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# **Convergence and partial regularity for weak solutions of some nonlinear elliptic equation: the supercritical case**

by

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**ABSTRACT.** – In this paper we prove a partial regularity result for stationary weak solutions of  $-\Delta u = u^\alpha$ , when  $\alpha$  is greater than the critical Sobolev exponent.

*Key words:* Partial regularity, nonlinear elliptic equation.

**RÉSUMÉ.** – On démontre dans cet article un résultat de régularité partielle pour les solutions faibles stationnaires de  $-\Delta u = u^\alpha$  lorsque l'exposant  $\alpha$  est supérieur à l'exposant critique de Sobolev.

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## **1. INTRODUCTION**

We consider the equation

$$-\Delta u = u^\alpha, \tag{1}$$

when  $\alpha$  is greater than the critical Sobolev exponent.

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In this paper, we give some results concerning the partial regularity of limits of sequences of regular solutions of (1) as well as some partial regularity result for positive weak solutions which are stationary.

Let  $\Omega$  be an open subset of  $\mathbb{R}^n$ ,  $u$  is said to be a positive weak solution of (1) in  $\Omega$  if  $u \geq 0$  a.e. and if, for all  $w \in C^\infty(\Omega)$  with compact support in  $\Omega$ , the following holds

$$\int_{\Omega} u \Delta w dx = - \int_{\Omega} u^\alpha w dx.$$

By definition, we will say that such a weak solution  $u$  is stationary if, in addition, it satisfies

$$\int_{\Omega} \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} \frac{\partial \phi^j}{\partial x_i} - \frac{1}{2} |\nabla u|^2 \frac{\partial \phi^i}{\partial x_i} + \frac{1}{\alpha + 1} u^{\alpha+1} \frac{\partial \phi^i}{\partial x_i} dx = 0, \quad (2)$$

for all regular vector field  $\phi$  having a compact support in  $\Omega$  (summation over indices  $i$  and  $j$  is understood).

For weak solutions belonging to  $H^1(\Omega) \cap L^{\alpha+1}(\Omega)$ , this equation is obtained by computing, for  $u_t(x) = u(x + t\phi(x))$ , the quantity

$$\frac{d}{dt} \mathcal{E}(u_t)|_{t=0} = 0.$$

Where the energy  $\mathcal{E}(u)$ , related to (1), is defined for all  $u \in H_0^1(\Omega) \cap L^{\alpha+1}(\Omega)$  by

$$\mathcal{E}(u) \equiv \frac{1}{2} \int_{\Omega} |\nabla u|^2 dx - \frac{1}{\alpha + 1} \int_{\Omega} u^{\alpha+1} dx.$$

If  $u$  satisfies the relation (2), this means that  $u$  is a critical point of  $\mathcal{E}(\cdot)$  with respect to variations on the parameterization of the domain.

In order to state our results we need to define the notion of singular set. Let  $u \in H^1(\Omega) \cap L^{\alpha+1}(\Omega)$  be a weak solution of (1) in  $\Omega$ . We denote by  $S$  the set of points  $x \in \Omega$  such that  $u$  is not bounded in any neighborhood  $V$  of  $x$  in  $\Omega$ . We recall that, if  $u$  is bounded in some neighborhood of  $x$ , then the classical regularity theory ensures us that  $u$  is regular in some neighborhood of  $x$ . Therefore  $S$  is the set of singularities of  $u$ . Moreover, considering the definition,  $S$  is a closed subset of  $\Omega$ .

Our partial regularity result reads as follows :

**THEOREM 1.** – *Let  $\alpha \geq \frac{n+2}{n-2}$  be given. If  $u \in H^1(\Omega) \cap L^{\alpha+1}(\Omega)$  is a positive weak solution of (1) which is stationary, then the Hausdorff dimension of the singular set of  $u$  is less than  $n - 2 \frac{\alpha+1}{\alpha-1}$ .*

The proof of this result relies on the monotonicity formula (see [10]) and an  $\epsilon$ -regularity result in Campanato spaces.

A weaker version of theorem 2 has already been derived in a former paper (see [10]). In that paper, it is shown that the result of theorem 2 holds for  $\alpha < \frac{n+1}{n-3}$ . The method used in the present paper is the same, in its spirit, but the final argument is taken from [1].

The result of theorem 1 is to be compared with a result of L. C. Evans on stationary Harmonic maps [4]. In that paper, the author proves that, in dimension  $n$ , all weak harmonic maps into spheres which are stationary have a singular set of Hausdorff dimension less than  $n - 2$ . More recently F. Bethuel has extended this results to arbitrary targets [1].

We now state our convergence result :

**THEOREM 2.** – *If  $u_k$  is a sequence of regular solutions of (1) bounded in  $H_0^1(\Omega)$  then there exists some closed subset  $\tilde{S} \subset \Omega$  and some subsequence of  $u_k$  which converges strongly to some  $u$  in  $C_{loc}^{2,\epsilon}(\Omega \setminus \tilde{S})$ . In addition, the Hausdorff dimension of  $\tilde{S}$  is less than or equal to  $n - 2\frac{\alpha+1}{\alpha-1}$ .*

An immediate corollary of this result is that the limit  $u$  is a weak solution of (1) which is regular except on a set of Hausdorff dimension less than or equal to  $n - 2\frac{\alpha+1}{\alpha-1}$ .

Moreover the result of the last theorem holds if one considers a sequence of stationary weak solutions bounded in  $H_0^1(\Omega)$ , instead of a sequence of regular solutions.

*Remark 1.* – Unless it is explicitly stated,  $c$  will denote a universal constant depending only on  $\alpha$  the exponent in equation (1) and  $n$  the dimension of the space.

## 2. A MONOTONICITY FORMULA

We define  $\mu = n - 2\frac{\alpha+1}{\alpha-1}$  and the rescaled energy

$$E_u(x, r) = \frac{\alpha - 1}{2} r^{-\mu} \int_{B(x,r)} |\nabla u|^2 dx + \frac{\alpha - 1}{\alpha + 1} r^{-\mu} \times \int_{B(x,r)} u^{\alpha+1} dx + \frac{d}{dr} \left\{ r^{-\mu} \int_{\partial B(x,r)} u^2 ds \right\} \quad (3)$$

In the above expression the derivative is to be understood in the sense of distributions. We have proved in [10] the following proposition :

**PROPOSITION 1** [10]. – *If  $u$  is a stationary positive weak solution of (1), then  $E_u(x, r)$ , defined above, is an increasing function of  $r$ . In addition  $E_u(x, r)$  is a positive continuous function of  $x \in \Omega$  and  $r > 0$ .*

The following lemma establishes a relation between the energy  $E_u(x, r)$  and a more familiar rescaled energy :

LEMMA 1 [10]. – *There exist  $c > 0$  and  $\tau_0 \in (0, 1)$  depending only on  $\alpha$  and  $n$ , such that, whenever  $E_u(x_0, r_0) \leq \epsilon$  for some  $x_0 \in \Omega$  and  $r_0 > 0$ , then*

$$\frac{1}{r^\mu} \int_{B(x_0, r)} u^{\alpha+1} dx \leq c\epsilon,$$

for all  $0 < r < \tau_0 r_0$ .

### 3. A CRITERION FOR BOUNDEDNESS

In order to prove our regularity results, we will need the following proposition :

PROPOSITION 2. – *There exists  $\epsilon > 0$  and  $\tau \in (0, 1)$  such that, if*

$$\frac{1}{r^\mu} \int_{B(x, r)} u^{\alpha+1} \leq \epsilon,$$

for all  $x$  in  $B(0, 1)$  and all  $r \leq 2$ , then  $u$  is bounded in  $B(0, \tau)$  by some constant depending only on  $\alpha$  and  $n$ .

*Proof.* – We give the definition of the space  $L^{p,q}(\Omega)$  :

$$L^{p,q}(\Omega) \equiv \left\{ v \in L^p(\Omega) / \sup_{x \in \Omega} \left( \sup_{r > 0} \left( r^{-q} \int_{B(x, r) \cap \Omega} v^p dx < +\infty \right) \right) \right\}.$$

(See [2], [3], [8] and also [5] for the general study of these spaces). We will denote by

$$\|v\|_{p,q,\Omega} = \sup_{x \in \Omega} \sup_{r > 0} \left( r^{-q} \int_{B(x, r) \cap \Omega} v^p dx \right)^{\frac{1}{p}}$$

the norm of  $v$  in  $L^{p,q}(\Omega)$ .

We define also the space  $\tilde{L}^{p,q}(\Omega)$  :

$$\tilde{L}^{p,q}(\Omega) \equiv \left\{ v \in L^p(\Omega) / \sup \left( r^{-q} \int_{B(x, r)} u^p dx, x \in \Omega, r > 0 \text{ and } B(x, r) \subset \Omega \right) < +\infty \right\}.$$

We will denote by

$$|v|_{p,q,\Omega} = \sup_{B(x,r) \subset \Omega} \left( r^{-q} \int_{B(x,r)} v^p dx \right)^{\frac{1}{p}},$$

the norm of  $v$  in  $\tilde{L}^{p,q}(\Omega)$ .

With the above notations, the assumption of proposition 2 can be written as

$$\|u\|_{\alpha+1,\mu,B(0,1)}^{\alpha+1} \leq \epsilon.$$

We decompose the proof of proposition 2 in many steps.

*Step 1.* – We are going to prove that if  $\epsilon$  is chosen small enough, then the following decay property is true :

LEMMA 2. – *Under the assumptions of proposition 2, there exists some  $\theta \in (0, 1)$  depending only on  $\alpha$  and  $n$ , such that the following holds :*

$$|u|_{\alpha,\lambda,B(x,\theta r)}^{\alpha} \leq \frac{1}{2} |u|_{\alpha,\lambda,B(x,r)}^{\alpha},$$

for all  $x \in B(0, 1)$ ,  $r > 0$  satisfying  $B(x, 2r) \subset B(0, 1)$ .

*Proof.* – In the whole proof, we assume that  $x$  and  $r$  are chosen to fulfill the inclusion  $B(x, 2r) \subset B(0, 1)$ . We defined  $\tilde{u} = u$  on  $B(x, r)$  and  $\tilde{u} = 0$  outside  $B(x, r)$ . The first step of the proof consists in proving some estimates on  $u$  using the Poisson kernel. We set

$$v(z) = c \int_{B(x,r)} |z - y|^{-n+2} u^{\alpha-1}(y) \tilde{u}(y) dy.$$

And first prove the estimate

$$\int_{B(x,r)} v^{\alpha} dz \leq cr^{\lambda} (\|u\|_{\alpha+1,\mu,B(x,2r)}^{\alpha-1} \|\tilde{u}\|_{\alpha,\lambda,B(x,2r)})^{\alpha}. \tag{4}$$

*Proof of the estimate.* – We define  $f = u^{\alpha-1} \tilde{u}$ . A simple application of Hölder inequality gives us

$$\|f\|_{\beta,\gamma,B(x,r)} \leq c \|\tilde{u}\|_{\alpha,\lambda,B(x,r)} \|u\|_{\alpha+1,\mu,B(x,r)}^{\alpha-1},$$

where  $\beta = \frac{\alpha^2 + \alpha}{\alpha^2 + 1}$  and  $\gamma = n - 2 \frac{(\alpha+1)\alpha^2}{(\alpha-1)(\alpha^2+1)}$ .

Using once more Hölder inequality it is easy to see that

$$\|f\|_{\frac{\alpha^2}{\alpha^2+1},\delta,B(x,r)} \leq c \|\tilde{u}\|_{\alpha,\lambda,B(x,r)} \|u\|_{\alpha+1,\mu,B(x,r)}^{\alpha-1},$$

where  $\delta = n - 2\frac{\alpha^3}{(\alpha-1)(\alpha^2+1)}$ . Let us compute for a.e.  $z \in B(x, r)$

$$v(z) = c \int_{B(x,r)} |z - y|^{-n+2} f(y) dy.$$

We can write this equality as

$$v(z) = c \int_{B(x,r)} |z - y|^{-n+2+a} f(y)^{\frac{1}{\alpha^2+1}} (f(y)^{\frac{\alpha^2}{\alpha^2+1}} |z - y|^{-a}) dy,$$

Where  $a > 0$  is to be chosen later. Since  $f^{\frac{\alpha^2}{\alpha^2+1}} \in L^{1,\delta}(B(x, r))$ , we have

$$\int_{B(x,r)} f^{\frac{\alpha^2}{\alpha^2+1}}(y) |z - y|^{-a} dy < cr^{\delta-a},$$

whenever  $a$  satisfies  $a < \delta$ . Using Hölder inequality, we obtain

$$\begin{aligned} v^\alpha(z) &\leq c \int_{B(x,r)} |z - y|^{(-n+2+a)\alpha-a} f^{\frac{\alpha^2+\alpha}{\alpha^2+1}}(y) dy \\ &\quad \times \left( \int_{B(x,r)} f^{\frac{\alpha^2}{\alpha^2+1}}(y) |z - y|^{-a} dy \right)^{\alpha-1}, \end{aligned} \tag{5}$$

as long as we choose  $a < \delta$ . If we choose  $a$  such that

$$\int_{B(x,2r)} |z - y|^{(-n+2+a)\alpha-a} dz < +\infty,$$

we can conclude that  $v \in L^\alpha(B(x, r))$ . Therefore  $a$  must satisfy the inequalities  $a < \delta$  and  $(n - 2 - a)\alpha + a < n$ . In order to be able to find some  $a$  satisfying the above inequalities, we must check that

$$n - 2\frac{\alpha}{\alpha - 1} < \delta = n - 2\frac{\alpha^3}{(\alpha - 1)(\alpha^2 + 1)}.$$

Which is true. Integrating inequality (5) over  $B(x, r)$  we find

$$\int_{B(x,r)} v^\alpha(z) dz \leq cr^\lambda (||u||_{\alpha+1,\mu,B(x,r)}^{\alpha-1} ||\tilde{u}||_{\alpha,\lambda,B(x,r)}^\alpha).$$

Which is the desired estimate.

Now, we turn to the second step of the proof of lemma 2. Let us decompose  $u$  over  $B(x, r)$  into two parts. We define  $w$  to be the solution of

$$\begin{cases} -\Delta w = 0 & \text{in } B(x, r) \\ w = u & \text{on } \partial B(x, r). \end{cases}$$

Then  $w$  is regular and Harmonic in  $B(x, r)$ . Therefore, for all  $y \in B(x, r/2)$  we can write

$$w(y) = c \frac{2^n}{r^n} \int_{B(y, r/2)} w(z) dz.$$

Integrating over  $B(x, \rho)$ , we obtain

$$\int_{B(x, \rho)} w^\alpha(z) dz \leq c \frac{\rho^n}{r^n} \int_{B(x, r)} w^\alpha(z) dz,$$

for all  $\rho < r/2$ . Now, the maximum principle leads to the estimate

$$\int_{B(x, \rho)} w^\alpha(z) dz \leq c \frac{\rho^n}{r^n} \int_{B(x, r)} u^\alpha(z) dz. \tag{6}$$

Since  $u \leq w + v$  over  $B(x, r)$ , we have

$$\int_{B(x, \rho)} u^\alpha(y) dy \leq c \left( \int_{B(x, \rho)} w^\alpha(y) dy + \int_{B(x, r)} v^\alpha(y) dy \right).$$

Thus, using (4) and (6), we get the estimate

$$\begin{aligned} & \int_{B(x, \rho)} u^\alpha(y) dy \\ & \leq c \left( \frac{\rho^n}{r^n} \int_{B(x, r)} u^\alpha(y) dy + r^\lambda (\|u\|_{\alpha+1, \mu, B(x, r)}^{\alpha-1} \|\tilde{u}\|_{\alpha, \lambda, B(x, r)}^\alpha) \right). \end{aligned} \tag{7}$$

We notice that

$$\|\tilde{u}\|_{\alpha, \lambda, B(x, r)} \leq |u|_{\alpha, \lambda, B(x, 2r)}.$$

So, we derive from (7) that

$$\begin{aligned} & \frac{1}{\rho^\lambda} \int_{B(x, \rho)} u^\alpha(y) dy \\ & \leq c \left( \frac{\rho^{n-\lambda}}{r^{n-\lambda}} |u|_{\alpha, \lambda, B(x, 2r)}^\alpha + \left(\frac{r}{\rho}\right)^\lambda (\|u\|_{\alpha+1, \mu, B(x, r)}^{\alpha-1} |u|_{\alpha, \lambda, B(x, 2r)}^\alpha) \right). \end{aligned}$$

In order to conclude, we may chose  $\rho = 2\theta r$ , with  $\theta > 0$  chosen in order to fulfill the inequalities

$$c(2\theta)^{n-\lambda} < 1/4 \quad \text{and} \quad 4\theta < 1.$$

Once this is done, we can choose  $\epsilon$  such that

$$c\left(\frac{1}{2\theta}\right)^\lambda (\|u\|_{\alpha+1,\mu}^{\alpha-1}, B(x, 4r))^\alpha \leq 1/4.$$

With the above choices, we get

$$\frac{1}{(2\theta r)^\lambda} \int_{B(x, 2\theta r)} u^\alpha(y) dy \leq \frac{1}{2} |u|_{\alpha,\lambda, B(x, 2r)}^\alpha \leq \frac{1}{2} |u|_{\alpha,\lambda, B(0,1)}^\alpha,$$

as  $B(x, r) \subset B(0, 1)$ . We conclude easily from the last inequality that

$$|u|_{\alpha,\lambda, B(0,\theta)}^\alpha \leq \frac{1}{2} |u|_{\alpha,\lambda, B(0,1)}^\alpha.$$

This ends the proof of lemma 2.

*Step 2.* – It is classical to see that, if the conclusion of lemma 2 holds, then there exists some  $\lambda_0 > \lambda$  such that  $u$  is bounded in  $L^{\alpha,\lambda_0}(B(0, 1/2))$  by some constant depending only on  $\alpha, n$  and  $\epsilon$ .

Using the method of [9], we get the following lemma, from which it is easy to derive the conclusion of proposition 2 :

LEMMA 3 [9]. – *Let  $u$  be a positive weak solution of (1), assume that  $u \in L^{\alpha,\lambda_0}(B(0, 1))$  for some  $\lambda_0 > \lambda$  then  $u$  is bounded in  $B(0, 1/2)$  by some constant depending on  $\alpha, n$  and the norm of  $u$  in  $L^{\alpha,\lambda_0}(B(0, 1))$ .*

*Proof.* – The proof is very similar to the proof of theorem 3 in [9] and many arguments of it are already used in the proof of lemma 2. Nevertheless, we give it here for sake of completeness.

Let us decompose  $u$  solution of (1) over  $B(0, 1)$  in two parts. We consider some open ball  $B(x, r)$  included in  $B(0, 1)$ . As in the proof of lemma 2, we get for all  $\rho < r/2$  the estimate

$$\int_{B(x,\rho)} u^\alpha dy \leq c \left( \frac{\rho^n}{r^n} \int_{B(x,r)} u^\alpha dy + \int_{B(x,r)} v^\alpha dy \right).$$

Now, by definition

$$v(y) = c \int_{B(x,r)} u^\alpha(z) |y - z|^{-n+2} dz.$$

Therefore, using the assumption  $u \in L^{\alpha,\lambda_0}(B(0, 1))$ , one can prove for  $v$  an estimate similar to (5). We write, for some  $a > 0$

$$v(z) = c \int_{B(x,r)} |x - z|^{-n+2+a} (u^\alpha(y) |x - y|^{-a}) dy.$$

Moreover, since  $u^\alpha \in L^{1,\lambda_0}(B(0, 1))$  we have

$$\int_{B(x,r)} u^\alpha(y)|x - y|^{-a} dy < +\infty,$$

if  $a$  satisfies  $a < \lambda_0$ . Using Hölder inequality, we get

$$v^\alpha(z) \leq c \int_{B(x,r)} |z - y|^{(-n+2+a)\alpha-a} u^\alpha(y) dy \times \left( \int_{B(x,r)} |z - y|^{-a} u^\alpha(y) dy \right)^{\alpha-1}, \tag{8}$$

as long as  $a < \lambda_0$ . So, if we choose  $a$  such that

$$\int_{B(x,2r)} |z - y|^{(-n+2+a)\alpha-a} dz < +\infty,$$

we can conclude that  $v \in L^\alpha(B(x, r))$ . Therefore, if  $\alpha < \frac{n-a}{n-a-2}$  then  $v \in L^\alpha(B(x, r))$ . Integrating inequality (8) over  $B(x, r)$ , we find

$$\int_{B(x,r)} v^\alpha dx \leq cr^{n+\lambda_0\alpha-(n-2)\alpha} \|u\|_{\alpha,\lambda_0,B(x,r)}^\alpha.$$

This leads to

$$\int_{B(x,\rho)} u^\alpha dy \leq c \left( \frac{\rho^n}{r^n} \int_{B(x,r)} u^\alpha dy + r^{n+\lambda_0\alpha-(n-2)\alpha} \|u\|_{\alpha,\lambda_0,B(x,r)}^\alpha \right).$$

We conclude that, if  $u \in L^{\alpha,\lambda_0}(B(0, 1))$  and if  $\rho < r/2$ , the inequality

$$\int_{B(x,\rho)} u^\alpha dy \leq c \left( \frac{\rho^n}{r^n} \int_{B(x,r)} u^\alpha dy + r^{n-\epsilon-(n-2-\epsilon)\alpha+\lambda_0} \right), \tag{9}$$

holds provided that  $\alpha < \frac{n-a}{n-a-2}$  and  $a < \lambda_0$  and  $B(x, r) \subset B(0, 1)$ . This inequality holds for  $\rho < r/2$  but increasing if necessary the constant  $c$  we can assume that it holds for all  $\rho < r$ .

Now, we use a lemma due to S. Campanato [3] (see [5] for a simple proof).

LEMMA 4 [3]. – Let  $0 < \lambda < n$  and  $c > 0$ , if  $\phi$  is a non decreasing function on  $\mathbb{R}$  such that for all  $\rho < r$  we have  $\phi(\rho) \leq c \left( \frac{\rho^n}{r^n} \phi(r) + r^\lambda \right)$ . Then there exists a positive constant  $C$  depending only on  $r, \phi(r)$  and  $c$  such that  $\phi(\rho) \leq C\rho^\lambda$ .

We assume that  $u \in L^{\alpha, \lambda_0}(B(0, 1))$  with  $\lambda_0 > \lambda$ . We choose  $4/5 < r_1 < 1$ . Using (9) and the last lemma, we prove that  $u \in L^{\alpha, \lambda_1}(B(0, r_1))$  for all  $\lambda_1 < \lambda_0 + (n - \lambda_0 - (n - 2 - \lambda_0)\alpha)$ . Let us notice that, as  $\lambda_0 > \lambda$ , we have  $\lambda_0 < \lambda_1$ . Going on by induction, we can define a sequence  $\lambda_i$  tending to  $n - 2$  and a sequence of radii  $4/5 < r_i < 1$  such that  $u \in L^{\alpha, \lambda_i}(B(0, r_i))$ . So  $u \in L^{\alpha, \mu}(B(0, 4/5))$  for all  $\mu < n - 2$ .

Now, using this fact we can prove as above that the Newtonian potential of  $u^\alpha$  belongs to  $L^p(B(0, 3/4))$  for all  $p > 1$ . Therefore, by standard elliptic estimates, that  $u$  is bounded in  $B(0, 1/2)$ . This ends the proof of the lemma and therefore the proof of proposition 2.

From the result of proposition 2, we may derive the following :

**COROLLARY 1.** – *Let us assume that  $E_u(x, r_0) \leq \epsilon$  for all  $x \in B(x_0, r_0)$ , where  $\epsilon$  is the constant given in proposition 2, then  $u$  is bounded near  $x_0$ .*

*Proof.* – First, using proposition 1 and lemma 1, we get that

$$\frac{1}{r^\mu} \int_{B(x, r)} u^{\alpha+1} \leq \epsilon,$$

for all  $x$  in  $B(x_0, r_0/2)$  and all  $r \leq r_0$ . Let us notice that equation (1) is invariant by scaling. More precisely, if  $u(x)$  is a weak solution of (1) then so is  $\delta^{\frac{2}{\alpha-1}} u(\delta x)$  for all  $\delta > 0$ . Using this invariance by scaling and the invariance by translation we can always assume that  $x_0 = 0$  and  $r_0/2 = 1$ .

Now, using proposition 2, we get the existence of some constant  $c > 0$ , independent of  $u$ , such that

$$\|u\|_{L^\infty(B(x_0, \tau r_0))} \leq c r_0^{-\frac{2}{\alpha-1}}.$$

Which is the desired result.

#### 4. THE PARTIAL REGULARITY THEORY

The proof of theorem 1 is now standard. We discard the points of  $\Omega$  where  $E_u(x, r)$  concentrates. We introduce the set

$$S \equiv \bigcap_{r>0} \{x \in \Omega / E_u(x, r) \geq \epsilon\},$$

where  $\epsilon$  is the constant given in proposition 2. It is easy to see that  $S$  is a closed set (this is proved using the monotonicity inequality). Moreover using proposition 1, we prove the lemma

LEMMA 5. – Let  $S$  be the set defined above then there exists some constant  $c_0 > 0$  which does not depend on  $\epsilon$  such that

$$S \subset \bigcap_{r>0} \left\{ x \in \Omega / \int_{B(x,r)} (u^{\alpha+1} + |\nabla u|^2) dx \geq c_0 \epsilon r^\mu \right\}.$$

*Proof.* – The claim we have to prove is the following :

$$S \subset \bigcap_{r>0} \left\{ x \in \Omega / \int_{B(x,r)} (u^{\alpha+1} + |\nabla u|^2) dx \geq c_0 \epsilon r^\mu \right\} \equiv \Sigma.$$

If  $x$  does not belong to  $\Sigma$ , then there exists some  $r > 0$  such that

$$\int_{B(x,r)} (u^{\alpha+1} + |\nabla u|^2) dx < c_0 \epsilon r^\mu.$$

Using the definition of  $E_u(x, r)$  given by (3), it is possible to obtain another formula for  $E_u(x, r)$  (see [10]), namely

$$\begin{aligned} E_u(x, r) &= r^{-\mu} \int_{B(x,r)} \frac{1}{2} |\nabla u|^2 dx - \frac{1}{\alpha + 1} r^{-\mu} \\ &\quad \times \int_{B(x,r)} u^{\alpha+1} dx + \frac{r^{-\mu-1}}{\alpha - 1} \int_{\partial B(x,r)} u^2 ds. \end{aligned}$$

Taking advantage of the fact that  $E_u(\sigma)$  is increasing and using the last formula, we get the estimate

$$\sigma^{\mu+1} E_u(x, \sigma) \leq c c_0 \epsilon r^{\mu+1} + c \int_{\partial B(x,\sigma)} u^2 ds,$$

for all  $\sigma \in [r/2, r]$ . Integrating this inequality between  $r/2$  and  $r$ , we find that

$$\int_{r/2}^r \sigma^{\mu+1} E_u(x, \sigma) d\sigma \leq c c_0 \epsilon r^{\mu+2} + c \int_{B(x,r)} u^2 dx.$$

Using Hölder inequality, we get

$$\int_{r/2}^r \sigma^{\mu+1} E_u(x, \sigma) d\sigma \leq c(c_0 + c_0^{\frac{2}{\alpha+1}}) \epsilon r^{\mu+2}.$$

Using proposition 1, we get the estimate

$$E_u(x, \sigma) < \epsilon,$$

for some  $\sigma \in [r/2, r]$ , if  $c_0$  is suitably chosen. This ends the the proof of lemma 5.

As  $u \in H^1(\Omega)$  and  $u \in L^{\alpha+1}(\Omega)$ , we conclude, considering the result of lemma 5, that the Hausdorff dimension of  $S$  is less than  $n - 2\frac{\alpha+1}{\alpha-1}$ . If  $x \in \Omega \setminus S$ , using the results of section 3, we know that  $u$  is regular in some neighborhood of  $x$ , so the singular set of  $u$  is exactly  $S$ . This ends the proof of theorem 1.

### 5. PARTIAL REGULARITY OF WEAK LIMITS

Using the results of section 3, we know that, if  $E_u(x, r_0)$  is small enough for all  $x \in B(x_0, r_0)$  (say  $E_u(x, r) < \epsilon$ ), we have a  $L^\infty$  bound on  $u$  which depends only on  $r_0$ ,  $n$  and  $\alpha$ .

The end of the proof of theorem 2 is now standard. As usual, *see* [12], we discard the points of  $\Omega$  where  $E_{u_i}(x, r)$  concentrates. We introduce the set

$$\tilde{S} \equiv \bigcap_{r>0} \{x \in \Omega / \liminf_{i \rightarrow \infty} E_{u_i}(x, r) \geq \epsilon\},$$

where  $\epsilon$  is the constant given above. It is easy to see that the following lemma holds :

LEMMA 6. –  $\tilde{S}$  is a closed set whose Hausdorff dimension is less than or equal to  $n - 2\frac{\alpha+1}{\alpha-1}$ .

*Proof.* – In order to prove this lemma, we first claim that, under the assumptions of theorem 2, for all  $x_0 \in \Omega$  and  $R > 0$  such that  $B(x_0, 4R) \subset \Omega$ , there exists some constant  $c > 0$  independent of  $i$  such that

$$\frac{1}{r^\mu} \int_{B(x,r)} u_i^{\alpha+1} \leq c \tag{10}$$

and

$$\frac{1}{r^\mu} \int_{B(x,r)} |\nabla u_i|^2 \leq c, \tag{11}$$

for all  $i$ , whenever  $x \in B(x_0, R)$  and  $r \leq R$  ( $c$  depends on the  $\sup_i \|u_i\|_{H^1_0(\Omega)}$ ).

*Proof of the claim.* – As in the proof of lemma 5, we integrate  $\sigma^{\mu+1} E_{u_i}(x, \sigma)$  between  $2R$  and  $4R$ . Then we use the fact that  $u_i$  is bounded in  $H^1(\Omega)$  and in  $L^{\alpha+1}(\Omega)$  as well as Hölder inequality in order to get the estimate

$$\int_{2R}^{4R} \sigma^{\mu+1} E_{u_i}(x, \sigma) d\sigma \leq c.$$

Where  $c$  does not depend on  $i$ . Using the result of proposition 1, we get a uniform bound on  $E_{u_i}(x, \sigma)$  for all  $r < 2R$ . For almost all  $x$ , we can integrate  $E_{u_i}(x, r)$  between 0 and  $r \leq 2R$  and obtain

$$\int_0^r \sigma^{-\mu} \int_{B(x, \sigma)} \left\{ \frac{\alpha - 1}{2} |\nabla u_i|^2 + \frac{\alpha - 1}{\alpha + 1} u_i^{\alpha+1} \right\} dx d\sigma + r^{-\mu} \int_{\partial B(x, r)} u_i^2 ds < cr.$$

In particular

$$\int_0^r \sigma^{-\mu} \int_{B(x, \sigma)} |\nabla u_i|^2 dx d\sigma < cr$$

and

$$\int_0^r \sigma^{-\mu} \int_{B(x, \sigma)} u_i^{\alpha+1} dx d\sigma < cr,$$

for all  $r \leq 2R$ . The estimates (10) and (11), for all  $r \leq R$ , follow easily from the last two inequalities.

We want to show that  $\tilde{S}$  is closed. Assume that this is not true and that we have a sequence  $x_j \in \tilde{S}$  converging to some point  $x_0$  which does not belong to  $\tilde{S}$ . This means that there exists some  $R > 0$  and some subsequence of  $u_i$  (that we will still denote by  $u_i$ ) such that

$$\lim_{i \rightarrow \infty} E_{u_i}(x_0, R) \leq \epsilon - \delta,$$

for some  $\delta > 0$ . Using proposition 1, we see that

$$\lim_{i \rightarrow \infty} E_{u_i}(x_0, r) \leq \epsilon - \delta,$$

for all  $r < R$ . Multiplying this inequality by  $r^{\mu+1}$ , and integrating it between  $[R/2, R]$ , we get the existence of some  $\delta > 0$  such that

$$\begin{aligned} \int_{R/2}^R r^{\mu+1} E_{u_i}(x_0, r) dr &= \frac{1}{2} \int_{R/2}^R r \int_{B(x_0, r)} |\nabla u_i|^2 dx dr \\ &- \frac{1}{\alpha + 1} \int_{R/2}^R r \int_{B(x_0, r)} u_i^{\alpha+1} dx dr \\ &+ \frac{1}{\alpha + 1} \int_{B(x_0, R) \setminus B(x_0, R/2)} u_i^2 dx \leq (\epsilon - \delta) \int_{R/2}^R r^{\mu+1} dr, \end{aligned} \tag{12}$$

for  $i$  large enough (here we have used, for  $E_u(x, r)$ , the formula given in the proof of lemma 5). As  $x_j \in \tilde{S}$ , we see that the following holds

$$\begin{aligned} \lim_{i \rightarrow \infty} \int_{R/2}^R r^{\mu+1} E_{u_i}(x_j, r) dr &= \lim_{i \rightarrow \infty} \left\{ \frac{1}{2} \int_{R/2}^R r \int_{B(x_j, r)} |\nabla u_i|^2 dx dr \right. \\ &\quad - \frac{1}{\alpha + 1} \int_{R/2}^R r \int_{B(x_j, r)} u_i^{\alpha+1} dx dr \\ &\quad \left. + \frac{1}{\alpha + 1} \int_{B(x_j, R) \setminus B(x_j, R/2)} u_i^2 dx \right\} \geq (\epsilon - \delta) \int_{R/2}^R r^{\mu+1} dr. \quad (13) \end{aligned}$$

Thanks to the bounds we have derived at the beginning of the proof of lemma 6 (see (10) and (11)), we see that the difference between the left hand side of (12) and the left hand side of (13) tends to 0 as  $x_j$  tends to  $x_0$  independently of  $i$ . Therefore, we get a contradiction. This ends the proof of lemma 6.

Finally, it can be shown, as in the proof of lemma 5, that the Hausdorff dimension of  $\tilde{S}$  is less than or equal to  $n - 2\frac{\alpha+1}{\alpha-1}$ . If  $x \in \Omega \setminus \tilde{S}$ , by what we have just seen, a subsequence of  $u_i$  is locally bounded near  $x$ , therefore we have strong convergence and the weak limit is regular. A classical diagonal process gives the result of theorem 2.

*Remark.* – The proof of theorem 2 can also be obtained, in the case of a sequence of regular solutions, using the method developed by R. Schoen in [12].

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