

ANNALES DE L'I. H. P., SECTION C

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Annales de l'I. H. P., section C, tome 12, n° 4 (1995), p. 415-424

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A note on weak approximation of minors

by

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ABSTRACT. – Let $A_{p,q}(\Omega)$, where $\Omega \subseteq \mathbb{R}^n$ is a bounded open domain, be a set of all mappings $u \in W^{1,p}(\Omega, \mathbb{R}^n)$ such that $\text{adj } Du \in L^q$. Among other results we prove that if $n - 1 \leq p < n$, $1 < q < n/(n - 1)$, then the subclass of $A_{p,q}$ mappings, which consists of mappings with bounded $(n - 1)$ -dimensional image, is dense in the sequential weak topology of $A_{p,q}$. We also extend this result to other $A_{p,q}$ type spaces.

1991 MSC Primary: 28A75, 73C50. **Secondary:** 28A05

Key words: Nonlinear elasticity, approximation of minors, Sobolev mappings, Suslin sets.

1. INTRODUCTION AND STATEMENT

Let $\Omega \subseteq \mathbb{R}^n$ be a bounded domain. We define

$$A_{p,q}(\Omega) = \{u : \Omega \rightarrow \mathbb{R}^n \mid u \in W^{1,p}(\Omega, \mathbb{R}^n), \text{adj } Du \in L^q\},$$

where the matrix $\text{adj } Du$ satisfies the identity $(\det Du)\text{Id} = Du \text{adj } Du$.

Spaces of this type arise in a natural way as classes on which variational functionals related to nonlinear elasticity are defined (see e.g. [1], [5], [6]).

* This work was partially supported by KBN grant no. 2 1057 91 01. This research was carried on while I was visiting C.M.L.A. E.N.S. de Cachan in 1993. I want to thank for support and hospitality.

[7] [8], [10], [16], [19], [21], [22], [24], [29]). It is very important to note that if $q > p/(n-1)$, then the space $A_{p,q}$ is not linear. The space $A_{p,q}$ is endowed both with strong and weak topology. We say that $u_k \rightarrow u$ strongly in $A_{p,q}$, if $u_k \rightarrow u$ strongly in $W^{1,p}$ and $\text{adj } Du_k \rightarrow \text{adj } Du$ strongly in L^q ; $u_k \rightarrow u$ weakly in $A_{p,q}$ if $u_k \rightarrow u$ weakly in $W^{1,p}$ and $\text{adj } Du_k \rightarrow \text{adj } Du$ weakly in L^q . All kinds of weak convergence, as usual, will be denoted by “harpoon” \rightarrow in place of “arrow”. Our main results are Theorem 1 and its generalisation, Theorem 3.

THEOREM 1. – *If $u \in A_{p,q}(\Omega)$, $\Omega \subseteq R^n$, $n-1 \leq p < n$, $1 < q < n/(n-1)$, then there exists a sequence $u^{(\nu)} \in A_{p,q}(\Omega)$, such that values of $u^{(\nu)}$ belong to a certain bounded $(n-1)$ -dimensional simplicial set (which depends on ν) and $u^{(\nu)} \rightarrow u$ weakly in $A_{p,q}(\Omega)$. In the other words the subset of $A_{p,q}$, which consists of mappings with bounded $(n-1)$ -dimensional range is dense in $A_{p,q}$ in the sequential weak topology.*

Remark. – Note that it is not possible to substitute in the above theorem weak convergence in $A_{p,q}$ by strong convergence. Otherwise we would have (after passing to a subsequence), $Du^{(\nu)} \rightarrow Du$ a.e. and hence $0 \equiv \det Du^{(\nu)} \rightarrow \det Du$ a.e.

COROLLARY 1. – *If u is as above, then there exists $u^{(\nu)} \in A_{p,q}(\Omega)$, such that $u^{(\nu)} \rightarrow u$ weakly in $A_{p,q}$ and $\det Du^{(\nu)} \equiv 0$.*

COROLLARY 2. – *If $u \in A_{n-1, \frac{n}{n-1}}(\Omega)$, then for each $q < n/(n-1)$ there exists a sequence $u^{(\nu)}$, such that $u^{(\nu)} \in A_{n-1,q}(\Omega)$, $u^{(\nu)} \rightarrow u$ weakly in $A_{n-1,q}$ and $\det Du^{(\nu)} \equiv 0$.*

The above corollaries are in contrast with the following theorem of Müller-Qi-Yan [24].

THEOREM 2 (Müller-Qi-Yan [24, Lemma 4.1]). – *Let $p \geq n-1$, $q \geq \frac{n}{n-1}$ and $u^{(\nu)} \rightarrow u$ weakly in $A_{p,q}(\Omega)$.*

1. *If $q > \frac{n}{n-1}$, then $\det Du^{(\nu)} \rightarrow \det Du$ in $L^{q(n-1)/n}$.*
2. *If $q = \frac{n}{n-1}$ and $\det Du^{(\nu)} \geq 0$, then $\det Du^{(\nu)} \rightarrow \det Du$ in $L^1(K)$ for all compact sets $K \subseteq \Omega$.*

The following theorem generalizes Theorem 1.

THEOREM 3. – *If $u \in W^{1,p}(\Omega, R^n)$, $p < n$, then there exists a sequence $u^{(\nu)} \in W^{1,p}(\Omega, R^n)$ such that values of $u^{(\nu)}$ belong to a certain bounded $[p]$ -dimensional simplicial set (this set depends on ν) and $u^{(\nu)} \rightarrow u$ weakly in $W^{1,p}$. Moreover if $\text{adj}_s Du \in L^{q_s}$, for some s, q_s*

such that $1 < q_s < ([p] + 1)/s$, then we can additionally obtain that $\text{adj}_s Du^{(\nu)} \rightharpoonup \text{adj}_s Du$ weakly in L^{q_s} .

Remarks. – 1) Up to the sign and order, $\text{adj}_s Du$ is a multidimensional matrix consisting of all $s \times s$ minors of Du (cf. [5]). By $[p]$ we denote the greatest integer, less than or equal to p .

2) By the same reason as in the case of Theorem 1, we cannot substitute weak convergence by strong convergence.

Theorems 1 and 3 are in some sense related to the results about the approximation of Sobolev mappings between manifolds $W^{1,p}(M, N)$ ([28], [3] [2], [15], [4], [13], [12]). At first sight the problem seems to be different because of a different nature of Sobolev mappings $W^{1,p}(M, N)$ and mappings $A_{p,q}(\Omega)$. However the careful study shows some deep connections between these problems. In particular the crucial (for us) method of retractions (Lemma 1) is a modification of the analogous method previously used in the context of approximation of Sobolev mappings between manifolds [4], [13].

To see further connections between the theory of $W^{1,p}(M, N)$ and $A_{p,q}(\Omega)$ mappings we refer to [14].

By Q^n we will denote a “general” n -dimensional cube. By C we will denote a general constant. It can change its value even in the same proof. Writing for example $C(n, p)$, we will show that this constant depends on n and p only.

2. PROOFS OF THEOREMS 1 AND 3

Before we proceed to the proofs of these theorems, we shall state and prove main technical lemma (Lemma 1 below). The following mapping is defined in $R^n \setminus \{x\}$:

$$\pi_{x,\varepsilon}(z) = \begin{cases} z & \text{when } |x - z| \geq \varepsilon \\ \frac{z - x}{|z - x|}\varepsilon + x & \text{when } |x - z| \leq \varepsilon. \end{cases}$$

In other words $\pi_{x,\varepsilon}$ is a mapping which is an identity on the complement of the ball $B^n(x, \varepsilon)$, and which is a projection along radii onto the boundary $S^{n-1}(x, \varepsilon)$ inside the ball. Evidently $\pi_{x,\varepsilon}$ is discontinuous at $z = x$. Moreover

$$|D_z(\pi_{x,\varepsilon})(z)| \leq C(n) \frac{\varepsilon}{|z - x|}, \tag{1}$$

for $z \in B^n(x, \varepsilon)$. In the sequel we will use the following notation. If $\bar{x} = (x_1, \dots, x_k)$ is a sequence of points of R^n , then we set

$$\pi_{\bar{x}, \varepsilon} = \pi_{x_1, \varepsilon} \circ \dots \circ \pi_{x_k, \varepsilon}.$$

In the proof of Theorems 1 and 3 the following lemma will play a fundamental role.

LEMMA 1. — *Let $u \in W^{1,p}(\Omega, R^n)$, $\Omega \subseteq R^n$, $p < n$. Let $A_1, \dots, A_k \subseteq R^n$ be a family of measurable sets such that $(\varepsilon/2)^n < |A_i| < \infty$ and $\text{dist}(A_i, A_j) \geq 2\varepsilon$ for all $i \neq j$. Then for almost all $\bar{x} = (x_1, \dots, x_k) \in A_1 \times \dots \times A_k$, $\pi_{\bar{x}, \varepsilon} \circ u \in W^{1,p}(\Omega)$. Moreover there exists $\bar{x} \in A_1 \times \dots \times A_k$ such that*

$$\int_{\Omega} |D(\pi_{\bar{x}, \varepsilon} \circ u)(z)|^p dz \leq C(n, p) \int_{\Omega} |Du(z)|^p dz,$$

where the constant $C(n, p)$ depends on n and p only.

Remark. — Assumption $\text{dist}(A_i, A_j) \geq 2\varepsilon$ guarantees that the sets $\mathcal{O}_{\varepsilon}(A_i) = \{x \in R^n \mid \text{dist}(x, A_i) < \varepsilon\}$ are pairwise disjoint.

Before we proceed to the proof of Lemma 1 we shall be concerned with some other lemmas.

We say that $f \in \text{ACL}(\Omega)$ if the function f is Borel measurable and absolutely continuous on almost all lines parallel to coordinate axes. Since absolutely continuous functions are almost everywhere differentiable, $f \in \text{ACL}(\Omega)$ has partial derivatives a.e. and hence the gradient ∇f is defined a.e. Now we say that $f \in \text{ACL}^p(\Omega)$ if $f \in L^p(\Omega) \cap \text{ACL}(\Omega)$ and $|\nabla f| \in L^p$. The following characterization of Sobolev space is due to Nikodym ([25], [20, Section 1.1.3]).

THEOREM 4 (Nikodym). — $W^{1,p}(\Omega) = \text{ACL}^p(\Omega)$.

Since maybe it is not evident how to understand this theorem, we shall comment it now. This theorem states that each $\text{ACL}^p(\Omega)$ function belongs to $W^{1,p}(\Omega)$ and the gradient ∇f , which is defined a.e. for $f \in \text{ACL}^p(\Omega)$ is just the distributional gradient. On the other hand, each element $f \in W^{1,p}(\Omega)$ (which is an equivalence class of functions equal except the set of measure zero) admit a Borel representative, which belongs to the space $\text{ACL}^p(\Omega)$.

LEMMA 2. — *Let $f : X \rightarrow Y$ be a mapping between separable metric spaces X and Y . If f is a Borel mapping, then the graph $G_f = \{(x, f(x)) \mid x \in X\} \subseteq X \times Y$ is a Borel set.*

Proof. – Let $\{A_i^{(n)}\}_{i=1}^\infty$ be a family of pairwise disjoint Borel subsets of Y such that $\bigcup_{i=1}^\infty A_i^{(n)} = Y$ and $\text{diam } A_i^{(n)} < 1/n$. Then the set

$$B_n = \bigcup_{i=1}^\infty (f^{-1}(A_i^{(n)}) \times A_i^{(n)}) \subseteq X \times Y$$

is Borel and hence $G_f = \bigcap_{n=1}^\infty B_n$ is also a Borel set.

We also need the following famous theorem of Lusin and Sierpiński [17], [18], [9, Thm. 2.2.13].

THEOREM 5 (Lusin-Sierpiński). – *Let $P : R^{n+k} \rightarrow R^n$ be an orthogonal projection. If $B \subseteq R^{n+k}$ is a Borel set, then $P(B) \subseteq R^n$ is H^n -measurable.*

Remark. – Note that this theorem is no longer valid if we assume that B is H^{n+k} measurable, instead of being Borel. Note also that it is not true in general that $P(B)$ is Borel even if B is Borel.

We will use the above Lusin-Sierpiński’s theorem in the proof of the following “Sard’s type” lemma.

LEMMA 3. – *Assume that $f : Q^n \rightarrow R^k$ is a Borel mapping. Then the following two conditions are equivalent:*

1. *For almost all segments $I \subseteq Q^n$, parallel to one of the coordinate axes, $H^k(f(I)) = 0$.*
2. *For almost all $x \in R^k$, the set $f^{-1}(x)$ is disjoint from almost all segments parallel to one of the coordinate axes.*

Remark. – For generalizations and further results see [11], [13].

Proof. – Let $P : Q^n \rightarrow Q^{n-1}$, $P(x_1, \dots, x_n) = (x_2, \dots, x_n)$ be the orthogonal projection in the direction parallel to the first coordinate axis. It induces the projection $\bar{P} : Q^n \times R^k \rightarrow Q^{n-1} \times R^k$, $\bar{P}(x, y) = (P(x), y)$. By Lemma 2, the graph $G_f \subseteq Q^n \times R^k$ is a Borel set and hence by Theorem 5, the set $\bar{P}(G_f) \subseteq Q^{n-1} \times R^k$ is $H^{n-1} \otimes H^k$ -measurable. Now it follows directly from Fubini’s theorem that the following three conditions are equivalent.

1. $(H^{n-1} \otimes H^k)(\bar{P}(G_f)) = 0$.
2. For almost all segments I , parallel to the axis x_1 , $H^k(f(I)) = 0$.
3. For almost all $x \in R^k$, $f^{-1}(x)$ is disjoint from almost all segments parallel to x_1 .

Of course one can repeat the above construction with x_1 replaced by x_2, x_3, \dots, x_n . Then the lemma follows easily.

The above lemma can be applied to $\text{ACL}(Q^n, R^k)$ mappings as it follows from the following elementary observation.

LEMMA 4. – *If $f : [0, 1] \rightarrow R^k$, $k \geq 2$ is absolutely continuous then $H^k(f([0, 1])) = 0$.*

Proof. – Otherwise, applying Fubini's theorem, we would find the one dimensional slice of the set $f([0, 1])$ with positive one dimensional measure, and such that this slice is the image of a certain subset of $[0, 1]$ with the measure zero. This contradicts the following well known fact: If $g : [0, 1] \rightarrow R$ is absolutely continuous and $E \subset [0, 1]$, $H^1(E) = 0$ then $H^1(g(E)) = 0$ (see [27, Theorem 7.18]).

Lemmas 3 and 4 lead to the fact that if $f \in \text{ACL}(Q^n, R^k)$, where $k \geq 2$, then for almost all $x \in R^k$, $f^{-1}(x)$ is disjoint from almost all lines parallel to an arbitrary coordinate axis. Hence $\pi_{x,\varepsilon} \circ f \in \text{ACL}(Q^n, R^k)$ for almost all $x \in R^k$. Now we are in position to prove Lemma 1.

Proof of Lemma 1. – As we have already seen, for almost all $\bar{x} \in A_1 \times \dots \times A_k$, $\pi_{\bar{x},\varepsilon} \circ u \in \text{ACL}(\Omega)$; hence the gradient $D(\pi_{\bar{x},\varepsilon} \circ u)$ is defined almost everywhere and now in order to prove $\pi_{\bar{x},\varepsilon} \circ u \in W^{1,p}(\Omega)$ it suffices to prove that $|D(\pi_{\bar{x},\varepsilon} \circ u)| \in L^p(\Omega)$ (see Theorem 4).

Evidently $\pi_{\bar{x},\varepsilon} \circ u(z) = u(z)$ for $z \in \Omega \setminus \bigcup_{i=1}^k u^{-1}(B^n(x_i, \varepsilon))$ and hence $D(\pi_{\bar{x},\varepsilon} \circ u)(z) = Du(z)$ for almost all $z \in \Omega \setminus \bigcup_{i=1}^k u^{-1}(B^n(x_i, \varepsilon))$. Now it is clear that the lemma will follow if we prove the inequality

$$\begin{aligned} & \int_{A_k} \dots \int_{A_1} \int_{u^{-1}\left(\bigcup_{i=1}^k B^n(x_i, \varepsilon)\right)} |D(\pi_{\bar{x},\varepsilon} \circ u)(z)|^p dz dx_1 \dots dx_k \\ & \leq C(n, p) |A_1| \dots |A_k| \int_{\Omega} |Du(z)|^p dz. \end{aligned} \quad (2)$$

Note that $\pi_{x_j, \varepsilon}|_{B^n(x_i, \varepsilon)} = \text{id}$ for $j \neq i$ and hence for almost all $z \in u^{-1}(B^n(x_i, \varepsilon))$

$$\begin{aligned} |D(\pi_{\bar{x},\varepsilon} \circ u)(z)| & \leq |D(\pi_{x_i, \varepsilon})(u(z))| |Du(z)| \\ & \leq C(n) \frac{\varepsilon}{|u(z) - x_i|} |Du(z)|. \end{aligned}$$

Denoting the left hand side of (2) by I we have

$$\begin{aligned} I &\leq C(n, p) \sum_{i=1}^k \int_{A_k} \\ &\quad \dots \int_{A_1} \int_{u^{-1}(B^n(x_i, \varepsilon))} |Du(z)|^p \left(\frac{\varepsilon}{|u(z) - x_i|} \right)^p dz dx_1 \dots dx_k \\ &\leq C(n, p) \sum_{i=1}^k \int_{u^{-1}(\mathcal{O}_\varepsilon(A_i))} |Du(z)|^p (|A_1| \dots |A_k| (\varepsilon^p |A_i|^{-p/n})) dz \\ &\leq C(n, p) |A_1| \dots |A_k| \int_{\Omega} |Du(z)|^p dz. \end{aligned}$$

In the last but one step we used the estimate $\int_A |x|^{-p} \leq C(n, p) |A|^{1-\frac{p}{n}}$. This ends the proof of (2) and hence that of the lemma.

Proof of Theorem 1. – First we will prove the theorem under the additional assumption that the image of u is bounded i.e. $u(x) \in Q^n$ for almost all $x \in \Omega$, where Q^n is a certain cube. Then, as we will see, the general case will follow from the density of bounded maps in the strong $A_{p,q}$ topology.

Divide the cube Q^n in a standard way, into ν^n identical, small cubes. Denote these cubes by $Q_i^{n,\nu}$, $i = 1, 2, \dots, \nu^n$. Let $\frac{1}{2}Q_i^{n,\nu}$ be a cube with the same centre as $Q_i^{n,\nu}$ and half the edglength. Let $x \in \frac{1}{2}Q_i^{n,\nu}$. We define the retraction

$$\pi_{i,x}^\nu : Q^n \setminus \{x\} \longrightarrow Q^n \setminus Q_i^{n,\nu}$$

as follows

$$\pi_{i,x}^\nu(z) = \begin{cases} z & \text{when } z \in Q^n \setminus Q_i^{n,\nu} \\ (z - x)\theta(x, z) + x & \text{when } z \in Q_i^{n,\nu}, \end{cases}$$

where $\theta(x, z) > 0$ is such that $\pi_{i,x}^\nu(z) \in \partial Q_i^{n,\nu}$. In the other words, inside $Q_i^{n,\nu}$, $\pi_{i,x}^\nu$ is a retraction along radii onto $\partial Q_i^{n,\nu}$, with “source” at x . Evidently $\pi_{i,x}^\nu$ is bilipschitz equivalent with $\pi_{x,\varepsilon}$ type mapping (with $\varepsilon = C\nu^{-1}$), so we can repeat the proof of Lemma 1, replacing A_i by $\frac{1}{2}Q_i^{n,\nu}$. This leads to the following corollary: There exists $x_i \in \frac{1}{2}Q_i^{n,\nu}$ for $i = 1, 2, \dots, \nu^n$, such that

$$u^{(\nu)} = \pi_{1,x_1}^\nu \circ \dots \circ \pi_{\nu^n, x_{\nu^n}}^\nu \circ u \in W^{1,p}(\Omega) \tag{3}$$

and

$$\int_{\Omega} |Du^{(\nu)}(z)|^p dz \leq C(n, p) \int_{\Omega} |Du(z)|^p dz. \tag{4}$$

Evidently the image of $u^{(\nu)}$ is $(n-1)$ -dimensional. Since for $\varphi \in C^1$, $|\text{adj } D(\varphi \circ u)(z)| \leq |D\varphi(u(z))|^{n-1} |\text{adj } Du(z)|$, then it is easy to see repeating arguments from the proof of Lemma 1 that we can choose $x_i \in \frac{1}{2}Q_i^{n,\nu}$ for $i = 1, 2, \dots, \nu^n$ in a way, such that in addition to (4)

$$\int_{\Omega} |\text{adj } Du^{(\nu)}(z)|^q dz \leq C(n, q) \int_{\Omega} |\text{adj } Du(z)|^q dz. \quad (5)$$

Now it suffices to prove that, after extracting a subsequence, $u^{(\nu)} \rightharpoonup u$ in $W^{1,p}$ and $\text{adj } Du^{(\nu)} \rightharpoonup \text{adj } Du$ in L^q . We have $|u(z) - u^{(\nu)}(z)| \leq C\nu^{-1}$ a.e., so $u^{(\nu)} \rightarrow u$ a.e. and hence in L^p (since u and $u^{(\nu)}$ are bounded).

The estimations (4), (5) show that sequences $Du^{(\nu)}$ and $\text{adj } Du^{(\nu)}$ are bounded in L^p and L^q respectively. Hence $u^{(\nu)} \rightharpoonup u$ in $W^{1,p}$ and moreover after taking a subsequence $\text{adj } Du^{(\nu)} \rightharpoonup H$ in L^q to a certain $H \in L^q$. By the theorem of Ball and Reshetnyak [1], [26], [5, Chapter 4, Thm. 2.6], [16], $\text{adj } Du^{(\nu)} \xrightarrow{*} \text{adj } Du$ in measures, and hence $\text{adj } Du = H$. This completes the proof in the case of bounded maps.

Since the constants in the inequalities (4) and (5) do not depend on the size of the cube Q^n , then the theorem will follow if we prove the density of bounded maps in the strong topology of $A_{p,q}$. To this end, let $P_R : R^n \rightarrow [-R, R]^n$ be a retraction with a Lipschitz constant 1; now it is easy to see that $P_R \circ u \xrightarrow{R \rightarrow \infty} u$ in $W^{1,p}$ and $\text{adj } D(P_R \circ u) \xrightarrow{R \rightarrow \infty} \text{adj } Du$ in L^q (in the proof of the second convergence we use an elementary inequality $|\text{adj } D(\varphi \circ u)| \leq |\text{Lip } \varphi|^{n-1} |\text{adj } Du|$). This completes the proof.

Proof of Theorem 3. – By the same reason as in the proof of Theorem 1, we can assume that u is bounded, i.e. $u(x) \in Q^n$ for a.e. x , where Q^n is a fixed cube.

At the beginning we prove the first part of the theorem, i.e. we will be concerned with the weak convergence of $u^{(\nu)}$, without taking care of minors $\text{adj}_s Du^{(\nu)}$.

In the first step, as in the proof of Theorem 1, we compose the mapping u with the discontinuous retraction onto $(n-1)$ -dimensional set which is the union of the boundaries of all cubes $Q_i^{n,\nu}$, $i = 1, 2, \dots, \nu^n$.

This way we obtain (cf. the proof of Theorem 1) a mapping $u_1^{(\nu)} \in W^{1,p}(\Omega, \bigcup_{i=1}^{\nu^n} \partial Q_i^{n,\nu})$ with

$$\int_{\Omega} |Du_1^{(\nu)}|^p dx \leq C(n, p) \int_{\Omega} |Du|^p dx. \quad (6)$$

$(u_1^{(\nu)})$ is the composition of u with a discontinuous retraction described above.) Since $u_1^{(\nu)} \xrightarrow{\nu \rightarrow \infty} u$ a.e., then it follows from (6) that $u_1^{(\nu)} \xrightarrow{\nu \rightarrow \infty} u$ in $W^{1,p}$. If $p \geq n - 1$, then the proof is completed. Hence we can assume that $p < n - 1$. The set $\partial Q_i^{n,\nu}$ consists of $2n$, $(n - 1)$ -dimensional cubes. Now we divide each such $(n - 1)$ -dimensional cube into a family of ν^{n-1} very small cubes (with edglength $C\nu^{-2}$). This leads to the decomposition

$$\partial Q_i^{n,\nu} = \bigcup_{j=1}^{2n\nu^{(n-1)}} Q_{i,j}^{n-1,\nu} \text{ and hence to the decomposition of } \bigcup_{i=1}^{\nu^n} \partial Q_i^{n,\nu}.$$

Now almost the same arguments as in the proof of (6) show that we can compose the mapping $u_1^{(\nu)}$ with the discontinuous retraction from $\bigcup_{i=1}^{\nu^n} \partial Q_i^{n,\nu}$ onto $(n - 2)$ -dimensional set $\bigcup_{i,j} \partial Q_{i,j}^{n-1,\nu}$. This way we obtain the mapping $u_2^{(\nu)} \in W^{1,p}(\Omega, \bigcup_{i,j} \partial Q_{i,j}^{n-1,\nu})$, with

$$\int_{\Omega} |Du_2^{(\nu)}|^p dx \leq C(n, p) \int_{\Omega} |Du_1^{(\nu)}|^p dx \leq C'(n, p) \int_{\Omega} |Du|^p dx.$$

As above $u_2^{(\nu)} \rightarrow u$ in $W^{1,p}$. If $p \geq n - 2$, then the proof is completed. If $p < n - 2$, then we can of course continue this construction and compose $u_2^{(\nu)}$ with retraction onto $(n - 3)$ -dimensional set. By induction, we can continue this construction up to the moment, we compose with retraction onto $[p]$ -dimensional set. This way we get $u_{n-[p]}^{(\nu)} \in W^{1,p}(\Omega)$, with values in $[p]$ -dimensional set. Moreover

$$\int_{\Omega} |Du_{n-[p]}^{(\nu)}|^p dx \leq C(n, p) \int_{\Omega} |Du|^p dx. \tag{7}$$

Since $u_{n-[p]}^{(\nu)} \xrightarrow{\nu \rightarrow \infty} u$ a.e., then by (7), $u_{n-[p]}^{(\nu)} \xrightarrow{\nu \rightarrow \infty} u$ in $W^{1,p}$. This ends the proof of the first part of Theorem 3.

If we know additionally that $\text{adj}_s Du \in L^{q_s}$ for some $1 < q_s < ([p] + 1)/s$, then as in the proof of Theorem 1, we can assume that additionally to (7), the following inequality holds

$$\int_{\Omega} |\text{adj}_s Du_{n-[p]}^{(\nu)}(x)|^{q_s} dx \leq C(n, s, q_s) \int_{\Omega} |\text{adj}_s Du(x)|^{q_s} dx.$$

Hence up to a subsequence, $\text{adj}_s Du_{n-[p]}^{(\nu)} \rightarrow H$ in L^{q_s} to a certain $H \in L^{q_s}$. The inequality $1 < q_s < ([p] + 1)/s$ implies that $p \geq s$, hence by the theorem of Ball-Reshetnyak (cf. the proof of Theorem 1), $\text{adj}_s Du_{n-[p]}^{(\nu)} \xrightarrow{*} \text{adj}_s Du$ in measures and hence $H = \text{adj}_s Du$. This ends the proof.

REFERENCES

- [1] J. BALL, Convexity conditions and existence theorems in nonlinear elasticity, *Arch. Rat. Mech. Anal.*, Vol. **63**, 1977, pp. 337-403.
- [2] F. BETHUEL, The approximation problem for Sobolev maps between two manifolds, *Acta Math.*, Vol. **167**, 1991, pp. 153-206.
- [3] F. BETHUEL and X. ZHENG, Density of smooth functions between two manifolds in Sobolev spaces, *J. Funct. Anal.*, Vol. **80**, 1988, pp. 60-75.
- [4] B. BOJARSKI, Geometric properties of Sobolev mappings, in: Pitman Res. Notes in Math., Vol. **211**, 1989, pp. 225-241.
- [5] B. DACOROGNA, *Direct Methods in the Calculus of Variations*, Springer-Verlag 1989.
- [6] M. ESTEBAN, A direct variational approach to Skyrme's model for meson fields, *Comm. Math. Phys.*, Vol. **105**, 1986, pp. 571-591.
- [7] M. ESTEBAN, A new setting for Skyrme's problem, in: Proc. Colloque problèmes variationnels (Paris, June 1988), Boston-Basel-Stuttgart 1990.
- [8] M. ESTEBAN and S. MÜLLER, Sobolev maps with the integer degree and applications to Skyrme's problem, *Proc. R. Soc. Lond.*, Vol. **436**, 1992, pp. 197-201.
- [9] H. FEDERER, *Geometric Measure Theory*, Springer-Verlag 1969.
- [10] M. GIAQUINTA, G. MODICA and J. SOUČEK, Cartesian currents and variational problems into spheres, *Annali Sc. Norm. Sup. Pisa*, Vol. **16**, 1989, pp. 393-485.
- [11] P. HAJŁASZ, A Sard type theorem for Borel mappings, *Colloq. Math.*, Vol. **67**, 1994, pp. 217-221.
- [12] P. HAJŁASZ, Approximation of Sobolev mappings, *Nonlinear Anal.*, Vol. **22**, 1994, pp. 1579-1591.
- [13] P. HAJŁASZ, Sobolev mappings, co-area formula and related topics (preprint).
- [14] P. HAJŁASZ, A note on approximation of Sobolev maps, (in preparation).
- [15] F. HÉLEIN, Approximation of Sobolev maps between an open set and Euclidean sphere, boundary data, and singularities, *Math. Ann.*, Vol. **285**, 1989, pp. 125-140.
- [16] T. IWANIEC and A. LUTOBORSKI, Integral estimates for null lagrangians, *Arch. Rat. Mech. Anal.*, Vol. **125**, 1993, pp. 25-80.
- [17] N. LUSIN, Sur les ensembles analytiques, *Fund. Math.*, Vol. **10**, 1927, pp. 1-95.
- [18] N. LUSIN and W. SIERPIŃSKI, Sur quelques propriétés des ensembles (A), *Bull. de l'Acad. de Cracovie*, 1918, p. 44.
- [19] J. MALÝ, L^p -approximation of Jacobians, *Comm. Math. Univ. Carolinae*, Vol. **32**, 1991, pp. 659-666.
- [20] V. MAZ'YA, *Sobolev Spaces*, Springer-Verlag 1985.
- [21] S. MÜLLER, Weak continuity of determinants and nonlinear elasticity, *C. R. Acad. Sci. Paris*, Vol. **307**, Série I, 1988, pp. 501-506.
- [22] S. MÜLLER, Higher integrability of determinants and weak convergence in L^1 , *J. Reine Angew. Math.*, Vol. **412**, 1990, pp. 20-34.
- [23] S. MÜLLER, Det= \det . A remark on the distributional determinant, *C. R. Acad. Sci. Paris*, Vol. **313**, 1990, pp. 13-17.
- [24] S. MÜLLER, T. QI and B. S. YAN, On a new class of elastic deformations not allowing for cavitation, *Ann. I.H.P. Anal. Nonl.*, Vol. **11**, 1994, pp. 217-243.
- [25] O. NIKODYM, Sur une classe de fonctions considérées le problème de Dirichlet, *Fund. Math.*, Vol. **21**, 1933, pp. 129-150.
- [26] Y. G. RESHETNYAK, On the stability of conformal mappings in multidimensional space, *Siberian Math. J.*, Vol. **8**, 1967, pp. 65-85.
- [27] W. RUDIN, *Real and Complex Analysis*, Third edition, Mc Graw-Hill, New York 1987.
- [28] R. SCHOEN and K. UHLENBECK, Approximation theorems for Sobolev mappings, (preprint).
- [29] V. ŠVERAK, Regularity properties of deformations with finite energy, *Arch. Rat. Mech. Anal.*, Vol. **100**, 1988, pp. 105-127.

(Manuscript received January 31, 1994;
Revised version received August, 1994.)