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On solutions of the exterior Dirichlet problem for the minimal surface equation

by

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ABSTRACT. — Uniqueness and existence results for boundary value problems for the minimal surface equation on exterior domains obtained by Langévin-Rosenberg and Krust in dimension two are generalized to arbitrary dimensions. A suitable n -dimensional version of the maximum principle at infinity is given.

Key words : Minimal surface equation, exterior domain problems, maximum principle at infinity.

RÉSUMÉ. — On présente des résultats d'unicité et d'existence pour l'équation des surfaces minimales sur un domaine extérieur de \mathbb{R}^n . On donne une généralisation du principe de maximum à l'infini, valable quel que soit n .

Classification A.M.S. : 49 F 10, 35 J 25, 35 B 50.

1. INTRODUCTION

Let $U \subset \mathbb{R}^n$ be a domain such that $K = \mathbb{R}^n \setminus U$ is compact. In this paper we consider solutions $u \in C^2(U)$ of the minimal surface equation

$$(E) \quad \operatorname{div} \left(\frac{\nabla u}{\sqrt{1 + |\nabla u|^2}} \right) = 0 \quad \text{in } U$$

which are *regular at infinity* in the sense that their graph has a welldefined asymptotic normal $v_\infty \in \{v \in S^n : v^{n+1} > 0\}$. Given $\varphi \in C^0(\partial U)$ a function $u \in C^0(\bar{U})$ is called a *solution of the exterior Dirichlet problem* if u satisfies (E) – in particular $u \in C^2(U)$ – and $u|_{\partial U} = \varphi$.

In case of a *bounded* domain U the solvability of the corresponding boundary value problem for *all* $\varphi \in C^0(\partial U)$ is equivalent to the mean curvature of ∂U being nonnegative [3]. Since this condition is necessarily violated for an exterior region, the existence problem is quite difficult. In [11] Osserman presented smooth functions on the unit circle which do not admit a bounded solution. Recently Krust [5] showed that the boundary data in Osserman’s examples would not even admit solutions having a vertical normal at infinity. Krust’s main result says that for $n=2$ all solutions having the same v_∞ form a *foliation*. From this he could derive the nonexistence statement using a symmetry argument. We will prove the above foliation property in arbitrary dimensions. We shall also give a simple proof different from [7] of the so-called *maximum principle at infinity*. Our argument is similar to the one given in [8] and suitably generalizes to the n -dimensional case.

Let us finally mention that $\Gamma = \text{graph } \varphi$ always bounds a minimal surface having a planar end by the work of Tomi and Ye [13] and the author [6]; here “minimal surface” refers either to a parametric solution ($n=2$) or to an embedded surface (possibly with singularities if $n \geq 7$).

2. ASYMPTOTIC EXPANSIONS
AND MAXIMUM PRINCIPLE AT INFINITY

We will use the following notation:

$$\begin{aligned} \omega_n &= \mathcal{H}^n(S^n) \\ px &= \xi \text{ for } x = (\xi, x^{n+1}) \in \mathbb{R}^{n+1} \\ r &= r(\xi) = |\xi|, \quad e_r = e_r(\xi) = \frac{\xi}{r} \text{ for } \xi \in \mathbb{R}^n \setminus \{0\} \end{aligned}$$

$$A(r, R) = \{ \xi \in \mathbb{R}^n : r < |\xi| < R \}, \quad A(r) = A(r, \infty)$$

U will always denote an open neighbourhood of infinity in \mathbb{R}^n . If $V \subset\subset U$ and ∂V is of class C^1 , u is a solution of (E) in U and φ is locally

Lipschitz continuous in U , then

$$\int_V \langle T(\nabla u), \nabla \varphi \rangle d\mathcal{L}^n = \int_{\partial V} \varphi \langle T(\nabla u), N \rangle d\mathcal{H}^{n-1} \tag{1}$$

where as usual $T(p) = (1 + |p|^2)^{-1/2} p$ for $p \in \mathbb{R}^n$ and N is the exterior unit normal along ∂V . Setting $w(p) = (1 + |p|^2)^{1/2}$, the ellipticity of (E) can be stated as follows:

$$\langle T(p_1) - T(p_2), p_1 - p_2 \rangle \geq (\max_{i=1,2} w(p_i))^{-3} |p_1 - p_2|^2 \quad \forall p_1, p_2 \in \mathbb{R}^n \tag{2}$$

A connected, oriented and embedded minimal surface $M^n \subset \mathbb{R}^{n+1}$ will be called *simple at infinity* if M has a welldefined normal $v_\infty \in S^n$ at infinity and M can be written as a graph over its asymptotic tangent plane outside some compact set. Assuming $v_\infty = e_{n+1}$ is the vertical direction, it is shown in [12] that the corresponding graph function has a *twice differentiable expansion*

$$u(\xi) = h + \alpha g(r) + O(r^{1-n}) \tag{3}$$

where $h \in \mathbb{R}$, $\alpha \in \mathbb{R}$ and g is the Newtonian potential in \mathbb{R}^n :

$$g(r) = \begin{cases} \log r & (n=2) \\ \frac{r^{2-n}}{2-n} & (n \geq 3) \end{cases}$$

For example the graph function of an n -dimensional catenoid is given by

$$|x^{n+1}| = c_a(r) = a \int_1^{r/a} (s^{2(n-1)} - 1)^{-1/2} ds \quad (a > 0) \tag{4}$$

and satisfies (3) with $\alpha = a^{n-1}$. If v_∞ is fixed, we will refer to h as the *height* and α as the *growth rate* (at infinity). The following result is due to Langévin and Rosenberg [7] in case $n = 2$.

THEOREM 1 (MAXIMUM PRINCIPLE AT INFINITY). — *Suppose $M_i (i=1, 2)$ are minimal surfaces which are simple and disjoint at infinity. If the M_i are at distance zero at infinity, then $n \geq 3$ and their growth rates are different.*

Proof. — We may assume that $M_i = \text{graph } u_i$ where $u_i \in C^0(\overline{A}(\mathbb{R}))$, $u_1 < u_2$ and $u_i = h_i + \alpha_i g(r) + O(r^{1-n})$. The assumptions imply $h_1 = h_2$ and $u_2 - u_1 \rightarrow 0$ uniformly as $\xi \rightarrow \infty$; for $n = 2$ we also had $\alpha_1 = \alpha_2$. The expansions yield

$$\langle T(\nabla u_i), e_r \rangle = \alpha_i r^{1-n} + O(r^{-n}).$$

Setting $d = \inf \{ u_2(\xi) - u_1(\xi) : |\xi| = R \} > 0$ we consider for any $\varepsilon \in (0, d)$ the test function

$$\varphi_\varepsilon := \begin{cases} 0 & \text{if } u_2 - u_1 \geq d \\ u_2 - u_1 - d & \text{if } \varepsilon < u_2 - u_1 < d \\ \varepsilon - d & \text{if } 0 < u_2 - u_1 < \varepsilon \end{cases}$$

Using φ_ε in (1) on $V=A(R, \rho)$ and letting $\rho \rightarrow \infty$ we obtain

$$(d-\varepsilon)\omega_{n-1}\alpha_i = \int_{\{\varepsilon < u_2 - u_1 < d\}} \langle T(\nabla u_i), \nabla u_1 - \nabla u_2 \rangle d\mathcal{L}^n \quad (i=1, 2).$$

Now subtract these two identities, apply (2) and let $\varepsilon \rightarrow 0$:

$$d\omega_{n-1}(\alpha_1 - \alpha_2) \geq \int_{\{u_2 - u_1 < d\}} (\max_{i=1,2} w(\nabla u_i))^{-3} |\nabla u_1 - \nabla u_2|^2 d\mathcal{L}^n.$$

Hence $\alpha_1 \leq \alpha_2$ is impossible. \square

COROLLARY 1. — *Let $u_i \in C^0(\bar{U})$ ($i=1, 2$) be two solutions of (E) having the same asymptotic normal and $u_1|_{\partial U} = u_2|_{\partial U}$. Let h_i and α_i be their heights and growth rates respectively. Then*

- (i) $\alpha_1 \geq \alpha_2 \Leftrightarrow u_1 \geq u_2$.
- (ii) *If $n \geq 3$, we also have: $h_1 \geq h_2 \Leftrightarrow u_1 \geq u_2$.*

Corollary 1 follows easily by looking at vertical translates of the graph of u_1 . Now let M be simple at infinity such that $v_\infty = e_{n+1}$ and suppose M is of class C^1 up to the boundary. Let v be the continuous unit normal on M determined by v_∞ , $M_R = \{x \in M : |px| < R\}$ and denote by η the exterior unit normal along ∂M_R in M . For sufficiently large $|\xi| = R$ we have

$$\eta(\xi) = \frac{e_r - \langle e_r, v \rangle v}{\sqrt{1 - \langle e_r, v \rangle^2}} = e_r + \alpha r^{1-n} e_{n+1} + O(r^{-n}),$$

$$(1 + |\nabla u|^2 - (\partial_r u)^2)^{1/2} = 1 + O(r^{2(1-n)}).$$

Applying the divergence theorem on M_R to the tangential component of a constant vector $v \in \mathbb{R}^{n+1}$ and letting $R \rightarrow \infty$ we obtain the “balancing formula” (compare [12], [4])

$$\alpha v_\infty = - \frac{1}{\omega_{n-1}} \int_{\partial M} \eta(x) d\mathcal{H}^{n-1}(x). \tag{5}$$

COROLLARY 2. — *Let $U \subset \mathbb{R}^n$ be an exterior domain with $\partial U \in C^1$. If $u_i \in C^1(\bar{U})$ ($i=1, 2$) are solutions of (E) having the same asymptotic normal and satisfying*

$$\langle T(\nabla u_1), N \rangle = \langle T(\nabla u_2), N \rangle \text{ along } \partial U,$$

then the difference $u_1 - u_2$ is a constant.

Proof. — Writing down (5) in terms of the graph functions, we obtain

$$\alpha_1 = \alpha_2 = - \frac{1}{\omega_{n-1} \langle v_\infty, e_{n+1} \rangle} \int_{\partial U} \langle T(\nabla u_i), N \rangle d\mathcal{H}^{n-1}.$$

Let $t_0 = \inf \{ t : u_2 + t \geq u_1 \}$. Then we have either $u_1 - u_2 \equiv t_0$ or $u_2 + t_0 > u_1$ in all of U . But the second case is impossible because of Thm 1 and the maximum principle at the boundary. \square

3. FOLIATION PROPERTY OF THE SOLUTIONS

The following result is a consequence of the interior maximum principle (see [11]).

LEMMA 1. — *If $M \subset \mathbb{R}^{n+1}$ is a compact minimal surface such that*

$$(\partial M \cap (A(\rho) \times \mathbb{R})) \subset \{ x : |px| = R, |x^{n+1}| \leq h_0 \} \text{ for some } \rho \in (0, R),$$

then

$$(M \cap (A(\rho) \times \mathbb{R})) \subset \{ x : |x^{n+1}| \leq h_0 + c_\rho(R) - c_\rho(r) \}.$$

Remark. — Let $n \geq 3$ and $u \in C^0(\bar{U})$ be a solution of (E) with $v_\infty = e_{n+1}$. Then if $B \subset \mathbb{R}^n$ is a closed ball of radius $\rho > 0$ containing ∂U , we conclude from the above lemma that

$$|u(\xi) - u(\infty)| \leq c_\rho(\infty) - c_\rho(r) \text{ in } U \setminus \text{int } B.$$

Setting $A = \partial B \cap \partial U$, we have in particular $\text{osc}(u) \leq 2\rho c_1(\infty)$. For exam-

ple there is no solution of the exterior Dirichlet problem having a vertical normal at infinity if $\varphi \in C^0(S^{n-1})$ is given with $\text{osc}(\varphi) > 2c_1(\infty)$.

Now let $U \subset \mathbb{R}^n$ be an exterior region, $K = \mathbb{R}^n \setminus U$, and suppose that $u^\pm \in C^0(\bar{U})$ are two solutions of (E) satisfying $u^- < u^+$ in U and $u^\pm|_{\partial U} = \varphi$.

LEMMA 2. — *Let $K \subset B_R(0) = B \subset \mathbb{R}^n$ and suppose $\psi \in C^2(\partial B)$ satisfies $u^- < \psi < u^+$ on ∂B . Setting $U_R = U \cap B$, there is a unique solution $u \in C^0(\bar{U}_R)$ of (E) satisfying $u|_{\partial U} = \varphi$, $u|_{\partial B} = \psi$. Moreover $u^- < u < u^+$ in U_R .*

Proof. — We refer to Haar's solution of the nonparametric Plateau problem [2] which is described in the book of Giusti [1]. Let us first consider the case that U has Lipschitz continuous boundary and $\max \{ \text{Lip}(u^\pm), U_R \} = l < \infty$. Then for any sufficiently large $k > l$, we can take $u^k \in C^0(\bar{U}_R)$ minimizing the area functional in the class $\{ u \in C^0(\bar{U}_R) : \text{Lip}(u) \leq k, u|_{\partial U} = \varphi, u|_{\partial B} = \psi \}$. The weak maximum principle [1], 12.5, yields $u^- \leq u^k \leq u^+$ in U_R . Now because of [1], 12.7, we know that

$$\text{Lip}(u^k) = \sup \left\{ \frac{|u^k(\xi) - u^k(\eta)|}{|\xi - \eta|} : \xi \in U_R, \eta \in \partial U_R \right\}.$$

Since $u^- \leq u^k \leq u^+$, for any $\eta \in \partial U$ and any $\xi \in U_R$ we have

$$|u^k(\xi) - u^k(\eta)| \leq l|\xi - \eta|.$$

On the other hand, it is easy to construct barriers in a neighbourhood of ∂B (see [1], pp. 142-144). Hence $\text{Lip}(u^k) < k$ for sufficiently large k , which means that u^k is a weak solution of (E) in U_R ; in fact because of the regularity theory ([1], 12.11) u^k is smooth. To treat the general case, we choose a regular value $\varepsilon > 0$ of $u^+ - u^-$. Letting

$$V_\varepsilon = \{ \xi \in U_R : u^+(\xi) - u^-(\xi) > \varepsilon \}, \quad \varphi_\varepsilon = u^+ \text{ on } \partial V_\varepsilon \setminus \partial B,$$

we can apply the argument above to obtain a solution $v_\varepsilon \in C^0(\bar{V}_\varepsilon)$ of (E) which coincides with u^+ on $\partial V_\varepsilon \setminus B$, and with ψ on ∂B . Since $u^- < v_\varepsilon < u^+$ in V_ε , the *a priori* estimates in [1] imply that $v_\varepsilon \rightarrow u$ locally uniformly in $C^2(U_R)$ as $\varepsilon \rightarrow 0$. Clearly u must attain the boundary values on ∂U . But on ∂B the same barriers apply to all the v_ε and hence $u \in C^0(\bar{U}_R)$ is a solution of our problem. Uniqueness follows easily from the interior maximum principle. \square

The following result is due to Krust [5] in the two dimensional case. In order to obtain the approximating solutions, he solved the parametric Plateau problem for a minimal annulus and referred to an embeddedness result of Meeks and Yau [9] together with the well-known argument of Kneser-Radó to show the graph property.

THEOREM 2. — *Let $U \subset \mathbb{R}^n$ be an exterior region and $\varphi \in C^0(\partial U)$. The set of solutions of the exterior Dirichlet problem with boundary data φ having the same asymptotic normal forms a (possibly empty) foliation.*

Proof. — Let us first consider the case $n \geq 3$. Suppose $u^\pm \in C^0(\bar{U})$ are two solutions with asymptotic normal v_∞ . Because of corollary 1, we may assume that $h^+ > h^-$ and $u^+ > u^-$ in U . Given any $h \in (h^-, h^+)$, we let

$$H = \{ x \in \mathbb{R}^{n+1} : \langle x, v_\infty \rangle = h \}, \\ \Gamma_R = \{ x \in H : |px| = R \}.$$

For any sufficiently large R there is a minimal graph $u_R \in C^0(\bar{U}_R)$ such that $u_R|_{\partial U} = \varphi$ and $\text{graph}(u_R)|_{\partial B_R(0)} = \Gamma_R$; moreover $u^- < u_R < u^+$ in U_R . As in lemma 2, we can let $R \rightarrow \infty$ to obtain a solution $u \in C^0(\bar{U})$ of the exterior Dirichlet problem with boundary data φ satisfying $u^- \leq u \leq u^+$ in U . Let π be the orthogonal projection onto H and let $B \subset H$ be an n -dimensional ball of radius $\rho > 0$ containing $\pi(\text{graph } \varphi)$. Applying lemma 1 to $\text{graph } u_R$ and then letting $R \rightarrow \infty$ we infer that

$$| \langle x, v_\infty \rangle - h | \leq c_\rho(\infty) - c_\rho(|\pi x|)$$

for any $x \in \text{graph } u$ satisfying $\pi x \notin \bar{B}$. In particular $\text{graph } u$ is at a height h at infinity. Now the gradient of u is bounded ([1], 13.6) and in fact converges to a limit (see [10], thm 6). This means that u is regular at infinity in the sense of the introduction and has asymptotic normal v_∞ . Thus we have shown that for any $h \in (h^-, h^+)$ there is a solution u_h with asymptotic height equal to h and moreover $u_h < u_{h'}$ for $h < h'$. Now let

$x = (\xi, x^{n+1}) \in U \times \mathbb{R}$ be given such that $u^-(\xi) < x^{n+1} < u^+(\xi)$. Then we let $h_1 = \sup \{ h : u_h(\xi) < x^{n+1} \}$, $h_2 = \inf \{ h : u_h(\xi) > x^{n+1} \}$. We see that $h_1 < h_2$ is impossible because otherwise we would have $u_{h_1}(\xi) < u_h(\xi) < u_{h_2}(\xi)$ for any $h \in (h_1, h_2)$. This proves the theorem if $n \geq 3$.

The case $n=2$ was treated in [5]; the main difference in this case is that one has to replace the parameter h by the growth rate α . Taking as $\Gamma_{\mathbb{R}}$ the intersection of the cylinder $\{x : |px| = R\}$ with a half catenoid of the desired growth rate centered around the axis $\mathbb{R}v_{\infty}$ one proceeds essentially in the same way as above. \square

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