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The Cauchy problem for a nonlinear Wheeler-DeWitt equation

by

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ABSTRACT. — In this paper we consider a nonlinear version of the simplified Wheeler-DeWitt equation which describes the minisuperspace model for the wave function ψ of a closed universe (*cf.* [3], [2], [1]). Following [1], where the linear case has been solved, we study this equation as an evolution equation in the scalar field $y \in \mathbf{R}$ with a scale factor $x \in]0, \mathbf{R}[$. We solve the Cauchy problem for the initial data $\psi(x, 0)$ and $\frac{\partial \psi}{\partial y}(x, 0)$ and we study some decay properties and blow-up situations. A particular nonlinear version has been proposed in [6].

RÉSUMÉ. — Dans cet article nous considérons une version non linéaire de l'équation de Wheeler-DeWitt simplifiée qui décrit le modèle de minisuperspace pour la fonction d'onde ψ d'un univers fermé (*cf.* [3], [2], [1]). Dans l'esprit de [1], où nous avons résolu le cas linéaire, nous étudions cette équation comme une équation d'évolution dans le champ scalaire $y \in \mathbf{R}$ avec le facteur d'échelle $x \in]0, \mathbf{R}[$. Nous résolvons le problème de Cauchy pour des données initiales $\psi(x, 0)$ et $\frac{\partial \psi}{\partial y}(x, 0)$ et nous étudions des propriétés de décroissance à l'infini et des situations d'explosion de la solution. Une version non linéaire particulière a été proposée dans [6].

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1. INTRODUCTION

We consider the following nonlinear model for the simplified Wheeler-DeWitt equation:

$$\frac{1}{x^2} \frac{\partial^2 \psi}{\partial y^2} - \frac{\partial^2 \psi}{\partial x^2} - \frac{p}{x} \frac{\partial \psi}{\partial x} + x^2 \psi - k^2 x^4 \psi + g x^{q-2} |\psi|^r \psi = 0 \quad (1.1)$$

where $y \in \mathbf{R}$ is the scalar field, $x \in]0, \mathbf{R}[$ ($\mathbf{R} > 0$) is a scale factor, $p \in \mathbf{R}$, $k^2 > 0$, $g \in \mathbf{R}$, $r \geq 1$ and $q \geq \frac{1}{2} r p$ are given constants (p reflects the factor-ordering ambiguity and k^2 is a cosmological constant) and $\psi:]0, \mathbf{R}[\times \mathbf{R} \rightarrow \mathbf{C}$ is the wave function of the universe for the minisuper-space model. The equation (1.1) is equivalent, in the sense of distributions, to the following one:

$$\frac{\partial^2 \psi}{\partial y^2} - x^2 \frac{\partial^2 \psi}{\partial x^2} - p x \frac{\partial \psi}{\partial x} + x^4 \psi - k^2 x^6 \psi + g x^q |\psi|^r \psi = 0 \quad (1.2)$$

which can be considered as an evolution equation in $y \in \mathbf{R}$.

Assuming that $\psi(x, 0)$ and $\frac{\partial \psi}{\partial y}(x, 0)$, belong to some suitable weighted Sobolev spaces (*cf.* [1]) we first prove the existence of a unique local solution for the correspondent Cauchy problem. Under the hypothesis

$$g > 0 \quad \text{and} \quad k^2 \mathbf{R}^2 \leq 1 \quad (\text{that is } x^4 - k^2 x^6 \geq 0), \quad (1.3)$$

we will prove that this solution is a global solution.

Furthermore, if $k^2 \mathbf{R}^2 \leq \frac{2}{3}$ (that is $\frac{d}{dx}(x^4 - k^2 x^6) \geq 0$), we obtain the decay property

$$\int_y^{y+1} \int_0^{\mathbf{R}} x^{q+p-2} |\psi(x, s)|^{r+2} dx ds \rightarrow 0 \quad (y \rightarrow +\infty).$$

Finally we point out that, if $g < 0$ and $k^2 \mathbf{R}^2 \leq 1$, and for special initial data, there exists $y_0 \in \mathbf{R}_+$ such that

$$\int_0^{\mathbf{R}} x^{p-2} |\psi(x, y)|^2 dx \rightarrow +\infty \quad (y \rightarrow y_0^-).$$

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2. SOME PROPERTIES OF THE ASSOCIATED LINEAR OPERATOR

Following [1] let us denote by $H_{p,0}^1$ the closure of $\mathcal{D}([0, R])$ in the space $\left\{ u \in L_p^2 \mid \frac{du}{dx} \in L_p^2 \right\}$ where $L_p^2([0, R])$ is the L^2 space for the measure $d\mu = x^p dx$. We recall that we have $H_{p,0}^1 \subset L_{p-2}^2$ if $p \neq 1$ and $H_{p,0}^1 \cap L_{p-2}^2 = \{ u \mid x^{p/2} u \in H_0^1 \}$ where $H_0^1 = H_{0,0}^1$ is the usual Sobolev space.

Let us put $H = H_{p,0}^1 \times L_{p-2}^2$,

$$B\psi = x^2 \frac{\partial^2 \psi}{\partial x^2} + px \frac{\partial \psi}{\partial x}, \quad V\psi = -x^4 \psi + k^2 x^6 \psi$$

$$D(A) = D(B) \times (H_{p,0}^1 \cap L_{p-2}^2) \quad \text{with} \quad D(B) = \{ u \in H_{p,0}^1 \mid Bu \in L_{p-2}^2 \}.$$

Now, with $\psi_1 = \frac{\partial \psi}{\partial y}$, the linear equation associated to (1.2) can be written as follows

$$\frac{\partial}{\partial y} \begin{pmatrix} \psi \\ \psi_1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ B & 0 \end{pmatrix} \begin{pmatrix} \psi \\ \psi_1 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ V & 0 \end{pmatrix} \begin{pmatrix} \psi \\ \psi_1 \end{pmatrix} \tag{2.1}$$

In [1] we proved that $A : D(A) \rightarrow H$ defined by

$$A \begin{pmatrix} \psi \\ \psi_1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ B & 0 \end{pmatrix} \begin{pmatrix} \psi \\ \psi_1 \end{pmatrix}$$

is skew-self-adjoint in H and that $D : H \rightarrow H$ defined by

$$D \begin{pmatrix} \psi \\ \psi_1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ V & 0 \end{pmatrix} \begin{pmatrix} \psi \\ \psi_1 \end{pmatrix}$$

is continuous. Furthermore

$$D(B) \cap L_{p-2}^2 = \{ u \in H_{p,0}^1 \cap L_{p-2}^2 \mid x^{p/2+1} u \in H^2 \} \\ = \{ u \mid x^{p/2} u \in H_0^1 \text{ and } x^{p/2+1} u \in H^2 \} \quad (\text{cf. [1]}).$$

We denote by $S(y)$, $y \in \mathbf{R}$, the strongly continuous group of operators in H with infinitesimal generator $T = A + D : D(A) \rightarrow H$.

We need the following

PROPOSITION 2.1. — Assume $p = 1$ (that is $H = H_{1,0}^1 \times L_{-1}^2$) and put

$$H_1 = (H_{1,0}^1 \cap L_{-1}^2) \times L_{-1}^2 \subset H, \quad S_1(y)|_{H_1}, \quad T_1 = T|_{D(T) \cap H_1}.$$

Then $S_1(y)$, $y \in \mathbf{R}$, is a strongly continuous group of operators in H_1 with infinitesimal generator T_1 .

Proof. — Let $(u_0, v_0) \in H_1 \cap D(T)$, $(u(y), v(y)) = S(y)(u_0, v_0)$.

Since $\frac{\partial}{\partial y}|u|^2 = 2 \Re u \bar{v}$ we obtain, for $\varepsilon > 0$,

$$\begin{aligned} \int_0^{\mathbf{R}} x^{-1+\varepsilon} |u(y)|^2 dx &\leq \int_0^{\mathbf{R}} x^{-1+\varepsilon} |u_0|^2 dx + 2 \int_0^y \int_0^{\mathbf{R}} \frac{|u(s)| |v(s)|}{x^{(1/2)-(\varepsilon/2)} x^{(1/2)-(\varepsilon/2)}} dx ds \\ &\leq \mathbf{R}^\varepsilon \int_0^{\mathbf{R}} x^{-1} |u_0|^2 dx + 2 \mathbf{R}^{\varepsilon/2} \int_0^y \|u(s)\|_{L^2_{-1+\varepsilon}} \|v(s)\|_{L^2_{-1}} ds. \end{aligned}$$

Hence, by applying Gronwall's inequality and Fatou's lemma ($\varepsilon \rightarrow 0$), we get $u(y) \in L^2_{-1}$ and

$$\|u(y)\|_{L^2_{-1}} \leq \|u_0\|_{L^2_{-1}} + \int_0^y \|v(s)\|_{L^2_{-1}} ds \quad (y \geq 0)$$

and this inequality can be extended, by density, to $(u_0, v_0) \in \mathbf{H}_1$. Since for $(u_0, v_0) \in \mathbf{H}_1$ we have

$$\|v(s)\|_{L^2_{-1}} \leq \|S(s)(u_0, v_0)\|_{\mathbf{H}} \leq M e^{\omega s} \|(u_0, v_0)\|_{\mathbf{H}}$$

with $M > 0$, $\omega \geq 0$. Hence for $(u_0, v_0) \in \mathbf{H}_1$ we get

$$\|u(y)\|_{L^2_{-1}} \leq c(y) (\|u_0\|_{L^2_{-1}} + \|u_0\|_{\mathbf{H}^1_{1,0}} + \|v_0\|_{L^2_{-1}}).$$

We also have $\|u(y) - u_0\|_{L^2_{-1}} \leq \int_0^y \|v(s)\|_{L^2_{-1}} ds$.

Now, let $\tilde{\mathbf{T}}$ be the infinitesimal generator of S_1 . It is easy to see that $\mathbf{D}(\tilde{\mathbf{T}}) \subset \mathbf{D}(\mathbf{T}) \cap \mathbf{H}_1$ and that, in $\mathbf{D}(\tilde{\mathbf{T}})$, $\tilde{\mathbf{T}} = \mathbf{T}_1$. Let $(u_0, v_0) \in \mathbf{D}(\mathbf{T}) \cap \mathbf{H}_1$.

We have $\frac{1}{y}(u(y) - u_0) \rightarrow v_0$ ($y \rightarrow 0^+$) in $\mathbf{H}^1_{1,0}$.

We want to prove that $\frac{1}{y}(u(y) - u_0) \rightarrow v_0$ ($y \rightarrow 0^+$) in L^2_{-1} .

If $y_n \rightarrow 0^+$, we obtain

$$\limsup_{n \rightarrow \infty} \left\| \frac{1}{y_n}(u(y_n) - u_0) \right\|_{L^2_{-1}} \leq \lim_{n \rightarrow \infty} \frac{1}{y_n} \int_0^{y_n} \|v(s)\|_{L^2_{-1}} ds = \|v_0\|_{L^2_{-1}}.$$

Therefore there exists a subsequence y_{n_k} such that

$$\frac{1}{y_{n_k}}(u(y_{n_k}) - u_0) \rightharpoonup \eta \in L^2_{-1} \quad (k \rightarrow \infty), \quad \text{weakly in } L^2_{-1}.$$

But, for $\varepsilon > 0$, $L^2_{-1} \hookrightarrow L^2_{-1+\varepsilon}$, $\mathbf{H}^1_{1,0} \hookrightarrow \mathbf{H}^1_{1+\varepsilon,0} \hookrightarrow L^2_{-1+\varepsilon}$, and so $\eta = v_0$.

Hence $\frac{1}{y_{n_k}}(u(y_{n_k}) - u_0) \rightarrow v_0$ ($k \rightarrow \infty$) in L^2_{-1} .

We conclude that $\mathbf{D}(\mathbf{T}) \cap \mathbf{H}_1 \subset \mathbf{D}(\tilde{\mathbf{T}})$. \square

3. THE CAUCHY PROBLEM FOR THE NONLINEAR EQUATION

In the remaining of the paper we put

$$X = \begin{cases} H = H_{p,0}^1 \times L_{p-2}^2, & \text{if } p \neq 1 \\ H_1 = (H_{1,0}^1 \cap L_{-1}^2) \times L_{-1}^2, & \text{if } p = 1 \end{cases} \quad (3.1)$$

$$U = \begin{cases} T = A + D, \text{ with domain } D(A), & \text{if } p \neq 1 \\ T_1 = T_{1,H_1}, & \text{if } p = 1 \end{cases} \quad (3.2)$$

and we will denote by $G(y)$, $y \in \mathbf{R}$, the strongly continuous group of operators generated by U in X .

Let

$$F \begin{pmatrix} \psi \\ \psi_1 \end{pmatrix} = \begin{pmatrix} 0 \\ -gx^q |\psi|^r \psi \end{pmatrix}, \quad \psi_1 = \frac{\partial \psi}{\partial y} \quad (3.3)$$

The nonlinear equation (1.2) can be written as follows

$$\frac{\partial}{\partial y} \begin{pmatrix} \psi \\ \psi_1 \end{pmatrix} = U \begin{pmatrix} \psi \\ \psi_1 \end{pmatrix} + F \begin{pmatrix} \psi \\ \psi_1 \end{pmatrix} \quad (3.4)$$

and, since $H_{p,0}^1 \cap L_{p-2}^2 = \{u \mid x^{p/2} u \in H_0^1\}$ and $H_0^1 \hookrightarrow L^\infty$, it is easy to prove that if $r \geq 1$ and $q \geq r \frac{p}{2}$, then F is a locally Lipschitz continuous function from X to X .

Furthermore X is an Hilbert space. Hence, by theorem 1.6 in [4] ch. 6, if we take

$$\begin{pmatrix} \psi_0 \\ \psi_{1,0} \end{pmatrix} \in D(U), \quad \text{that is } \psi_0 \in D(B) \cap L_{p-2}^2, \quad \psi_{1,0} \in H_{1,0}^1 \cap L_{p-2}^2,$$

then there exists a unique local mild solution $\begin{pmatrix} \psi \\ \psi_1 \end{pmatrix}$ of the corresponding Cauchy problem for the equation (3.4) which is a local strong solution, that is

$$\left. \begin{aligned} & \begin{pmatrix} \psi \\ \psi_1 \end{pmatrix} \in \mathcal{C}]-\varepsilon, \varepsilon[; D(U) \cap \mathcal{C}^1]-\varepsilon, \varepsilon[; X, \\ & \begin{pmatrix} \psi \\ \psi_1 \end{pmatrix} \text{ satisfies (3.4) for } y \in]-\varepsilon, \varepsilon[\quad \text{and} \quad \begin{pmatrix} \psi \\ \psi_1 \end{pmatrix} (0) = \begin{pmatrix} \psi_0 \\ \psi_{1,0} \end{pmatrix}. \end{aligned} \right\} (3.5)$$

In order to obtain a result on the global existence of solution we need the following

LEMMA 3.1. — Assume $\begin{pmatrix} \psi_0 \\ \psi_{1,0} \end{pmatrix} \in D(U)$ and let $\begin{pmatrix} \psi \\ \psi_1 \end{pmatrix}$ be the corresponding local solution for the Cauchy problem. We have

$$E(y) = \frac{1}{2} \int_0^{\mathbf{R}} x^{p-2} \left| \frac{\partial \psi}{\partial y} \right|^2 dx + \frac{1}{2} \int_0^{\mathbf{R}} x^p \left| \frac{\partial \psi}{\partial x} \right|^2 dx + \frac{1}{2} \int_0^{\mathbf{R}} x^{p+2} |\psi|^2 dx \\ - \frac{k^2}{2} \int_0^{\mathbf{R}} x^{p+4} |\psi|^2 dx + \frac{g}{r+2} \int_0^{\mathbf{R}} x^{p+q-2} |\psi|^{r+2} dx \\ \equiv E(0) = E, \quad y \in]-\varepsilon, \varepsilon[. \quad (3.6)$$

Proof. — If we multiply the equation (1.2) by $x^{p-2} \frac{\partial \bar{\psi}}{\partial y}$, take the real part and integrate in x (over $]0, \mathbf{R}[$) we obtain $\frac{d}{dy} E(y) = 0$, since

$$\left(x^p \frac{\partial \psi}{\partial x} \frac{\partial \bar{\psi}}{\partial y} \right) (y) \in W_0^{1,1}. \quad \square$$

Hence, we have

THEOREM 3.1. — Assume (1.3) and $\begin{pmatrix} \psi_0 \\ \psi_{1,0} \end{pmatrix} \in D(U)$.

Then there exists a unique global solution $\begin{pmatrix} \psi \\ \psi_1 \end{pmatrix}$ of the corresponding Cauchy problem for the equation (1.2) in the sense of (3.5) (for $y \in]-\infty, +\infty[$).

Proof. — By (3.6) we have

$$\int_0^{\mathbf{R}} x^{p-2} |\psi_1|^2 dx + \int_0^{\mathbf{R}} x^p \left| \frac{\partial \psi}{\partial x} \right|^2 dx \leq E.$$

In the special case $p=1$, this implies, reasoning as in the first part of the proof of proposition 2.1, that

$$\|\psi(y)\|_{L^2_1} \leq \|\psi_0\|_{L^2_1} + \int_0^y \|\psi_1(s)\|_{L^2_1} ds \leq \|\psi_0\|_{L^2_1} + E y (y \geq 0). \quad \square$$

Remark. — The theorem 3.1 can be extended to more general situations, for example, if gx^q is replaced by a function $g(x, y) \geq 0$, g continuous, $\frac{\partial g}{\partial y}$ continuous and such that

$$g(x, y) \leq c(y) x^q, \quad \left| \frac{\partial g}{\partial y}(x, y) \right| \leq \min(c_1(y) x^q, c_2(y) g(x, y)),$$

with $c, c_1 \in L^\infty_{\text{loc}}(\mathbf{R})$, $c_2 \in L^1_{\text{loc}}(\mathbf{R} \setminus \{0\})$, $c, c_1, c_2 \geq 0$. Furthermore, the condition $k^2 \mathbf{R}^2 \leq 1$ can be replaced by $\mathbf{R}^4 - k^2 \mathbf{R}^6 + \frac{1}{4}(p-1)^2 > 0$ if $p \neq 1$.

4. DECAY PROPERTIES AND A BLOW-UP CASE

THEOREM 4.1. - Let us assume $g > 0$ and $k^2 R^2 \leq \frac{2}{3}$ and let

$$\begin{pmatrix} \psi_0 \\ \psi_{1,0} \end{pmatrix} \in D(U) \text{ and}$$

$$\psi \in \mathcal{C}(\mathbf{R}; D(\mathbf{B}) \cap L^2_{p-2}) \cap \mathcal{C}^1(\mathbf{R}; H^1_{p,0} \cap L^2_{p-2}) \cap \mathcal{C}^2(\mathbf{R}; L^2_{p-2})$$

be the unique global solution of the corresponding Cauchy problem for the equation (1.2). Then we have

$$(2 - 3k^2 R^2) \int_0^y \int_0^R x^{p+2} |\psi|^2 dx ds + g \frac{2q - rp + r}{2(r+2)} \int_0^y \int_0^R x^{p+q-2} |\psi|^{r+2} dx ds \leq 4E, \quad (4.1)$$

$\forall y \in \mathbf{R}_+$, where E is defined by (3.6).

Proof. - Let $v = x^{p/2} \psi$. We have $v \in \mathcal{C}^1(\mathbf{R}; H^1_0) \cap \mathcal{C}^2(\mathbf{R}; L^2_{-2})$, $xv \in \mathcal{C}(\mathbf{R}; H^2)$ and the equation (1.2) takes the form

$$\frac{\partial^2 v}{\partial y^2} - x^2 \frac{\partial^2 v}{\partial x^2} + \frac{p}{4}(p-2)v + x^4 v - k^2 x^6 v + gx^{q-(1/2)pr} |v|^r v = 0. \quad (4.2)$$

Let us multiply the equation (4.2) by $\frac{1}{2} \frac{\bar{v}}{x^2} - \frac{1}{x} \frac{\partial \bar{v}}{\partial x}$ take the real part and integrate over $]0, R[$. Taking in account that

$$\Re \int_0^R \frac{\partial^2 v}{\partial y^2} \frac{\bar{v}}{x^2} dx = \Re \frac{d}{dy} \int_0^R \frac{\partial v}{\partial y} \frac{\bar{v}}{x^2} dx - \int_0^R \frac{1}{x^2} \left| \frac{\partial v}{\partial y} \right|^2 dx$$

and

$$\Re \int_0^R \frac{\partial^2 v}{\partial y^2} \frac{\partial \bar{v}}{\partial x} \frac{1}{x} dx = \Re \frac{d}{dy} \int_0^R \frac{\partial v}{\partial y} \frac{\partial \bar{v}}{\partial x} \frac{1}{x} dx - \frac{1}{2} \int_0^R \frac{1}{x} \frac{\partial}{\partial x} \left| \frac{\partial v}{\partial y} \right|^2 dx$$

and integrating by parts we obtain (recall that if $u \in H^1$ and $u(0) = 0$ then $x^{-(1/2)} u \rightarrow 0 (x \rightarrow 0^+)$):

$$\begin{aligned} 0 &= \frac{1}{2} \frac{d}{dy} \Re \int_0^R \frac{1}{x^2} \frac{\partial v}{\partial y} \bar{v} dx - \frac{d}{dy} \Re \int_0^R \frac{1}{x} \frac{\partial v}{\partial y} \frac{\partial \bar{v}}{\partial x} dx \\ &+ \int_0^R (2x^2 - 3k^2 x^4) |v|^2 dx + g \frac{2q - rp + r}{2(r+q)} \int_0^R x^{q-(1/2)pr-2} |v|^{r+2} dx \\ &+ \frac{1}{2} \mathbf{R} \left| \frac{\partial v}{\partial x} \right|^2 (\mathbf{R}) - \frac{1}{2} \lim_{x \rightarrow 0^+} \left(x \left| \frac{\partial v}{\partial x} \right|^2 \right) + \frac{1}{4} \lim_{x \rightarrow 0^+} \frac{\partial |v|^2}{\partial x}. \quad (4.3) \end{aligned}$$

But

$$x \frac{\partial v}{\partial x} \in H^1 \quad \text{and} \quad x \frac{\partial v}{\partial x} = \frac{\partial}{\partial x}(xv) - \frac{xv}{x} \rightarrow 0 (x \rightarrow 0^+).$$

Hence

$$x \left| \frac{\partial v}{\partial x} \right|^2 \rightarrow 0 (x \rightarrow 0^+)$$

and

$$\frac{\partial |v|^2}{\partial x} = 2 \Re e \left[(x^{-1/2} \bar{v}) \left(x^{1/2} \frac{\partial v}{\partial x} \right) \right] \rightarrow 0 (x \rightarrow 0^+)$$

From (4.3) we obtain for $\psi = x^{-(p/2)} v$ and with $k^2 R^2 \leq \frac{2}{3}$:

$$\begin{aligned} \theta &= (2 - 3k^2 R^2) \int_0^R x^{p+2} |\psi|^2 dx + g \frac{2q - rp + r}{2(r+2)} \int_0^R x^{p+q-2} |\psi|^{r+2} dx \\ &\leq -\frac{1}{2} \frac{d}{dy} \Re e \int_0^R x^{p-2} \frac{\partial \psi}{\partial y} \bar{\psi} dx + \frac{d}{dy} \Re e \int_0^R x^{p-1} \frac{\partial \psi}{\partial y} \frac{\partial \bar{\psi}}{\partial x} dx \\ &\quad + \frac{d}{dy} \Re e \frac{p}{2} \int_0^R x^{p-2} \frac{\partial \psi}{\partial x} \bar{\psi} dx. \end{aligned}$$

Hence, for $y > 0$,

$$\begin{aligned} &\int_0^y \theta(s) ds \\ &\leq \left[\int_0^R x^{p-2} \left| \frac{\partial \psi}{\partial y} \right|^2 dx + \frac{1}{2} \int_0^R x^p \left| \frac{\partial \psi}{\partial x} \right|^2 dx + \frac{1}{2} \frac{(p-1)^2}{4} \int_0^R x^{p-2} |\psi|^2 dx \right] (y) \\ &\quad + \left[\int_0^R x^{p-2} |\psi_{1,0}|^2 dx + \frac{1}{2} \int_0^R x^p \left| \frac{\partial \psi_0}{\partial x} \right|^2 dx \right. \\ &\quad \left. + \frac{1}{2} \frac{(p-1)^2}{4} \int_0^R x^{p-2} |\psi_0|^2 dx \right] \leq 4E, \end{aligned}$$

since for $p \neq 1$ we have, by Hardy's inequality (cf. (2.1) in [1]),

$$\frac{(p-1)^2}{4} \int_0^R x^{p-2} |\psi|^2 dx \leq \int_0^R x^p \left| \frac{\partial \psi}{\partial x} \right|^2 dx. \quad \square$$

COROLLARY. — Under the hypothesis of theorem 4.1 we have

$$\int_y^{y+1} \int_0^R x^{p+q-2} |\psi|^{2+r} dx ds \rightarrow 0 (y \rightarrow +\infty).$$

Remark. — By the reversibility in y we have similar results for $y \in \mathbf{R}_-$.

Finally we can point out a blow-up situation. The proof of the following result is similar to the one in the example of X. 13 in [5] for the nonlinear wave equation and so we omit it.

PROPOSITION 4.1. — Assume $g < 0$, $k^2 R^2 \leq 1$ and let $\begin{pmatrix} \psi_0 \\ \psi_{1,0} \end{pmatrix} \in D(U)$ be real and such that $E \leq 0$ and $\int_0^R x^{p-2} \psi_0 \psi_{1,0} dx > 0$. Let ψ be the local solution of the corresponding Cauchy problem for the equation (1.2). Then there exists $y_0 \in \mathbf{R}_+$ such that

$$\lim_{y \rightarrow y_0} \int_0^R x^{p-2} \psi^2(x, y) dx = +\infty.$$

Remark. — Like in the quoted example for the nonlinear wave equation, it is easy to check that $y_0 \leq \frac{2}{r} \left(\int_0^R x^{p-2} \psi_0^2 dx \right) \left(\int_0^R x^{p-2} \psi_0 \psi_{1,0} dx \right)^{-1}$.

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