liminf_{h\to+\infty}[F(t\delta_x)+F(s_h\delta_{x_h})]=g(x, t)+\hat{g}(x, s).\end{aligned}$$

By the definition of g^∞, this implies that $g^\infty(x, s)\leqq\hat{g}(x, s)$.

Let us prove now condition (H_5). From the definition of φ and from (H_4) we get $g \leqq g^\infty \leqq \varphi_{f,\mu} = \varphi$ on $\Omega \times \mathbf{R}^n$; hence, by the homogeneity of φ, it is $g^0 \leqq \varphi$ on $\Omega \times \mathbf{R}^n$. Then we have just to prove that the set

$$N = \{ x \in \Omega : g^0(x, s) < \varphi(x, s) \text{ for some } s \in \mathbf{R}^n \}$$

is at most countable, which is equivalent, by Lemma 3.1 of Bouchitté & Buttazzo [3], to show that $\nu(N) = 0$ for every positive non-atomic measure $\nu \in \mathcal{M}(\Omega)$. By (3.6) it will be enough to show that

$$\int_K \varphi(x, s)\, d\nu \leqq \int_K g^0(x, s)\, d\nu \tag{3.7}$$

for every positive non-atomic measure $\nu \in \mathcal{M}(\Omega)$, every compact subset K of Ω, and every $s \in \mathbf{R}^n$. Fix $\varepsilon > 0$, for every $h \in \mathbf{N}$ let $(B_{i,h})_{i \in I_h}$ be a finite partition of K into Borel sets whose diameter is less than $1/h$, and let $s_{i,h} = s\,\nu(B_{i,h})$. If D is the set defined in (2.2), we may find $x_{i,h} \in B_{i,h} \setminus D$ such that

$$g^0(x_{i,h}, s) \leqq \varepsilon + g^0(x, s) \qquad \text{for every} \quad x \in B_{i,h} \setminus D;$$

moreover, since $g(x_{i,h}, .)$ is subadditive and D is at most countable (*see* Remark 2.4), we have

$$g(x_{i,h}, s_{i,h}) \leqq g^0(x_{i,h}, s_{i,h}) = \nu(B_{i,h})\, g^0(x_{i,h}, s) \leqq \int_{B_{i,h}} [\varepsilon + g^0(x, s)]\, d\nu.$$

It is easy to see that the sequence

$$\lambda_h = \sum_{i \in I_h} s_{i,h}\, \delta_{x_{i,h}}$$

weakly* converges to $1_K s \nu$, so that, by the lower semicontinuity of F,

$$\begin{aligned}
\int_K f\left(x, s\frac{d\nu}{d\mu}\right) d\mu + \int_K \varphi(x, s)\, d\nu^s &= F(1_K s\nu) \leqq \liminf_{h \to \infty} F(\lambda_h) \\
= \liminf_{h \to \infty} \sum_{i \in I_h} g(x_{i,h}, s_{i,h}) &\leqq \liminf_{h \to +\infty} \sum_{i \in I_h} \nu(B_{i,h})\, g^0(x_{i,h}, s) \\
&\leqq \liminf_{h \to \infty} \sum_{i \in I_h} \int_{B_{i,h}} [\varepsilon + g^0(x, s)]\, d\nu = \varepsilon\nu(K) + \int_K g^0(x, s)\, d\nu.
\end{aligned}$$

Letting $\varepsilon \to 0$ and substituting s with $ts\,(t > 0)$, we get

$$\int_K \frac{1}{t} f\left(x, ts\frac{d\nu}{d\mu}\right) d\mu + \int_K \varphi(x, s)\, d\nu^s \leqq \int_K g^0(x, s)\, d\nu,$$

and passing to the limit as $t \to +\infty$, using the homogeneity of f^∞,

$$\int_K f^\infty(x, s)\frac{d\nu}{d\mu}\, d\mu + \int_K \varphi(x, s)\, d\nu^s \leqq \int_K g^0(x, s)\, d\nu.$$

Recalling condition (H_2) already proved, we obtain (3.7), and so the proof of Theorem 2.3 is completely achieved. ■

Proof of Theorem 2.6. – Let $\lambda_h \to \lambda$ and set $\lambda'_h = 1_{\Omega \setminus D} \lambda_h$ and $\lambda''_h = 1_D \lambda_h$. Possibly passing to subsequences, we may assume that $\lambda'_h \to \lambda'$ and $\lambda''_h \to \lambda''$ with $\lambda' + \lambda'' = \lambda$, and that $F(\lambda_h)$ has a limit. Denote by $x_1, x_2, \ldots$ the elements of D; since every x_k is isolated, the set D is closed, so that λ'' is supported by D. Moreover,

$$\lambda''_h(x_k) \to \lambda''(x_k) \qquad \text{for every} \quad k \in \mathbf{N}.$$

By the lower semicontinuity of g we have

$$\liminf_{h \to +\infty} F(\lambda''_h) = \liminf_{h \to +\infty} \sum_{k \in \mathbf{N}} g(x_k, \lambda''_h(x_k)) \geqq \sum_{k \in \mathbf{N}} g(x_k, \lambda''(x_k)) = F(\lambda'').$$

Let us denote by $G : \mathscr{M}(\Omega; \mathbf{R}^n) \to [0, +\infty]$ the functional

$$G(\lambda) = \int_\Omega f\left(x, \frac{d\lambda}{d\mu}\right) d\mu + \int_{\Omega \setminus A_\lambda} \varphi(x, \lambda^s) + \int_{A_\lambda} g^\infty(x, \lambda(x))\, d\#;$$

it is easy to check that $g^\infty(x, .)$ is subadditive for all $x \in \Omega$ and that $(g^\infty)^0 = \varphi = \varphi_{f, \mu}$ on $(\Omega \setminus N) \times \mathbf{R}^n$, so that all assumptions of the lower semicontinuity Theorem 3.3 of Bouchitté & Buttazzo [3] are fulfilled. Moreover, $G(\lambda'_h) = F(\lambda'_h)$ because $g = g^\infty$ on $(\Omega \setminus D) \times \mathbf{R}^n$. Therefore

$$\liminf_{h \to +\infty} F(\lambda'_h) = \liminf_{h \to +\infty} G(\lambda'_h) \geqq G(\lambda'),$$

so that

$$\begin{aligned}
\liminf_{h \to +\infty} F(\lambda_h) &\geqq \liminf_{h \to +\infty} F(\lambda'_h) + \liminf_{h \to +\infty} F(\lambda''_h) \\
&\geqq G(\lambda') + F(\lambda'') = G(1_{\Omega \setminus D} \lambda') + G(1_D \lambda') + F(1_D \lambda'') \\
&= F(1_{\Omega \setminus D} \lambda) + \int_D [g^\infty(x, \lambda'(x)) + g(x, \lambda''(x))]\, d\# \\
&\geqq F(1_{\Omega \setminus D} \lambda) + \int_D g(x, \lambda'(x) + \lambda''(x))\, d\# \\
&= F(1_{\Omega \setminus D} \lambda) + F(1_D \lambda) = F(\lambda). \qquad ■
\end{aligned}$$

Proof of Proposition 2.9. – Let us recall that, by the measurable projection theorem, the set A is analytic, hence μ-measurable. Let M be a Borel subset of Ω such that

$$\mu_1(M) = 0, \qquad 1_{\Omega \setminus M} \mu \ll \mu_1, \qquad 1_M \mu \perp \mu_1,$$

and set $m = 1_M \mu + \mu_1$. Let $\lambda \in \mathscr{M}(\Omega; \mathbf{R}^n)$ and take its decomposition $\lambda = um + \nu$ with respect to m, with $u \in L^1(\Omega; \mathbf{R}^n; m)$ and $\nu \perp m$. Then we have

$$\lambda = \left(u \frac{dm}{d\mu}\right) \mu + \nu + u \left(m - \frac{dm}{d\mu} \mu\right),$$

where the measures μ, ν, $m-\dfrac{dm}{d\mu}\mu$ are mutually singular. Therefore

$$F(\lambda)=\int_\Omega f\left(x,u\frac{dm}{d\mu}\right)d\mu+\int_{\Omega\setminus A_\lambda}\varphi\left(x,\nu+um-u\frac{dm}{d\mu}\mu\right)$$
$$+\int_{A_\lambda}g(x,\lambda(x))\,d\#$$
$$=\int_\Omega \bar{f}(x,u)\,dm+\int_{\Omega\setminus A_\lambda}\varphi(x,\nu)+\int_{A_\lambda}g(x,\lambda(x))\,d\#,$$

where

$$\bar{f}(x,s)=f\left(x,s\frac{dm}{d\mu}\right)\frac{d\mu}{dm}+\varphi(x,s)\left[1-\frac{dm}{d\mu}\frac{d\mu}{dm}\right].$$

Analogously, denoting by F_1 the functional in (2.4), we have

$$F_1(\lambda)=\int_\Omega \bar{f}_1(x,u)\,dm+\int_{\Omega\setminus A_\lambda}\varphi_1(x,\nu)+\int_{A_\lambda}g_1(x,\lambda(x))\,d\#,$$

where

$$\bar{f}_1(x,s)=f_1\left(x,s\frac{dm}{d\mu_1}\right)\frac{d\mu_1}{dm}+\varphi_1(x,s)\left[1-\frac{dm}{d\mu_1}\frac{d\mu_1}{dm}\right].$$

Since $f^\infty(x,.)=\varphi(x,.)$ for μ-a.e. $x\in\Omega$, we have $\bar{f}^\infty(x,.)=\varphi(x,.)$ for m-a.e. $x\in\Omega$; analogously, from $f_1^\infty(x,.)=\varphi_1(x,.)$ for μ_1-a.e. $x\in\Omega$, we have $\bar{f}_1^\infty(x,.)=\varphi_1(x,.)$ for m-a.e. $x\in\Omega$. Now, by Proposition 3.2 of Bouchitté & Buttazzo [3] we have that $F\equiv F_1$ if and only if conditions (i) and (ii) hold, and

$$\bar{f}(x,.)=\bar{f}_1(x,.)\qquad\text{for}\quad m\text{-a.e. } x\in\Omega. \tag{3.8}$$

By the definition of m, (3.8) splits into

$$f(x,.)=\varphi_1(x,.)=\varphi(x,.)\qquad\text{for}\quad \mu\text{-a.e. } x\in M \tag{3.9}$$

$$f\left(x,.\frac{d\mu_1}{d\mu}\right)\frac{d\mu}{d\mu_1}+\varphi(x,.)\left[1-\frac{d\mu_1}{d\mu}\frac{d\mu}{d\mu_1}\right]=f_1(x,.)$$
$$\text{for}\quad \mu_1\text{-a.e. } x\in\Omega\setminus M. \tag{3.10}$$

It is easy to see that (3.9) is equivalent to $1_A\mu\ll\mu_1$, that is (iii). On the other hand, if α is the function in (iii), (3.10) becomes

$$\begin{cases} f\left(x,.\dfrac{1}{\alpha(x)}\right)\alpha(x)=f_1(x,.) & \text{if}\quad \alpha(x)\neq 0 \\ \varphi(x,.)=f_1(x,.) & \text{if}\quad \alpha(x)=0\end{cases}\qquad \text{for } \mu_1\text{-a.e. } x\in\Omega\setminus M,$$

that is (iv). ■

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