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## **On solutions of the exterior Dirichlet problem for the minimal surface equation**

by

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**ABSTRACT.** — Uniqueness and existence results for boundary value problems for the minimal surface equation on exterior domains obtained by Langévin-Rosenberg and Krust in dimension two are generalized to arbitrary dimensions. A suitable  $n$ -dimensional version of the maximum principle at infinity is given.

*Key words* : Minimal surface equation, exterior domain problems, maximum principle at infinity.

**RÉSUMÉ.** — On présente des résultats d'unicité et d'existence pour l'équation des surfaces minimales sur un domaine extérieur de  $\mathbb{R}^n$ . On donne une généralisation du principe de maximum à l'infini, valable quel que soit  $n$ .

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*Classification A.M.S.* : 49 F 10, 35 J 25, 35 B 50.

## 1. INTRODUCTION

Let  $U \subset \mathbb{R}^n$  be a domain such that  $K = \mathbb{R}^n \setminus U$  is compact. In this paper we consider solutions  $u \in C^2(U)$  of the minimal surface equation

$$(E) \quad \operatorname{div} \left( \frac{\nabla u}{\sqrt{1 + |\nabla u|^2}} \right) = 0 \quad \text{in } U$$

which are *regular at infinity* in the sense that their graph has a welldefined asymptotic normal  $v_\infty \in \{v \in S^n : v^{n+1} > 0\}$ . Given  $\varphi \in C^0(\partial U)$  a function  $u \in C^0(\bar{U})$  is called a *solution of the exterior Dirichlet problem* if  $u$  satisfies (E) – in particular  $u \in C^2(U)$  – and  $u|_{\partial U} = \varphi$ .

In case of a *bounded* domain  $U$  the solvability of the corresponding boundary value problem for *all*  $\varphi \in C^0(\partial U)$  is equivalent to the mean curvature of  $\partial U$  being nonnegative [3]. Since this condition is necessarily violated for an exterior region, the existence problem is quite difficult. In [11] Osserman presented smooth functions on the unit circle which do not admit a bounded solution. Recently Krust [5] showed that the boundary data in Osserman's examples would not even admit solutions having a vertical normal at infinity. Krust's main result says that for  $n=2$  all solutions having the same  $v_\infty$  form a *foliation*. From this he could derive the nonexistence statement using a symmetry argument. We will prove the above foliation property in arbitrary dimensions. We shall also give a simple proof different from [7] of the so-called *maximum principle at infinity*. Our argument is similar to the one given in [8] and suitably generalizes to the  $n$ -dimensional case.

Let us finally mention that  $\Gamma = \text{graph } \varphi$  always bounds a minimal surface having a planar end by the work of Tomi and Ye [13] and the author [6]; here "minimal surface" refers either to a parametric solution ( $n=2$ ) or to an embedded surface (possibly with singularities if  $n \geq 7$ ).

## 2. ASYMPTOTIC EXPANSIONS AND MAXIMUM PRINCIPLE AT INFINITY

We will use the following notation:

$$\omega_n = \mathcal{H}^n(S^n)$$

$$px = \xi \text{ for } x = (\xi, x^{n+1}) \in \mathbb{R}^{n+1}$$

$$r = r(\xi) = |\xi|, \quad e_r = e_r(\xi) = \frac{\xi}{r} \text{ for } \xi \in \mathbb{R}^n \setminus \{0\}$$

$$A(r, R) = \{\xi \in \mathbb{R}^n : r < |\xi| < R\}, \quad A(r) = A(r, \infty)$$

$U$  will always denote an open neighbourhood of infinity in  $\mathbb{R}^n$ . If  $V \subset\subset U$  and  $\partial V$  is of class  $C^1$ ,  $u$  is a solution of (E) in  $U$  and  $\varphi$  is locally

Lipschitz continuous in U, then

$$\int_V \langle T(\nabla u), \nabla \varphi \rangle d\mathcal{L}^n = \int_{\partial V} \varphi \langle T(\nabla u), N \rangle d\mathcal{H}^{n-1} \tag{1}$$

where as usual  $T(p) = (1 + |p|^2)^{-1/2} p$  for  $p \in \mathbb{R}^n$  and  $N$  is the exterior unit normal along  $\partial V$ . Setting  $w(p) = (1 + |p|^2)^{1/2}$ , the ellipticity of (E) can be stated as follows:

$$\langle T(p_1) - T(p_2), p_1 - p_2 \rangle \geq (\max_{i=1,2} w(p_i))^{-3} |p_1 - p_2|^2 \quad \forall p_{1,2} \in \mathbb{R}^n \tag{2}$$

A connected, oriented and embedded minimal surface  $M^n \subset \mathbb{R}^{n+1}$  will be called *simple at infinity* if  $M$  has a welldefined normal  $v_\infty \in S^n$  at infinity and  $M$  can be written as a graph over its asymptotic tangent plane outside some compact set. Assuming  $v_\infty = e_{n+1}$  is the vertical direction, it is shown in [12] that the corresponding graph function has a *twice differentiable expansion*

$$u(\xi) = h + \alpha g(r) + O(r^{1-n}) \tag{3}$$

where  $h \in \mathbb{R}$ ,  $\alpha \in \mathbb{R}$  and  $g$  is the Newtonian potential in  $\mathbb{R}^n$ :

$$g(r) = \begin{cases} \log r & (n=2) \\ \frac{r^{2-n}}{2-n} & (n \geq 3) \end{cases}$$

For example the graph function of an  $n$ -dimensional catenoid is given by

$$|x^{n+1}| = c_a(r) = a \int_1^{r/a} (s^{2(n-1)} - 1)^{-1/2} ds \quad (a > 0) \tag{4}$$

and satisfies (3) with  $\alpha = a^{n-1}$ . If  $v_\infty$  is fixed, we will refer to  $h$  as the *height* and  $\alpha$  as the *growth rate* (at infinity). The following result is due to Langévin and Rosenberg [7] in case  $n = 2$ .

**THEOREM 1 (MAXIMUM PRINCIPLE AT INFINITY).** — *Suppose  $M_i (i=1, 2)$  are minimal surfaces which are simple and disjoint at infinity. If the  $M_i$  are at distance zero at infinity, then  $n \geq 3$  and their growth rates are different.*

*Proof.* — We may assume that  $M_i = \text{graph } u_i$  where  $u_i \in C^0(\overline{A(\mathbb{R})})$ ,  $u_1 < u_2$  and  $u_i = h_i + \alpha_i g(r) + O(r^{1-n})$ . The assumptions imply  $h_1 = h_2$  and  $u_2 - u_1 \rightarrow 0$  uniformly as  $\xi \rightarrow \infty$ ; for  $n = 2$  we also had  $\alpha_1 = \alpha_2$ . The expansions yield

$$\langle T(\nabla u_i), e_r \rangle = \alpha_i r^{1-n} + O(r^{-n}).$$

Setting  $d = \inf \{ u_2(\xi) - u_1(\xi) : |\xi| = R \} > 0$  we consider for any  $\varepsilon \in (0, d)$  the test function

$$\varphi_\varepsilon := \begin{cases} 0 & \text{if } u_2 - u_1 \geq d \\ u_2 - u_1 - d & \text{if } \varepsilon < u_2 - u_1 < d \\ \varepsilon - d & \text{if } 0 < u_2 - u_1 < \varepsilon \end{cases}$$

Using  $\varphi_\varepsilon$  in (1) on  $V = A(R, \rho)$  and letting  $\rho \rightarrow \infty$  we obtain

$$(d - \varepsilon)\omega_{n-1} \alpha_i = \int_{\{\varepsilon < u_2 - u_1 < d\}} \langle T(\nabla u_i), \nabla u_1 - \nabla u_2 \rangle d\mathcal{L}^n \quad (i = 1, 2).$$

Now subtract these two identities, apply (2) and let  $\varepsilon \rightarrow 0$ :

$$d\omega_{n-1}(\alpha_1 - \alpha_2) \geq \int_{\{u_2 - u_1 < d\}} (\max_{i=1,2} w(\nabla u_i))^{-3} |\nabla u_1 - \nabla u_2|^2 d\mathcal{L}^n.$$

Hence  $\alpha_1 \leq \alpha_2$  is impossible.  $\square$

**COROLLARY 1.** — *Let  $u_i \in C^0(\bar{U})$  ( $i = 1, 2$ ) be two solutions of (E) having the same asymptotic normal and  $u_1|_{\partial U} = u_2|_{\partial U}$ . Let  $h_i$  and  $\alpha_i$  be their heights and growth rates respectively. Then*

- (i)  $\alpha_1 \geq \alpha_2 \Leftrightarrow u_1 \geq u_2$ .
- (ii) *If  $n \geq 3$ , we also have:  $h_1 \geq h_2 \Leftrightarrow u_1 \geq u_2$ .*

Corollary 1 follows easily by looking at vertical translates of the graph of  $u_1$ . Now let  $M$  be simple at infinity such that  $v_\infty = e_{n+1}$  and suppose  $M$  is of class  $C^1$  up to the boundary. Let  $v$  be the continuous unit normal on  $M$  determined by  $v_\infty$ ,  $M_R = \{x \in M : |px| < R\}$  and denote by  $\eta$  the exterior unit normal along  $\partial M_R$  in  $M$ . For sufficiently large  $|\xi| = R$  we have

$$\eta(\xi) = \frac{e_r - \langle e_r, v \rangle v}{\sqrt{1 - \langle e_r, v \rangle^2}} = e_r + \alpha r^{1-n} e_{n+1} + O(r^{-n}),$$

$$(1 + |\nabla u|^2 - (\partial_r u)^2)^{1/2} = 1 + O(r^{2(1-n)}).$$

Applying the divergence theorem on  $M_R$  to the tangential component of a constant vector  $v \in \mathbb{R}^{n+1}$  and letting  $R \rightarrow \infty$  we obtain the “balancing formula” (compare [12], [4])

$$\alpha v_\infty = - \frac{1}{\omega_{n-1}} \int_{\partial M} \eta(x) d\mathcal{H}^{n-1}(x). \tag{5}$$

**COROLLARY 2.** — *Let  $U \subset \mathbb{R}^n$  be an exterior domain with  $\partial U \in C^1$ . If  $u_i \in C^1(\bar{U})$  ( $i = 1, 2$ ) are solutions of (E) having the same asymptotic normal and satisfying*

$$\langle T(\nabla u_1), N \rangle = \langle T(\nabla u_2), N \rangle \text{ along } \partial U,$$

*then the difference  $u_1 - u_2$  is a constant.*

*Proof.* — Writing down (5) in terms of the graph functions, we obtain

$$\alpha_1 = \alpha_2 = - \frac{1}{\omega_{n-1} \langle v_\infty, e_{n+1} \rangle} \int_{\partial U} \langle T(\nabla u_i), N \rangle d\mathcal{H}^{n-1}.$$

Let  $t_0 = \inf \{ t : u_2 + t \geq u_1 \}$ . Then we have either  $u_1 - u_2 \equiv t_0$  or  $u_2 + t_0 > u_1$  in all of  $U$ . But the second case is impossible because of Thm 1 and the maximum principle at the boundary.  $\square$

### 3. FOLIATION PROPERTY OF THE SOLUTIONS

The following result is a consequence of the interior maximum principle (see [11]).

LEMMA 1. — *If  $M \subset \mathbb{R}^{n+1}$  is a compact minimal surface such that*

$$(\partial M \cap (A(\rho) \times \mathbb{R})) \subset \{ x : |px| = R, |x^{n+1}| \leq h_0 \} \text{ for some } \rho \in (0, R),$$

then

$$(M \cap (A(\rho) \times \mathbb{R})) \subset \{ x : |x^{n+1}| \leq h_0 + c_\rho(R) - c_\rho(r) \}.$$

Remark. — Let  $n \geq 3$  and  $u \in C^0(\bar{U})$  be a solution of (E) with  $v_\infty = e_{n+1}$ . Then if  $B \subset \mathbb{R}^n$  is a closed ball of radius  $\rho > 0$  containing  $\partial U$ , we conclude from the above lemma that

$$|u(\xi) - u(\infty)| \leq c_\rho(\infty) - c_\rho(r) \text{ in } U \setminus \text{int } B.$$

Setting  $A = \partial B \cap \partial U$ , we have in particular  $\text{osc}(u) \leq 2\rho c_1(\infty)$ . For exam-

ple there is no solution of the exterior Dirichlet problem having a vertical normal at infinity if  $\varphi \in C^0(S^{n-1})$  is given with  $\text{osc}(\varphi) > 2c_1(\infty)$ .

Now let  $U \subset \mathbb{R}^n$  be an exterior region,  $K = \mathbb{R}^n \setminus U$ , and suppose that  $u^\pm \in C^0(\bar{U})$  are two solutions of (E) satisfying  $u^- < u^+$  in  $U$  and  $u^\pm|_{\partial U} = \varphi$ .

LEMMA 2. — *Let  $K \subset B_R(0) = B \subset \mathbb{R}^n$  and suppose  $\psi \in C^2(\partial B)$  satisfies  $u^- < \psi < u^+$  on  $\partial B$ . Setting  $U_R = U \cap B$ , there is a unique solution  $u \in C^0(\bar{U}_R)$  of (E) satisfying  $u|_{\partial U} = \varphi$ ,  $u|_{\partial B} = \psi$ . Moreover  $u^- < u < u^+$  in  $U_R$ .*

Proof. — We refer to Haar's solution of the nonparametric Plateau problem [2] which is described in the book of Giusti [1]. Let us first consider the case that  $U$  has Lipschitz continuous boundary and  $\max \{ \text{Lip}(u^\pm, U_R) \} = l < \infty$ . Then for any sufficiently large  $k > l$ , we can take  $u^k \in C^0(\bar{U}_R)$  minimizing the area functional in the class  $\{ u \in C^0(\bar{U}_R) : \text{Lip}(u) \leq k, u|_{\partial U} = \varphi, u|_{\partial B} = \psi \}$ . The weak maximum principle [1], 12.5, yields  $u^- \leq u^k \leq u^+$  in  $U_R$ . Now because of [1], 12.7, we know that

$$\text{Lip}(u^k) = \sup \left\{ \frac{|u^k(\xi) - u^k(\eta)|}{|\xi - \eta|} : \xi \in U_R, \eta \in \partial U_R \right\}.$$

Since  $u^- \leq u^k \leq u^+$ , for any  $\eta \in \partial U$  and any  $\xi \in U_R$  we have

$$|u^k(\xi) - u^k(\eta)| \leq l|\xi - \eta|.$$

On the other hand, it is easy to construct barriers in a neighbourhood of  $\partial B$  (see [1], pp. 142-144). Hence  $\text{Lip}(u^k) < k$  for sufficiently large  $k$ , which means that  $u^k$  is a weak solution of (E) in  $U_R$ ; in fact because of the regularity theory ([1], 12.11)  $u^k$  is smooth. To treat the general case, we choose a regular value  $\varepsilon > 0$  of  $u^+ - u^-$ . Letting

$$V_\varepsilon = \{ \xi \in U_R : u^+(\xi) - u^-(\xi) > \varepsilon \}, \quad \varphi_\varepsilon = u^+ \quad \text{on } \partial V_\varepsilon \setminus \partial B,$$

we can apply the argument above to obtain a solution  $v_\varepsilon \in C^0(\bar{V}_\varepsilon)$  of (E) which coincides with  $u^+$  on  $\partial V_\varepsilon \setminus B$ , and with  $\psi$  on  $\partial B$ . Since  $u^- < v_\varepsilon < u^+$  in  $V_\varepsilon$ , the *a priori* estimates in [1] imply that  $v_\varepsilon \rightarrow u$  locally uniformly in  $C^2(U_R)$  as  $\varepsilon \rightarrow 0$ . Clearly  $u$  must attain the boundary values on  $\partial U$ . But on  $\partial B$  the same barriers apply to all the  $v_\varepsilon$  and hence  $u \in C^0(\bar{U}_R)$  is a solution of our problem. Uniqueness follows easily from the interior maximum principle.  $\square$

The following result is due to Krust [5] in the two dimensional case. In order to obtain the approximating solutions, he solved the parametric Plateau problem for a minimal annulus and referred to an embeddedness result of Meeks and Yau [9] together with the well-known argument of Kneser-Radó to show the graph property.

**THEOREM 2.** — *Let  $U \subset \mathbb{R}^n$  be an exterior region and  $\varphi \in C^0(\partial U)$ . The set of solutions of the exterior Dirichlet problem with boundary data  $\varphi$  having the same asymptotic normal forms a (possibly empty) foliation.*

*Proof.* — Let us first consider the case  $n \geq 3$ . Suppose  $u^\pm \in C^0(\bar{U})$  are two solutions with asymptotic normal  $v_\infty$ . Because of corollary 1, we may assume that  $h^+ > h^-$  and  $u^+ > u^-$  in  $U$ . Given any  $h \in (h^-, h^+)$ , we let

$$H = \{ x \in \mathbb{R}^{n+1} : \langle x, v_\infty \rangle = h \}, \\ \Gamma_R = \{ x \in H : |px| = R \}.$$

For any sufficiently large  $R$  there is a minimal graph  $u_R \in C^0(\bar{U}_R)$  such that  $u_R|_{\partial U} = \varphi$  and  $\text{graph}(u_R)|_{\partial B_R(0)} = \Gamma_R$ ; moreover  $u^- < u_R < u^+$  in  $U_R$ . As in lemma 2, we can let  $R \rightarrow \infty$  to obtain a solution  $u \in C^0(\bar{U})$  of the exterior Dirichlet problem with boundary data  $\varphi$  satisfying  $u^- \leq u \leq u^+$  in  $U$ . Let  $\pi$  be the orthogonal projection onto  $H$  and let  $B \subset H$  be an  $n$ -dimensional ball of radius  $\rho > 0$  containing  $\pi(\text{graph } \varphi)$ . Applying lemma 1 to  $\text{graph } u_R$  and then letting  $R \rightarrow \infty$  we infer that

$$| \langle x, v_\infty \rangle - h | \leq c_\rho(\infty) - c_\rho(|\pi x|)$$

for any  $x \in \text{graph } u$  satisfying  $\pi x \notin \bar{B}$ . In particular  $\text{graph } u$  is at a height  $h$  at infinity. Now the gradient of  $u$  is bounded ([1], 13.6) and in fact converges to a limit (see [10], thm 6). This means that  $u$  is regular at infinity in the sense of the introduction and has asymptotic normal  $v_\infty$ . Thus we have shown that for any  $h \in (h^-, h^+)$  there is a solution  $u_h$  with asymptotic height equal to  $h$  and moreover  $u_h < u_{h'}$  for  $h < h'$ . Now let

$x = (\xi, x^{n+1}) \in U \times \mathbb{R}$  be given such that  $u^-(\xi) < x^{n+1} < u^+(\xi)$ . Then we let  $h_1 = \sup \{h : u_h(\xi) < x^{n+1}\}$ ,  $h_2 = \inf \{h : u_h(\xi) > x^{n+1}\}$ . We see that  $h_1 < h_2$  is impossible because otherwise we would have  $u_{h_1}(\xi) < u_h(\xi) < u_{h_2}(\xi)$  for any  $h \in (h_1, h_2)$ . This proves the theorem if  $n \geq 3$ .

The case  $n=2$  was treated in [5]; the main difference in this case is that one has to replace the parameter  $h$  by the growth rate  $\alpha$ . Taking as  $\Gamma_{\mathbb{R}}$  the intersection of the cylinder  $\{x : |px| = R\}$  with a half catenoid of the desired growth rate centered around the axis  $\mathbb{R} \vee_{\infty}$  one proceeds essentially in the same way as above.  $\square$

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