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The symmetry of minimizing harmonic maps from a two dimensional domain to the sphere

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ABSTRACT. – We show that minimizing harmonic maps from an annulus in $\mathbb{R}^2$ to the sphere in $\mathbb{R}^3$ that agree on the boundary with the map $u_0(x, y)=(x/r, y/r, 0)$ – where $r^2=x^2+y^2$ – must be radially symmetric. This result combined with previous results of Jäger-Kaul [JK], Brezis-Coron [BC] and of Bethuel-Brezis-Coleman-Hélein [BBCH] shows that for any symmetrical domain in $\mathbb{R}^2$ and any symmetrical boundary data with image lying in a closed hemisphere, minimizing harmonic maps must be radially symmetric. We also give an example showing that this no longer has to be true when the boundary data has its image lying in a neighborhood – however small it may be – of a closed hemisphere.

Key words : Liquid crystals, harmonic maps.

RÉSUMÉ. – On montre que les applications harmoniques minimisantes d'un domaine bidimensionnel vers la sphère sont nécessairement symétriques dès que leur trace est symétrique et à valeur dans un hémisphère fermé. On montre également que ceci devient faux lorsque la trace est à valeurs dans un voisinage arbitrairement petit d'un hémisphère.

A.M.S. Classification: 35.

INTRODUCTION

F. Bethuel, H. Brezis, B. D. Coleman and F. Hélein have studied in a recent paper [BBCH] minimizing harmonic maps from an annulus

$$\Omega_\rho = \{ (x, y) \in \mathbb{R}^2 / \rho^2 < x^2 + y^2 < 1 \}$$

to the unit sphere

$$S^2 = \{ (x, y, z) \in \mathbb{R}^3 / x^2 + y^2 + z^2 = 1 \}$$

that agree on the boundary $\partial\Omega_\rho$ with the map $u_0(x, y) = (x/r, y/r, 0)$, where $r^2 = x^2 + y^2$. These maps are the minimizers of the problem

$$\min_{v \in \mathscr{E}_\rho} \iint_{\Omega_\rho} |\nabla v|^2,$$

where $\mathscr{E}_\rho = \{ v \in H^1(\Omega_\rho, S^2) / v_{|\partial\Omega_\rho} = u_{0\,|\partial\Omega_\rho} \}$. They have obtained the following results:

– For $\rho > e^{-\pi}$, u_0 is the only minimizer.

– For $\rho \leqq e^{-\pi}$, there is a unique minimizer in the class of radially symmetric maps (or *radial* maps), and it differs from u_0 when $\rho < e^{-\pi}$.

A radial map u is a map of the form

$$u(x, y) = (x/r \sin(\varphi(r)), y/r \sin(\varphi(r)), \cos(\varphi(r))),$$

where $r^2 = x^2 + y^2$ and φ is a real valued function.

We start by proving the following theorem

THEOREM 1. – *Let* $0 < \rho < 1$, *and suppose u is a minimizer for the problem*

$$\min_{v \in \mathscr{E}_\rho} \iint_{\Omega_\rho} |\nabla v|^2,$$

then u has radial symmetry.

Putting together previous results in [JK], [BC], [BBCH] and our theorem, we can state the following

THEOREM 2. – *Let* Ω *be a symmetric domain in* $\mathbb{R}^2$ *and let* $\psi : \partial\Omega \to S^2$ *be a radial boundary data with image lying in the closed upper hemisphere* $S^2_+ = \{ (x, y, z) \in \mathbb{R}^3 / z \geqq 0 \}$.

Then any minimizing harmonic map agreeing with ψ *on the boundary* $\partial\Omega$ *is radial.*

Remark 1. – It was already known from [JK] that this is true when the boundary data has values in a compact subset of the *open* upper hemisphere.

Remark 2. – The result of Theorem 2 fails to be true if one replaces S^2_+ by $K_\alpha = \{ (x, y, z) \in S^2 / z \geqq -\alpha \}$. We show at the end of this paper that for any $\alpha>0$, one can find a boundary data with values in K_α for which minimizers must break the symmetry.

1. PROOF OF THEOREM 1

The proof goes as follows: first, by an argument of I. Shafrir, we can suppose that the boundary data on the inner part of the annulus is free. Then we define for any minimizing harmonic map u a symmetrized map $\tilde{u}$. This map has the property that if u is not radial, it has strictly less energy than u on *part of the annulus*. In the rest of the annulus we are able to still reduce the energy by gluing $\tilde{u}$ with an appropriate conformal map.

Reduction of the problem

Let

$$\mathscr{E}_r=\{v\in H^1(\Omega_r, S^2)/v_{|\partial\Omega_r}=u_{0\,|\,\partial\Omega_r}\},$$

and for any $v\in\mathscr{E}_r$, and $r<s<1$ let $E_s(v)=\displaystyle\iint_{\Omega_s}|\nabla v|^2$.

We can state the

LEMMA 1 (I. Shafrir). — *If* $u\in\mathscr{E}_r$ *and* $E_r(u)=\min\limits_{v\in\mathscr{E}_r} E_r(v)$, *then for all* $v\in H^1(\Omega_{\sqrt{r}}, S^2)$ *such that* $v(x, y)=(x, y, 0)$ *whenever* $x^2+y^2=1$, *we have*

$$E_{\sqrt{r}}(u)\leqq E_{\sqrt{r}}(v).$$

Remark 3. — This means that the restriction of u to $\Omega_{\sqrt{r}}$ is minimizing in the bigger space $\mathscr{F}_{\sqrt{r}}$ of maps with finite energy that agree with u_0 only on the *outer boundary of* $\Omega_{\sqrt{r}}$.

Proof of Lemma 1. — For any $u\in\mathscr{E}_{r_0}$, we set

$$u_1(x, y)=\begin{cases} u(x, y) & \text{if } r<x^2+y^2\leqq 1\\ u\left(\dfrac{rx}{x^2+y^2}, \dfrac{ry}{x^2+y^2}\right) & \text{if } r^2\leqq x^2+y^2\leqq r\end{cases}$$

$$u_2(x, y)=\begin{cases} u\left(\dfrac{rx}{x^2+y^2}, \dfrac{ry}{x^2+y^2}\right) & \text{if } r<x^2+y^2\leqq 1\\ u(x, y) & \text{if } r^2\leqq x^2+y^2\leqq r\end{cases}$$

It is easily seen, since the map $(x, y) \to \left(\frac{rx}{x^2+y^2}, \frac{ry}{x^2+y^2}\right)$ is conformal, that

$$E_r(u_1)=2\iint_{\Omega_{\sqrt{r}}} |\nabla u|^2, \qquad E_r(u_2)=2\iint_{\Omega_r \setminus \Omega_{\sqrt{r}}} |\nabla u|^2,$$

therefore $2\,E_r(u)=E_r(u_1)+E_r(u_2)$. Moreover, $u \in \mathscr{E}_r$ implies that $u_1, u_2 \in \mathscr{E}_r$.

We now prove the lemma. Suppose u is a minimizer for E_r over $\mathscr{E}_r$, we must have $E_r(u)=E_r(u_1)=E_r(u_2)$ for if it were not true either u_1 or u_2 would have strictly less energy than u – a contradiction. So u_1 and u_2 are both minimizers.

If now v belongs to $\mathscr{F}_r$, we have $E_{\sqrt{r}}(v) \geqq E_{\sqrt{r}}(u)$; indeed define

$$v_1(x, y)=\begin{cases} v(x, y) & \text{if } r<x^2+y^2 \leqq 1 \\ v\left(\frac{rx}{x^2+y^2}, \frac{ry}{x^2+y^2}\right) & \text{if } r^2 \leqq x^2+y^2 \leqq r \end{cases}$$

we have $v_1 \in \mathscr{E}_r$ and since u_1 is a minimizer, $E_r(v_1) \geqq E_r(u_1)$. But $E_r(u_1)=2\,E_{\sqrt{r}}(u)$ and $E_r(v_1)=2\,E_{\sqrt{r}}(v)$, so that $E_{\sqrt{r}}(v) \geqq E_{\sqrt{r}}(u)$. Therefore the restriction of u to $\Omega_{\sqrt{r}}$ is a minimizer over $\mathscr{F}_{\sqrt{r}}$. ■

Hence proving Theorem 1 reduces to proving the following

THEOREM 1′. – *Let* $0<\rho<1$, *and suppose u is a minimizer for the problem*

$$\min_{v \in \mathscr{F}_\rho} \iint_{\Omega_\rho} |\nabla v|^2,$$

then u has radial symmetry.

Indeed by lemma 1, and if $u \in \mathscr{E}_\rho$ is such that

$$E_\rho(u)=\min_{v \in \mathscr{E}_\rho} \iint_{\Omega_\rho} |\nabla v|^2,$$

then Theorem 1′ would assert that u is radial *on the smaller annulus* $\Omega_{\sqrt{\rho}}$. But by a classical result of Morrey, u is analytic in Ω_ρ so that u must be radial everywhere. We now proceed to prove Theorem 1′.

Symmetrization

From now on, ρ is a fixed inner radius, and u is a minimizer for E_ρ over the space $\mathscr{F}_\rho$, we further assume that u is *not* radial. We are going to construct a radial v with strictly less energy than u and the theorem will be proved. First we define the symmetrized function of u, namely $\tilde{u}$.

For $r>0$, set $\gamma_r=\{(x, y)\in\mathbb{R}^2/x^2+y^2=r^2\}$, and

$$\sigma_r=\iint_{\Omega_r}\left|u.\left(\frac{\partial u}{\partial x}\wedge\frac{\partial u}{\partial y}\right)\right|,$$

that is, σ_r is the area – counted positively – spanned by $u(x, y)$ when (x, y) spans $\Omega_1\setminus\Omega_r$. Now $\tilde{u}$ is defined as the only map satisfying the following requirements:

– $\tilde{u}$ is radial,

– $\tilde{u}$ has its image lying in S^2_+,

– for any $\rho\leqq r\leqq 1$, $\tilde{u}(\gamma_r)$ is the perimeter of a surface having area $2\pi-\sigma_r$ if $\sigma_r\leqq 2\pi$, and $u_{|\gamma_r}=(0, 0, 1)$ otherwise.

Note that the fact that $\tilde{u}$ is radial implies that $\tilde{u}(\gamma_r)$ is in fact a circle parallel to the equator $S^1=\{(x, y, 0)\in\mathbb{R}^3/x^2+y^2=1\}$.

Let us now show that for some $\varepsilon>0$, $\tilde{u}$ has strictly less energy than u on the annulus $\Omega_{1-\varepsilon}$. For this we need to split the energy in the following way: let (s, θ) be polar coordinates in $\mathbb{R}^2$. For any $\rho\leqq r\leqq 1$, we get

$$N_r(u)=\int_{\gamma_r}|u_s|^2,\qquad T_r(u)=\frac{1}{r}\int_{\gamma_r}|u_\theta|^2,$$

where u_s and u_θ denote the partial derivatives of u with respect to the variables s and θ. We define in a similar way $\tilde{T}_r$ and $\tilde{N}_r$, the tangential and normal energies of $\tilde{u}$ on the circle γ_r. Note that T_r, N_r, $\tilde{T}_r$, $\tilde{N}_r$ are continuous functions of r. We have

$$\left.\begin{aligned}E_r(u)&=\iint_{\Omega_r}|\nabla u|^2=\int_r^1 T_s+N_s\,ds,\\ E_r(\tilde{u})&=\iint_{\Omega_r}|\nabla\tilde{u}|^2=\int_r^1\tilde{T}_s+\tilde{N}_s\,ds.\end{aligned}\right\}\tag{1}$$

On the other hand, from the definition of σ_r and $\tilde{u}$, we have

$$\left|\frac{d\sigma_r}{dr}\right|\leqq\int_{\gamma_r}\left|u.\left(u_s\wedge\frac{u_\theta}{r}\right)\right|,\qquad\left|\frac{d\sigma_r}{dr}\right|=\left|\int_{\gamma_r}\tilde{u}.\left(\tilde{u}_s\wedge\frac{\tilde{u}_\theta}{r}\right)\right|.\tag{2}$$

We need two lemmas:

LEMMA 2. – *For all $\rho\leqq r\leqq 1$ we have*

$$\left|\frac{d\sigma_r}{dr}\right|\leqq(T_r N_r)^{1/2},\qquad\left|\frac{d\sigma_r}{dr}\right|=(\tilde{T}_r\tilde{N}_r)^{1/2}.$$

Moreover, if $u_{|\gamma_r}=\tilde{u}_{|\gamma_r}$ then the first inequality is strict. More precisely, equality in the inequality implies that u is radial.

Proof. — We first prove the inequality. We know by (2) that

$$\left|\frac{d\sigma_r}{dr}\right| \leqq \int_{\gamma_r} |u_s| \left|\frac{u_\theta}{r}\right| \leqq (\mathrm{T}_r \mathrm{N}_r)^{1/2},$$

by Cauchy-Schwarz inequality. For $\tilde{u}$, equality holds in all inequalities since by radiality $\tilde{u}_s \perp \tilde{u}_\theta$ and $|\tilde{u}_s|$, $|\tilde{u}_\theta|$ are constant on γ_r.

To prove the last assertion, let us suppose that $u_{|\gamma_r} = \tilde{u}_{|\gamma_r}$ and that equality holds in the above inequalities for u also. It is not difficult to see that this implies that $\tilde{u}$ and u coincide as well as their first derivatives on the circle γ_r. Now for some $\varepsilon > 0$, one can find a radial harmonic map defined on $\Omega_{r-\varepsilon} \setminus \Omega_r$ that agrees with $\tilde{u}$ as well as its first derivatives on γ_r: it suffices to find a local solution to an ordinary differential equation of second order with initial conditions. Call this map v and set

$$w(x, y) = \begin{cases} u(x, y) & \text{if } \ r < x^2 + y^2 \leqq 1 \\ v(x, y) & \text{if } \ (r-\varepsilon)^2 \leqq x^2 + y^2 \leqq r^2. \end{cases}$$

Since w is a C^1 harmonic map, it is analytic and thus it is identically equal to u. Hence u is radial. ■

LEMMA 3. — *For any* $\rho \leqq r \leqq 1$, *we have*

$$\tilde{\mathrm{T}}_r \leqq \mathrm{T}_r.$$

Moreover, equality holds if and only if for some rotation or anti rotation R *of* $\mathbb{R}^3$ *we have* $\mathrm{R} \circ u_{|\gamma_r} \equiv \tilde{u}_{|\gamma_r}$.

Proof. — This is a consequence of an isoperimetric inequality. $\tilde{u}(\gamma_r)$ is by definition the boundary of a surface having area $2\pi - \sigma_r$ while we claim that $u(\gamma_r)$ — when it is a smooth path — is the boundary of two disjoint surfaces both having area greater than $2\pi - \sigma_r$. Suppose this is true, then since $\tilde{u}(\gamma_r)$ is a circle it is by isoperimetric inequality a curve of shorter length then $u(\gamma_r)$. Call $\tilde{l}$ and l these two lengths, we have

$$\int_{\gamma_r} \left|\frac{\tilde{u}_\theta}{r}\right| = \tilde{l} \leqq l \leqq \int_{\gamma_r} \left|\frac{u_\theta}{r}\right|.$$

The first inequality comes from the radiality of $\tilde{u}$. But $|\tilde{u}_\theta|$ is constant on γ_r and so by Cauchy-Schwartz inequality the left hand side is equal to $(\tilde{\mathrm{T}}_r \times 2\pi r)^{1/2}$ while the right hand side is less than $(\mathrm{T}_r \times 2\pi r)^{1/2}$. The last assertion of the lemma follows by looking at the cases where equality holds.

Now we prove that $u(\gamma_r)$ — when it is a smooth path — is the boundary of two disjoint surfaces both having area greater than $2\pi - \sigma_r$. Via stereographic projection, we can define the winding number of $u(\gamma_r)$ with respect to a point in S^2, and we call $\Sigma_e(r)$ [resp. $\Sigma_0(r)$] the set of points in S^2 for which this number is even (resp. odd). We have $u(\gamma_r) = \partial\Sigma_e(r) = \partial\Sigma_0(r)$.

Because of the boundary data, we know that $|\Sigma_e(1)|=|\Sigma_0(1)|=2\pi$, and if x doesn't belong to $u(\Omega_1\setminus\Omega_r)$, then $x\in\Sigma_e(1)$ iff $x\in\Sigma_e(r)$: indeed the path $u(\gamma_r)$ depends continuously on r so that if $u(\gamma_1)$ is deformed into $u(\gamma_r)$ without touching x, the winding number of both paths with respect to x is the same. Now we can conclude that

$$|\Sigma_e(r)|\geqq 2\pi-\sigma_\tau,\qquad |\Sigma_0(r)|\geqq 2\pi-\sigma_r,$$

and our claim is proved when $u(\gamma_r)$ is a smooth path. If it isn't, we get the result by approximation. ■

To summarize we have the following: u and $\tilde{u}$ agree on γ_1 so Lemma 1 gives us $T_1 N_1>\tilde{T}_1\tilde{N}_1$ but $T_1=\tilde{T}_1$ and so $N_1>\tilde{N}_1$. By continuity of N_r, $\tilde{N}_r$, we know that for some $\varepsilon>0$ and for any $1-\varepsilon<r\leqq 1$ we have $N_r>\tilde{N}_r$. On the other hand $\tilde{T}_r\leqq T_r$ always holds. Therefore by (1) we have

$$E_{1-\varepsilon}(\tilde{u})<E_{1-\varepsilon}(u).$$

From now on we call δ the difference between these two energies.

Set

$$r_0=\inf\left\{s\in[\rho,1];\text{ there is a radial } v\in\mathscr{F}_s \text{ s.t.}\begin{cases}\text{(i) } v_{|\partial\Omega_s}=\tilde{u}_{|\partial\Omega_s}\\ \text{(ii) } E_s(v)+\delta\leqq E_s(u)\end{cases}\right\}.\qquad(3)$$

Then we have $r_0\leqq 1-\varepsilon$ for some $\varepsilon>0$. We now show that there is a $v\in\mathscr{F}_{r_0}$ satisfying conditions (i) and (ii) above with $s=r_0$ – that is the inf is achieved. Indeed, from the definition of r_0, there is a sequence (r_i) of real numbers decreasing to r_0, and a sequence (v_i) of radial maps satisfying (i) and (ii) above with $s=r_i$. For each i we set

$$\tilde{v}_i(x,y)=\begin{cases}v_i(x,y) & \text{if } r_i^2\leqq x^2+y^2\leqq 1\\ \tilde{u}(x,y) & \text{if } r_0^2\leqq x^2+y^2\leqq r_i^2\end{cases}.$$

Then $\tilde{v}_i$ is radial, agrees with $\tilde{u}$ on $\partial\Omega_{r_0}$, and

$$E_{r_0}(\tilde{v}_i)=E_{r_i}(v_i)+(E_{r_0}(\tilde{u})-E_{r_i}(\tilde{u})).$$

It is then easily seen that the sequence $(\tilde{v}_i)$ will converge in $H^1(\Omega_{r_0},S^2)$ to a map v which is as desired.

Now if $r_0=\rho$, Theorem 1′ is proved: v is a radial map in $\mathscr{F}_\rho$ with strictly less energy then u. If not, we proceed to extend v with a conformal map.

Continuation

Let r_0 and v be as in the previous section and set, for $\varepsilon>0$,

$$v_\varepsilon(x,y)=\begin{cases}v(x,y) & \text{if } r_0^2\leqq x^2+y^2\leqq 1\\ \tilde{u}(x,y) & \text{if } (r_0-\varepsilon)^2\leqq x^2+y^2\leqq r_0^2.\end{cases}$$

Then by (3), v_ε has more energy than u on $\Omega_{r_0-\varepsilon}\setminus\Omega_{r_0}$, and so

$$\int_{r_0-\varepsilon}^{r} \mathrm{T}_s+\mathrm{N}_s\,ds<\int_{r_0-\varepsilon}^{r} \tilde{\mathrm{T}}_s+\tilde{\mathrm{N}}_s\,ds.$$

Letting ε go to 0 we get $\mathrm{T}_{r_0}+\mathrm{N}_{r_0}\leqq\tilde{\mathrm{T}}_{r_0}+\tilde{\mathrm{N}}_{r_0}$. But we know that $\mathrm{T}_{r_0}\geqq\tilde{\mathrm{T}}_{r_0}$ so that $\mathrm{N}_{r_0}\leqq\tilde{\mathrm{N}}_{r_0}$ and then $\mathrm{T}_{r_0}-\mathrm{N}_{r_0}\leqq\tilde{\mathrm{T}}_{r_0}-\tilde{\mathrm{N}}_{r_0}$.

In fact this is a strict inequality, for suppose $\mathrm{T}_{r_0}=\tilde{\mathrm{T}}_{r_0}$ and $\mathrm{N}_{r_0}=\tilde{\mathrm{N}}_{r_0}$ then for some rotation or anti rotation R of $\mathbb{R}^3$ we have $\tilde{u}_{|\gamma_{r_0}}=\mathrm{R}\circ u_{|\gamma_{r_0}}$ (*see* Lemma 3). Now we can adapt the proof of Lemma 2 to conclude that $\mathrm{R}\circ u$ is radial. But then $\mathrm{R}\circ u$ and u coincide on γ_1 so that R is either the identity or the reflection across the xy plane. In both cases u is radial.

From $\mathrm{T}_{r_0}+\mathrm{N}_{r_0}\leqq\tilde{\mathrm{T}}_{r_0}+\tilde{\mathrm{N}}_{r_0}$ and $\mathrm{T}_{r_0}\mathrm{N}_{r_0}\geqq\tilde{\mathrm{T}}_{r_0}\tilde{\mathrm{N}}_{r_0}$ we conclude that $(\mathrm{T}_{r_0}-\mathrm{N}_{r_0})^2\leqq(\tilde{\mathrm{T}}_{r_0}-\tilde{\mathrm{N}}_{r_0})^2$. Finally, and this is what we were getting at

$$\tilde{\mathrm{T}}_{r_0}>\tilde{\mathrm{N}}_{r_0}. \tag{4}$$

This fact will allow us to extend v.

Let $\hat{u}$ be the conformal mapping such that

- $\hat{u}_{|\gamma_{r_0}}=\tilde{u}_{|\gamma_{r_0}}$
- $\displaystyle\int_{\gamma_{r_0}} \hat{u}\cdot\left(\hat{u}_s\wedge\frac{\hat{u}_\theta}{r_0}\right)\geqq 0.$

This $\hat{u}$ is given, for an appropriate $\lambda\in\mathbb{R}$ by

$$\hat{u}(x,y)=\frac{1}{1+\lambda^2x^2+\lambda^2y^2}(2\lambda x,\,2\lambda y,\,1-\lambda^2x^2-\lambda^2y^2).$$

From now on, $\tilde{v}$ will denote the gluing together of v and $\hat{u}$ that is $\tilde{v}$ is equal to v on Ω_{r_0} and to $\hat{u}$ on $\Omega_\rho\setminus\Omega_{r_0}$. The reason for using a conformal mapping is that it spans a given area with less energy than any other map. More precisely, define for any $\rho\leqq r\leqq r_0$

$$\hat{\sigma}(r)=\iint_{\Omega_r\setminus\Omega_{r_0}}\left|\hat{u}\cdot\left(\hat{u}_s\wedge\frac{\hat{u}_\theta}{r}\right)\right|,$$

and define similarly $\tilde{\sigma}(r)$, the area (counted positively) spanned by $\tilde{u}(x,y)$ when (x,y) spans $\Omega_r\setminus\Omega_{r_0}$. Then

$$\tilde{\sigma}(r)\leqq 2(\mathrm{E}_r(\tilde{u})-\mathrm{E}_{r_0}(\tilde{u})),\qquad \hat{\sigma}(r)=2(\mathrm{E}_r(\hat{u})-\mathrm{E}_{r_0}(\hat{u})). \tag{5}$$

This comes from the inequality $|\alpha\wedge\beta|\leqq\frac{1}{2}(|\alpha|^2+|\beta|^2)$ – which holds for any two vectors α, β – applied pointwise to the derivatives of $\tilde{u}$ and $\hat{u}$; equality holds for conformal maps. Note that from the definition of $\tilde{u}$, the area – counted positively – spanned by $u(x,y)$ when (x,y) spans $\Omega_r\setminus\Omega_{r_0}$ is equal to $\tilde{\sigma}(r)$, so we also have

$$\tilde{\sigma}(r)\leqq 2(\mathrm{E}_r(u)-\mathrm{E}_{r_0}(u)). \tag{6}$$

Now we have $\hat{\sigma}(r_0)=\tilde{\sigma}(r_0)=0$, and

$$\left|\frac{d\tilde{\sigma}}{dr}(r_0)\right|=\int_{\gamma_{r_0}}\left|\tilde{u}.\left(\tilde{u}_s\wedge\frac{\tilde{u}_\theta}{r_0}\right)\right|,$$

note that on the righthand side, the absolute value could as well be placed outside the integral since $\tilde{u}$ is radial. We have a similar formula for $\left|\frac{d\hat{\sigma}}{dr}(r_0)\right|$. From these two formulas, using the radiality of $\tilde{u}$ and $\hat{u}$ and the conformality of $\hat{u}$, we get:

$$\left|\frac{d\tilde{\sigma}}{dr}(r_0)\right|=(\tilde{\mathrm{T}}_{r_0}\tilde{\mathrm{N}}_{r_0})^{1/2},\qquad \left|\frac{d\hat{\sigma}}{dr}(r_0)\right|=(\tilde{\mathrm{T}}_{r_0}\tilde{\mathrm{T}}_{r_0})^{1/2}.$$

Thus we can use (4) to conclude that

$$\left|\frac{d\tilde{\sigma}}{dr}(r_0)\right|>\left|\frac{d\hat{\sigma}}{dr}(r_0)\right|. \tag{7}$$

Thus for $\varepsilon>0$ small enough and $r_0-\varepsilon<s<r_0$, we have $\hat{\sigma}(s)<\tilde{\sigma}(s)$. Let s_0 be the smallest number in $[\rho, r_0]$ for which the last inequality is true for all $s_0<s<r_0$. We see by (5) and (6) that

$$\mathrm{E}_{s_0}(\hat{u})-\mathrm{E}_{r_0}(\hat{u})\leqq\mathrm{E}_{s_0}(u)-\mathrm{E}_{r_0}(y),$$

and so

$$\mathrm{E}_{s_0}(\tilde{v})+\delta\leqq\mathrm{E}_{s_0}(u).$$

Then if $s_0=\rho$, the proof is over because then $\tilde{v}$ is a radial map in $\mathscr{F}_\rho$ with strictly less energy than u.

But this must be true: if s_0 were greater than ρ, then we would have $\hat{\sigma}(s_0)=\tilde{\sigma}(s_0)$. In turn this would mean that $\hat{u}$ and $\tilde{u}$ agree on γ_{s_0}. Now $\tilde{v}$ would agree with conditions (i) and (ii) of (3), with $s=s_0$. But this is impossible since $s_0<r_0$. The proof of Theorem 1′ is complete. ∎

Remark. – After we announced our Theorem 1 (*see* [S]), a simpler proof has been found by S. Kaniel [K]. It relies on a different symmetrization and does not require a continuation argument.

2. PROOF OF THEOREM 2, AND AN EXAMPLE

Proof of Theorem 2

We have a given domain Ω in $\mathbb{R}^2$ invariant under rotations of the plane, and a boundary data $\varphi:\partial\Omega\to \mathrm{S}^2_+$, having radial symmetry. We want to show that a minimizing harmonic map with boundary data φ must have radial symmetry. But a connected component of Ω is either conformal through a dilatation to the unit disk $D^2=\{(x, y)\in\mathbb{R}^2/x^2+y^2<1\}$ or to some annulus Ω_ρ with $\rho>0$. Moreover the Dirichlet integral is invariant under conformal transformations so that it suffices to prove the theorem in these two cases.

For the case where Ω is the disk, the result has been proved in [BC], so we are left with the case of an annulus Ω_ρ.

Now if φ has values in the *open* hemisphere, the result follows from [JK], *see* also [BBCH]. If φ is the restriction to $\partial\Omega_\rho$ of the map u_0, our Theorem 1 gives the result. The only case left is therefore one where the restriction of φ to one of the connected components of $\partial\Omega_\rho$ – we can assume without loss of generality that it is the outer boundary γ_1 – is equal to u_0, while on the other component – say the inner boundary γ_ρ –, φ is given by $\forall (x, y)\in\gamma_\rho$

$$\varphi(x, y)=(\mathrm{R}\,x/\rho,\ \mathrm{R}\,y/\rho,\ \sqrt{1-\mathrm{R}^2}),$$

for some $0\leqq \mathrm{R}<1$.

In this case, arguments similar to those in [BBCH] tell us that there is a radial minimizing harmonic map having such boundary data, we call it v, with image lying in S^2_+. Now v can be extended to a radial harmonic map $\tilde{v}$ defined on a bigger annulus $\Omega_{\rho-\varepsilon}$ whose image still lies in S^2_+. Hence by [BBCH] $\tilde{v}$ is minimizing. Suppose u is another minizing harmonic map on Ω_ρ with φ as boundary data, then we can set

$$w(x, y)=\begin{cases} u(x, y) & \text{if } \rho^2\leqq x^2+y^2\leqq 1\\ \tilde{v}(x, y) & \text{if } (\rho-\varepsilon)^2\leqq x^2+y^2\leqq\rho^2.\end{cases}$$

It is obvious that w is minimizing, hence analytic. Since $\tilde{v}$ also is, $w\equiv\tilde{v}$ and $u\equiv v$. Therefore v is the only minimizing harmonic map with boundary data φ and the proof of Theorem 2 is completed.

An example

We show in this section that Theorem 2 cannot be improved in an obvious way. More precisely, set for any positive real α,

$$\mathrm{K}_\alpha=\{(x, y, z)\in S^2/z\geqq-\alpha\}.$$

Then for any $\alpha>0$, there is a radius $\rho>0$ and a radial boundary data $\varphi:\partial\Omega_\rho\to\mathrm{K}_\alpha$ such that minimizing harmonic maps having φ as boundary data cannot be radial.

Indeed, fix $\alpha>0$ and set $\forall (x, y)\in\gamma_1$, $\varphi(x, y)=(0, 0, 1)$, and $\forall (x, y)\in\gamma_\rho$, $\varphi(x, y)=(\lambda x, \lambda y, -\alpha)$, where $\lambda=1/\rho\sqrt{1-\alpha^2}$. Then if u is a radial map on Ω_ρ having boundary data φ, u must cover all of K_α and therefore

$$\iint_{\Omega_\rho} |\nabla u|^2 \geqq 2(2\pi+\beta),$$

where $\beta>0$ and $2\pi+\beta$ is the area of K_α. This is true for any ρ, therefore we will choose ρ later on.

On the other hand, for r small enough, we can find a map v (which will not be radial) such that (i) $v_{|\gamma_1} \equiv (0, 0, 1)$, (ii) $v_{|\gamma_r} \equiv (0, 0, -1)$, and

$$\iint_{\Omega_r} |\nabla v|^2 < \beta/2.$$

(*See* [BG].)

Moreover if ρ is chosen small enough, then by deforming slightly a conformal map we may construct a map w such that (i) $w_{|\gamma_r} \equiv (0, 0, -1)$, (ii) $v_{|\gamma_\rho} = \varphi_{|\gamma_\rho}$, and

$$\iint_{\Omega_\rho \setminus \Omega_r} |\nabla w|^2 < 2(2\pi-\beta)+\beta/2,$$

note that $2\pi-\beta$ is the area of $S^2 \setminus K_\alpha$.

Then we can glue v and w to get a map defined on Ω_ρ with strictly less energy then any radial map agreeing with it on the boundary of Ω_ρ.

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