ANNALES DE L'I. H. P., SECTION C

G. CITTI

C^∞ regularity of solutions of the Levi equation

Annales de l'I. H. P., section C, tome 15, nº 4 (1998), p. 517-534
<http://www.numdam.org/item?id=AIHPC_1998__15_4_517_0>

Ann. Inst. Henri Poincaré,

Vol. 15, n° 4, 1998, p. 517-534 *Analyse non linéaire*

C^∞ regularity of solutions of the Levi equation

by

G. CITTI

Dipartimento di Matematica, Univ. di Bologna,
P.zza di Porta S. Donato 5, 40127, Bologna, Italy,
E-mail: citti@dm.unibo.it

ABSTRACT. – We will prove the C^∞ regularity of the classical solutions of the equation

$$\mathcal{L}u = q\frac{(1+|gradu|^2)^{3/2}}{1+u_t^2}$$

where

$$\mathcal{L}u = u_{xx} + u_{yy} + 2\frac{u_y - u_x u_t}{1+u_t^2}u_{xt} - 2\frac{u_x + u_y u_t}{1+u_t^2}u_{yt} + \frac{u_x^2+u_y^2}{1+u_t^2}u_{tt},$$

$q \in C^\infty(\Omega)$ and $q(\xi) \neq 0$ for every $\xi \in \Omega$. This is a second order quasilinear equation, whose characteristic form has zero determinant at every point, and for every function u. However we will write it as a sum of squares of nonlinear vector fields, and we will extablish the result by means of a suitable freezing method. © Elsevier, Paris

RÉSUMÉ. – Nous prouvons la régularité C^∞ des solutions classiques de l'équation

$$\mathcal{L}u = q\frac{(1+|gradu|^2)^{3/2}}{1+u_t^2}$$

où

$$\mathcal{L}u = u_{xx} + u_{yy} + 2\frac{u_y - u_x u_t}{1+u_t^2}u_{xt} - 2\frac{u_x + u_y u_t}{1+u_t^2}u_{yt} + \frac{u_x^2+u_y^2}{1+u_t^2}u_{tt},$$

$q \in C^{\infty}(\Omega)$ et $q(\xi) \neq 0$ pour tout $\xi \in \Omega$. Il s'agit d'une équation quasilinéaire du second ordre, dont le déterminant du symbole principal est nul en tout point ξ, et pour toute fonction u. Nous écrivons l'équation comme une somme de carrés de champs de vecteurs, et nous prouvons le résultat en employant une méthode « freesing ».

1. INTRODUCTION

In this note we study the regularity of the solutions of the equation

$$\mathcal{L}u = q\frac{\left(1+|\nabla u|^2\right)^{3/2}}{1+u_t^2} \quad \text{in } \Omega \subset R^3 \tag{1}$$

where

$$\mathcal{L}u = u_{xx} + u_{yy} + 2\frac{u_y - u_x u_t}{1+u_t^2}u_{xt} - 2\frac{u_x + u_y u_t}{1+u_t^2}u_{yt} + \frac{u_x^2+u_y^2}{1+u_t^2}u_{tt}, \tag{2}$$

and $q \in C^{\infty}(\Omega)$. Here we have denoted (x, y, t) a point of R^3, u_x the first derivative with respect to x, and ∇ the euclidean gradient of u. (1) is the Levi equation, and it describes the curvature of a hypersurface in R^4 (*see* for example [9] for some more details on the geometrical meaning of the equation). It is a quasilinear equation, whose characteristic form is semidefinite positive and has least eingenvalue identically 0, for every u and every $(x, y, t) \in \Omega$. Hence elliptic theory does not apply.

When $q = 0$, the following existence and regularity result was extablished by Bedford and Gaveau there exist only two point p_1 and p_2 where the tangent space to the graph of ϕ is a complex line in C^2, and these two point are elliptic. If Ω is pseudoconvex in $\mathbb{C} \times \mathbb{R}$, $\phi \in C^{m+5}(\partial\Omega)$, then the problem

$$\begin{cases} \mathcal{L}(u) = 0 & \text{in } \Omega \\ u = \phi & \text{on } \partial\Omega \end{cases} \tag{3}$$

has a solution in $C^{m+\alpha}(\Omega\backslash\{p_1, p_2\}) \cap Lip(\bar{\Omega})$ (*see* [1]).

More recently Slodkowsky and Tomassini proved that, if Ω is pseudoconvex, and q satisfies a geometric hypothesis related to the Levi curvature of $\partial\Omega \times R$, the Dirichlet problem associated to equation (1), has at least a viscosity solution $u \in Lip(\bar{\Omega})$ (*see* [8]). However nothing was known about the regularity of the solution.

On the other hand the author in [2] studied the simplified equation

$$\mathcal{L}u = q\frac{1+|\nabla u|^2}{1+u_t^2} \quad \text{in } \Omega \subset R^3, \tag{4}$$

when $q \neq 0$ for all $\xi \in \Omega$, and proved that if $\alpha > \frac{1}{2}$ and u is a solution of class $C^{2,\alpha}(\Omega)$ of (4), then it is of class $C^\infty(\Omega)$.

Here we show that the same technique can be adapted to prove that

THEOREM 1.1. – *If* $\alpha > \frac{1}{2}$, $q(\xi) \neq 0$ *for every* $\xi \in \Omega$ *and* u *is a solution of class* $C^{2,\alpha}(\Omega)$ *of* (1) *then* u *is of class* $C^\infty(\Omega)$.

If u is a fixed C^1 function, and

$$X = \begin{pmatrix} 1 \\ 0 \\ \dfrac{u_y - u_x u_t}{1+u_t^2} \end{pmatrix} \quad Y = \begin{pmatrix} 0 \\ 1 \\ -\dfrac{u_x + u_y u_t}{1+u_t^2} \end{pmatrix}, \tag{5}$$

then $\mathcal{L}$ can be formally written

$$\mathcal{L}u = X^2 u + Y^2 u - c(u)\partial_t u, \tag{6}$$

where

$$c(u) = X\Big(\frac{u_y - u_x u_t}{1+u_t^2}\Big) - Y\Big(\frac{u_x + u_y u_t}{1+u_t^2}\Big).$$

Besides

$$[X, Y] = -\frac{\mathcal{L}u}{1+(\partial_t u)^2}\partial_t \tag{7}$$

so that, if u is a solution of (1), and $q(\xi) \neq 0$ $\forall \xi \in \Omega$, then

$$X, \ Y \ \text{and} \ [X, Y] \ \text{are linearly independent at every point.} \tag{8}$$

In term of these vector fields the second member in (4) becomes

$$\mathcal{M}u = q(1 + |Xu| + |Yu|^2) \tag{9}$$

so that it is a quasilinear second order equation, related to the vector fields X and Y. In term of the same vector fields the second member in (1) becomes

$$\mathcal{M}u = q(1 + |Xu| + |Yu|^2)^{3/2}(1 + |\partial_t u|^2)^{1/2}. \tag{10}$$

Since ∂_t is proportional to $[X, Y]$, it has to be considered a second order operator and $\mathcal{M}$ is fully nonlinear. Then equation (1) will be written

$$\mathcal{L}u - \mathcal{M}u = 0, \tag{11}$$

and it will be treated as a second order, fully non linear equation associated to the vector fields X and Y.

Linear operator sum of squares of C^∞ vector fields which satisfies condition (8), have been intensively studied (*see* for example [4], [5], [6]). A very particular non linear problem of the form

$$F(\xi, u, Xu, Yu) = 0$$

has been considered by Xu, but in his case Xu and Yu are linear first order C^∞ vector fields. Here, on the contrary X and Y are non linear vector fields, who have the regularity of the gradient of u, so that we have to adapt to this situation the results in [2].

We first consider a simple non linear operator, which has the same structure as $\mathcal{L}u - \mathcal{M}u$. If a, b, e, f g are continuous functions on an open set Ω, will denote:

$$X = \begin{pmatrix} 1 \\ 0 \\ a \end{pmatrix} \quad \text{and} \quad Y = \begin{pmatrix} 0 \\ 1 \\ b \end{pmatrix}, \tag{12}$$

then we will consider the operators formally defined as

$$Lu = X^2u + Y^2u - (Xa + Yb)\partial_t u, \tag{13}$$

$$Mu = e\partial_t u + f(1 + |\partial_t u|^2)^{1/2}, \tag{14}$$

$$\Lambda u = Lu + Mu$$

and we will study the equation $\Lambda u = g$.

In this setting Xu and Yu are the analogous of the first derivatives for the elliptic operators, and $C^1_\Lambda(\Omega)$ denotes the set of functions such that Xu and Yu are continuous. More generally, if the coefficients of X and Y are of class $C^{k-1}_\Lambda(\Omega)$, a function u is in $C^k_\Lambda(\Omega)$ if Xu and $Yu \in C^{k-1}_\Lambda(\Omega)$. If

$$X,\ Y \text{ and } [X, Y] \text{ are linearly independent at every point} \tag{15}$$

it is possible to introduce a distance d naturally associated to X and Y, (*see* [6] for the definition in the regular case), and it is equivalent to

$$\tilde{d}(\xi,\xi_0) = (((x-x_0)^2 + (y-y_0)^2)^2 + (t - t_0 - a(\xi_0)(x-x_0) - b(\xi_0)(y-y_0))^2)^{1/4},$$

in the sense that there exist two constants C_1 and C_2 such that

$$\forall \xi, \xi_0 \in \Omega \quad C_1 d(\xi,\xi_0) \le \tilde{d}(\xi,\xi_0) \le C_2 d(\xi,\xi_0). \tag{16}$$

A function u is said of class $C^\alpha_\Lambda(\Omega)$ with $0 < \alpha \le 1$ if

$$|u(\xi) - u(\xi_0)| \le C d^\alpha(\xi,\xi_0) \quad \forall \xi, \xi_0 \in \Omega. \tag{17}$$

With these notations we will prove that

THEOREM 1.2. – *Assume that a and b are of class $C^{k+1,\alpha}_{\Lambda,loc}(\Omega)$, e and $f \in C^{k,\alpha}_{\Lambda,loc}(\Omega)$ and* (8) *holds. If u is a solution of class $C^{2,\alpha}_{\Lambda,loc}(\Omega)$ of the equation $Lu + Mu = g \in C^{k,\alpha}_{\Lambda,loc}(\Omega)$, with $\alpha > 1/2$, then $u \in C^{k+2,\alpha}_{\Lambda,loc}(\Omega)$.*

The proof follows the same steps as in [2]: we call frozen operator of Λ the operator whose coefficients are the first order Taylor expansion of the coefficients of Λ, and X_{ξ_0} and Y_{ξ_0} the related vector fields. The operator L_{ξ_0} obtained in this way is a linear hypoelliptic second order operator, which has already been intensively studied. Hence it is possible we write a representation formula for functions of class $C^{2,\alpha}_\Lambda$, in term of its fundamental solution (*see* section 2). In section 3 we differentiate explicitly this formula, and prove the regularity Theorem 1.2. We also get some technical regularity results in the directions X_{ξ_0} and Y_{ξ_0}. Using these theorems, and arguing as in [2] we first deduce that $u \in C^{3,\alpha}_{\Lambda,loc}(\Omega)$ and $\partial_t u \in C^{2,\alpha}_{\Lambda,loc}(\Omega)$ (in proposition 3.1 we will give a short scheme of the proof). Then applying an iterative procedure and theorem 1.2 we deduce that $u \in C^\infty(\Omega)$.

We are deeply indebited with A. Montanari, for bringing to our attention a mistake in a previous version of the paper, and suggesting us how to correct it.

2. REPRESENTATION FORMULA

In the sequel X Y will always be the vector fields introduced in (12), L and M the corresponding operators, and a, b, e and f are of class $C^{1,\alpha}_\Lambda(\Omega)$. Then every $v \in C^{1,\alpha}_\Lambda(\Omega)$ has the following Taylor development:

$$v(\xi) = P^1_{\xi_0} v(\xi) + O(d^{1+\alpha}(\xi,\xi_0)), \tag{18}$$

where

$$P^1_{\xi_0} v(\xi) = v(\xi_0) + Xv(\xi_0)(x - x_0) + Yv(\xi_0)(y - y_0)$$

(*see* [2]). Hence it is natural to call frozen operator of L

$$L_{\xi_0} = X^2_{\xi_0} + Y^2_{\xi_0} - (Xa + Yb)(\xi_0)\partial_t, \tag{19}$$

where

$$X_{\xi_0} = \begin{pmatrix} 1 \\ 0 \\ P^1_{\xi_0} a \end{pmatrix} \quad \text{and} \quad Y_{\xi_0} = \begin{pmatrix} 0 \\ 1 \\ P^1_{\xi_0} b \end{pmatrix}, \tag{20}$$

while the frozen operator of M will be

$$M_{\xi_0} v = \left(e(\xi_0) + \frac{f(\xi_0)\partial_t v(\xi_0)}{(1 + \partial_t v(\xi_0)^2)^{1/2}} \right) \partial_t v. \tag{21}$$

The resulting operator $\Lambda_{\xi_0} = L_{\xi_0} + M_{\xi_0}$ is a linear operator, sum of squares of nihilpotent C^∞ vector fields, hence it is well known how to associate to it a homogeneous nihilpotent Lie group such that X_{ξ_0} and Y_{ξ_0} are left invariant with respect to the traslations, and homogeneous of order 1 with respect to the dilations. In this particular case the canonical change of variables is

$$\begin{aligned} &\phi_{\xi_0}(x, y, t) \\ &= \Bigg(x, y, \frac{2\big(2t - Xa(\xi_0)x^2 - Yb(\xi_0)y^2 - (Xb(\xi_0) + Ya(\xi_0))xy\big) -}{Ya(\xi_0) - Xb(\xi_0)} \\ &\qquad - \frac{4(a(\xi_0) - Ya(\xi_0)y_0 - Xa(\xi_0)x_0)x}{Ya(\xi_0) - Xb(\xi_0)} \\ &\qquad - \frac{4(b(\xi_0) - Xb(\xi_0)x_0 - Xb(\xi_0)y_0)y}{Ya(\xi_0) - Xb(\xi_0)} \Bigg) \end{aligned}$$

and it changes L_{ξ_0} into the Kohn Laplacian on the Heisenberg group.

Consequently for every ξ_0 the control distance naturally associated to Λ_{ξ_0} is explicitly known, and the fundamental solution is equivalent to a power of d. Precisely

$$\Gamma_{\xi_0}(\xi, \zeta) \simeq d_{\xi_0}^{-N+2}(\xi, \zeta),$$

where $N = 4$, and it is the homogeneous dimension of R^3, in the sense that the Lebesgue measure of the sphere of the metric d is

$$|B(\xi, r)| = r^N.$$

Because of the homogeneity of X_{ξ_0} and Y_{ξ_0} we have

$$X_{\xi_0}\Gamma_{\xi_0}(\xi,\zeta) \simeq d_{\xi_0}^{-3}(\xi,\zeta) \quad Y_{\xi_0}\Gamma_{\xi_0}(\xi,\zeta) \simeq d_{\xi_0}^{-3}(\xi,\zeta),$$

while

$$\partial_t\Gamma_{\xi_0}(\xi,\zeta) \simeq d_{\xi_0}^{-4}(\xi,\zeta).$$

Finally an easy computation shows that there exist constants C_0 e C_1 such that

$$C_0 d_{\xi_0}(\xi,\xi_0) \leq \tilde{d}(\xi,\xi_0) \leq C_1 d_{\xi_0}(\xi,\xi_0), \tag{22}$$

where $\tilde{d}(\xi,\xi_0)$ has been defined in the introduction. Hence, because of (16) also $d_{\xi_0}(\xi,\xi_0)$ provides an estimate of $d(\xi,\xi_0)$.

Let us now prove the representation formula, in terms of Γ_{ξ_0}. In order to do so, we fix three open sets Ω, Ω_1 and Ω_2 such that $\Omega_2 \subset\subset \Omega_1 \subset\subset \Omega$, and a function $\phi \in C_0^\infty(\Omega)$ such that $\phi \equiv 1$ in Ω_1, and we study only $v_{|\Omega_2} = v\phi_{|\Omega_2}$.

THEOREM 2.1. – *If $v \in C_\Lambda^{2,\alpha}(\Omega)$, then for every ξ and $\xi_0 \in \Omega_2$ $v(\xi) = v\phi(\xi)$ can be represented in the following way:*

$$\phi v(\xi) = A(\xi,\xi_0) + B(\xi,\xi_0) + \partial_t v(\xi_0)C(\xi,\xi_0) + E(\xi,\xi_0) + \\ + F_A(\xi,\xi_0) + F_B(\xi,\xi_0),$$

where

$$A(\xi,\xi_0) = \int \Gamma_{\xi_0}(\xi,\zeta)\Lambda v(\zeta)\phi(\zeta)d\zeta,$$

$$\begin{aligned} B(\xi,\xi_0) = {} & 2\int \Gamma_{\xi_0}(\xi,\zeta)\big(Xa(\zeta) - Xa(\xi_0)\big)\phi(\zeta)\big(\partial_t v(\zeta) - \partial_t v(\xi_0)\big)d\zeta - \\ & -2\int X_{\xi_0}\Gamma_{\xi_0}(\xi,\zeta)\big(a(\zeta) - P_{\xi_0}^1 a(\zeta)\big)\phi(\zeta)\big(\partial_t v(\zeta) - \partial_t v(\xi_0)\big)d\zeta - \\ & -2(Xb - Ya)(\xi_0)\int (y_\xi - y_\zeta)\partial_t\Gamma_{\xi_0}(\xi,\zeta) \\ & \times\big(a(\zeta) - P_{\xi_0}^1 a(\zeta)\big)\phi(\zeta)\big(\partial_t v(\zeta) - \partial_t v(\xi_0)\big)d\zeta - \\ & -\int \partial_t\Gamma_{\xi_0}(\xi,\zeta)\big(a(\zeta) - P_{\xi_0}^1 a(\zeta)\big)^2\phi(\zeta)\big(\partial_t v(\zeta) - \partial_t v(\xi_0)\big)d\zeta + \\ & +2\int \Gamma_{\xi_0}(\xi,\zeta)\big(Yb(\zeta) - Yb(\xi_0)\big)\phi(\zeta)\big(\partial_t v(\zeta) - \partial_t v(\xi_0)\big)d\zeta - \end{aligned}$$

$$-2\int Y_{\xi_0}\Gamma_{\xi_0}(\xi,\zeta)\big(b(\zeta)-P^1_{\xi_0}b(\zeta)\big)\phi(\zeta)\big(\partial_t v(\zeta)-\partial_t v(\xi_0)\big)d\zeta-$$

$$-2(Ya-Xb)(\xi_0)\int(x_\xi-x_\zeta)\partial_t\Gamma_{\xi_0}(\xi,\zeta)$$

$$\times\big(b(\zeta)-P^1_{\xi_0}b(\zeta)\big)\phi(\zeta)\big(\partial_t v(\zeta)-\partial_t v(\xi_0)\big)d\zeta-$$

$$-\int\partial_t\Gamma_{\xi_0}(\xi,\zeta)\big(b(\zeta)-P^1_{\xi_0}b(\zeta)\big)^2\phi(\zeta)\big(\partial_t v(\zeta)-\partial_t v(\xi_0)\big)d\zeta,$$

$$C(\xi,\xi_0)=-2\int\Gamma_{\xi_0}(\xi,\zeta)\big(a(\zeta)-P^1_{\xi_0}a(\zeta)\big)X_{\xi_0}\phi(\zeta)d\zeta-$$

$$-2\int\Gamma_{\xi_0}(\xi,\zeta)\big(b(\zeta)-P^1_{\xi_0}b(\zeta)\big)Y_{\xi_0}\phi(\zeta)d\zeta-$$

$$-\int\Gamma_{\xi_0}(\xi,\zeta)\big(a(\zeta)-P^1_{\xi_0}a(\zeta)\big)^2\partial_t\phi(\zeta)d\zeta$$

$$-\int\Gamma_{\xi_0}(\xi,\zeta)\big(b(\zeta)-P^1_{\xi_0}b(\zeta)\big)^2\partial_t\phi(\zeta)d\zeta,$$

$$E(\xi,\xi_0)=2\int\Gamma_{\xi_0}(\xi,\zeta)Xv(\zeta)X_{\xi_0}\phi(\zeta)d\zeta+$$

$$+2\int\Gamma_{\xi_0}(\xi,\zeta)Yv(\zeta)Y_{\xi_0}\phi(\zeta)d\zeta+\int\Gamma_{\xi_0}(\xi,\zeta)v(\zeta)L_{\xi_0}\phi(\zeta)d\zeta+$$

$$+\int\Gamma_{\xi_0}(\xi,\zeta)\big(a(\zeta)-P^1_{\xi_0}a(\zeta)\big)^2\partial_t v(\zeta)\partial_t\phi(\zeta)d\zeta+$$

$$+\int\Gamma_{\xi_0}(\xi,\zeta)\big(b(\zeta)-P^1_{\xi_0}b(\zeta)\big)^2\partial_t v(\zeta)\partial_t\phi(\zeta)d\zeta,$$

$$F_A(\xi,\xi_0)=-\partial_t v(\xi_0)\int\Gamma_{\xi_0}(\xi,\zeta)\big(e(\zeta)-e(\xi_0)\big)\phi(\zeta)d\zeta-$$

$$-\frac{1}{(1+\partial_t v(\xi_0)^2)^{1/2}}\int\Gamma_{\xi_0}(\xi,\zeta)f(\zeta)\phi(\zeta)d\zeta-$$

$$-\frac{\partial_t v(\xi_0)^2}{(1+\partial_t v(\xi_0)^2)^{1/2}}\int\Gamma_{\xi_0}(\xi,\zeta)\big(f(\zeta)-f(\xi_0)\big)\phi(\zeta)d\zeta+$$

$$+\left(e(\xi_0)+f(\xi_0)\frac{\partial_t v(\xi_0)}{(1+\partial_t v(\xi_0)^2)^{1/2}}\right)\int\Gamma_{\xi_0}(\xi,\zeta)v(\zeta)\partial_t\phi(\zeta)d\zeta,$$

$$F_B(\xi,\xi_0)=-\int\Gamma_{\xi_0}(\xi,\zeta)\big(e(\zeta)-e(\xi_0)\big)(\partial_t v(\zeta)-\partial_t v(\xi_0))\phi(\zeta)d\zeta-$$

$$-\int\Gamma_{\xi_0}(\xi,\zeta)f(\zeta)\Big((1+\partial_t v(\zeta)^2)^{1/2}-(1+\partial_t v(\xi_0)^2)^{1/2}-$$

$$-\frac{\partial_t v(\xi_0)}{(1+\partial_t v(\xi_0)^2)^{1/2}}(\partial_t v(\zeta)-\partial_t v(\xi_0))\Big)\phi(\zeta)d\zeta-$$

$$-\frac{\partial_t v(\xi_0)}{(1+\partial_t v(\xi_0)^2)^{1/2}}\int \Gamma_{\xi_0}(\xi,\zeta)(f(\zeta)-f(\xi_0))(\partial_t v(\zeta)-\partial_t v(\xi_0))\phi(\zeta)d\zeta.$$

Proof of Theorem. 2.1. – Due to a density argument we only have to prove the Theorem for smooth v, a, b, e and f. By definition of fundamental solution

$$v\phi(\xi)=\int \Gamma_{\xi_0}(\xi,\zeta)\Lambda_{\xi_0}(v\phi)(\zeta)d\zeta=$$
$$=\int \Gamma_{\xi_0}(\xi,\zeta)\Lambda v(\zeta)\phi(\zeta)d\zeta-$$
$$-\int \Gamma_{\xi_0}(\xi,\zeta)\big(Lv(\zeta)\phi(\zeta)-L_{\xi_0}(v\phi)(\zeta)\big)d\zeta-$$
$$-\int \Gamma_{\xi_0}(\xi,\zeta)\big(Mv(\zeta)\phi(\zeta)-M_{\xi_0}(v\phi)(\zeta)\big)d\zeta. \tag{23}$$

The second term has been already studied in [2], where we proved that

$$-\int \Gamma_{\xi_0}(\xi,\zeta)\big(Lv(\zeta)\phi(\zeta)-L_{\xi_0}(v\phi)(\zeta)\big)d\zeta=$$
$$=B(\xi,\xi_0)+\partial_t v(\xi_0)C(\xi,\xi_0)+E(\xi,\xi_0).$$

Since $\phi\equiv 1$ in a neighborhood of ξ_0, then $M_{\xi_0}(v\phi)=\big(e(\xi_0)+\frac{f(\xi_0)\partial_t v(\xi_0)}{(1+\partial_t v(\xi_0)^2)^{1/2}}\big)(\partial_t v\phi+\partial_t\phi v)$. Hence the third term can be evaluated as follows:

$$-\int \Gamma_{\xi_0}(\xi,\zeta)\big(Mv\phi(\zeta)-M_{\xi_0}(v\phi)(\zeta)\big)d\zeta=$$
$$=-\int \Gamma_{\xi_0}(\xi,\zeta)\Big(e(\zeta)-e(\xi_0)\Big)\partial_t v(\zeta)\phi(\zeta)d\zeta-$$
$$-\int \Gamma_{\xi_0}(\xi,\zeta)\Big(f(\zeta)(1+\partial_t v(\zeta)^2)^{1/2}-$$
$$-f(\xi_0)\frac{\partial_t v(\xi_0)}{(1+\partial_t v(\zeta)^2)^{1/2}}\partial_t v(\zeta)\Big)\phi(\zeta)d\zeta+$$
$$+\Big(e(\xi_0)+f(\xi_0)\frac{\partial_t v(\xi_0)}{(1+\partial_t v(\xi_0)^2)^{1/2}}\Big)\int \Gamma_{\xi_0}(\xi,\zeta)v(\zeta)\partial_t\phi(\zeta)d\zeta=$$
$$=-\int \Gamma_{\xi_0}(\xi,\zeta)\big(e(\zeta)-e(\xi_0)\big)(\partial_t v(\zeta)-\partial_t v(\xi_0))\phi(\zeta)d\zeta-$$

$$-\partial_t v(\xi_0)\int \Gamma_{\xi_0}(\xi,\zeta)\big(e(\zeta)-e(\xi_0)\big)\phi(\zeta)d\zeta-$$

$$-\int \Gamma_{\xi_0}(\xi,\zeta)f(\zeta)\Big((1+\partial_t v(\zeta)^2)^{1/2}-(1+\partial_t v(\xi_0)^2)^{1/2}-$$

$$-\frac{\partial_t v(\xi_0)}{(1+\partial_t v(\xi_0)^2)^{1/2}}(\partial_t v(\zeta)-\partial_t v(\xi_0))\Big)\phi(\zeta)d\zeta-$$

$$-\int \Gamma_{\xi_0}(\xi,\zeta)f(\zeta)\Big((1+\partial_t v(\xi_0)^2)^{1/2}-\frac{\partial_t v(\xi_0)^2}{(1+\partial_t v(\xi_0)^2)^{1/2}}\Big)\phi(\zeta)d\zeta-$$

$$-\int \Gamma_{\xi_0}(\xi,\zeta)\big(f(\zeta)-f(\xi_0)\big)\frac{\partial_t v(\xi_0)}{(1+\partial_t v(\xi_0)^2)^{1/2}}\partial_t v(\zeta)\phi(\zeta)d\zeta+$$

$$+\Big(e(\xi_0)+f(\xi_0)\frac{\partial_t v(\xi_0)}{(1+\partial_t v(\xi_0)^2)^{1/2}}\Big)\int \Gamma_{\xi_0}(\xi,\zeta)v(\zeta)\partial_t\phi(\zeta)d\zeta=$$

$$=-\int \Gamma_{\xi_0}(\xi,\zeta)\big(e(\zeta)-e(\xi_0)\big)(\partial_t v(\zeta)-\partial_t v(\xi_0))\phi(\zeta)d\zeta-$$

$$-\partial_t v(\xi_0)\int \Gamma_{\xi_0}(\xi,\zeta)\big(e(\zeta)-e(\xi_0)\big)\phi(\zeta)d\zeta-$$

$$-\int \Gamma_{\xi_0}(\xi,\zeta)f(\zeta)\Big((1+\partial_t v(\zeta)^2)^{1/2}-(1+\partial_t v(\xi_0)^2)^{1/2}-$$

$$-\frac{\partial_t v(\xi_0)}{(1+\partial_t v(\xi_0)^2)^{1/2}}(\partial_t v(\zeta)-\partial_t v(\xi_0))\Big)\phi(\zeta)d\zeta-$$

$$-\frac{1}{(1+\partial_t v(\xi_0)^2)^{1/2}}\int \Gamma_{\xi_0}(\xi,\zeta)f(\zeta)\phi(\zeta)d\zeta-$$

$$-\frac{\partial_t v(\xi_0)}{(1+\partial_t v(\xi_0)^2)^{1/2}}\int \Gamma_{\xi_0}(\xi,\zeta)\big(f(\zeta)-f(\xi_0)\big)(\partial_t v(\zeta)-\partial_t v(\xi_0))\phi(\zeta)d\zeta-$$

$$-\frac{\partial_t v(\xi_0)^2}{(1+\partial_t v(\xi_0)^2)^{1/2}}\int \Gamma_{\xi_0}(\xi,\zeta)\big(f(\zeta)-f(\xi_0)\big)\phi(\zeta)d\zeta+$$

$$+\Big(e(\xi_0)+f(\xi_0)\frac{\partial_t v(\xi_0)}{(1+\partial_t v(\xi_0)^2)^{1/2}}\Big)\int \Gamma_{\xi_0}(\xi,\zeta)v(\zeta)\partial_t\phi(\zeta)d\zeta.$$

THEOREM 2.2. – *Assume that* $v\in C^{2,\alpha}_\Lambda(\Omega)$ *and* $\partial_t v\in C^{1,\alpha}_\Lambda(\Omega)$. *Then for every* $\xi,\xi_0\in\Omega_2$ *we have:*

$$\phi v(\xi)=A_1(\xi,\xi_0)+B_1(\xi,\xi_0)+\partial_t v(\xi_0)C(\xi,\xi_0)+D_1(\xi,\xi_0)+E_1(\xi,\xi_0)+$$

$$+F_{1,A}(\xi,\xi_0)+F_{1,B}(\xi,\xi_0),$$

where we have denoted

$$A_1(\xi,\xi_0) = \int \Gamma_{\xi_0}(\xi,\zeta)\big(\Lambda v(\zeta) - 2aX\partial_t v(\xi_0) - 2bY\partial_t v(\xi_0)\big)\phi(\zeta)d\zeta,$$

$$B_1(\xi,\xi_0) = 2\int \Gamma_{\xi_0}(\xi,\zeta)\big(Xa(\zeta) - Xa(\xi_0)\big)\phi(\zeta)\big(\partial_t v(\zeta) - P^1_{\xi_0}\partial_t v(\zeta)\big)d\zeta -$$

$$-2\int X_{\xi_0}\Gamma_{\xi_0}(\xi,\zeta)\big(a(\zeta) - P^1_{\xi_0}a(\zeta)\big)\phi(\zeta)\big(\partial_t v(\zeta) - P^1_{\xi_0}\partial_t v(\zeta)\big)d\zeta -$$

$$-2(Xb - Ya)(\xi_0)\int (y_\xi - y_\zeta)\partial_t\Gamma_{\xi_0}(\xi,\zeta)$$

$$\times\big(a(\zeta) - P^1_{\xi_0}a(\zeta)\big)\phi(\zeta)\big(\partial_t v(\zeta) - P^1_{\xi_0}\partial_t v(\zeta)\big)(\zeta)d\zeta -$$

$$-\int \partial_t\Gamma_{\xi_0}(\xi,\zeta)\big(a(\zeta) - P^1_{\xi_0}a(\zeta)\big)^2\phi(\zeta)\big(\partial_t v(\zeta) - P^1_{\xi_0}\partial_t v(\zeta)\big)d\zeta +$$

$$+2\int \Gamma_{\xi_0}(\xi,\zeta)\big(Yb(\zeta) - Yb(\xi_0)\big)\phi(\zeta)\big(\partial_t v(\zeta) - P^1_{\xi_0}\partial_t v(\zeta)\big)d\zeta -$$

$$-2\int Y_{\xi_0}\Gamma_{\xi_0}(\xi,\zeta)\big(b(\zeta) - P^1_{\xi_0}b(\zeta)\big)\phi(\zeta)\big(\partial_t v(\zeta) - P^1_{\xi_0}\partial_t v(\zeta)\big)d\zeta -$$

$$-2(Ya - Xb)(\xi_0)\int (x_\xi - x_\zeta)\partial_t\Gamma_{\xi_0}(\xi,\zeta)$$

$$\times\big(b(\zeta) - P^1_{\xi_0}b(\zeta)\big)\phi(\zeta)\big(\partial_t v(\zeta) - P^1_{\xi_0}\partial_t v(\zeta)\big)d\zeta -$$

$$-\int \partial_t\Gamma_{\xi_0}(\xi,\zeta)\big(b(\zeta) - P^1_{\xi_0}b(\zeta)\big)\phi(\zeta)\big(\partial_t v(\zeta) - P^1_{\xi_0}\partial_t v(\zeta)\big)d\zeta,$$

$$C(\xi,\xi_0) = -2\int \Gamma_{\xi_0}(\xi,\zeta)\big(a(\zeta) - P^1_{\xi_0}a(\zeta)\big)X_{\xi_0}\phi(\zeta)d\zeta -$$

$$-2\int \Gamma_{\xi_0}(\xi,\zeta)\big(b(\zeta) - P^1_{\xi_0}b(\zeta)\big)Y_{\xi_0}\phi(\zeta)d\zeta -$$

$$-\int \Gamma_{\xi_0}(\xi,\zeta)\big(a(\zeta) - P^1_{\xi_0}a(\zeta)\big)^2\partial_t\phi(\zeta)d\zeta$$

$$-\int \Gamma_{\xi_0}(\xi,\zeta)\big(b(\zeta) - P^1_{\xi_0}b(\zeta)\big)^2\partial_t\phi(\zeta)d\zeta,$$

$$D_1(\xi,\xi_0) = t(x - x_0)X\partial_t v(\xi_0) - t(y - y_0)Y\partial_t v(\xi_0),$$

$$E_1(\xi,\xi_0) = 2\int \Gamma_{\xi_0}(\xi,\zeta)$$

$$\times\big(Xv(\zeta) - tX\partial_t v(\xi_0) - aX\partial_t v(\xi_0)(x - x_0)\big)X_{\xi_0}\phi(\zeta)d\zeta$$

$$+2\int \Gamma_{\xi_0}(\xi,\zeta)\big(Yv(\zeta)-tY\partial_t v(\xi_0)-bY\partial_t v(\xi_0)(y-y_0)\big)Y_{\xi_0}\phi(\zeta)d\zeta+$$

$$+\int \Gamma_{\xi_0}(\xi,\zeta)\big(v(\zeta)-X\partial_t v(\xi_0)t(x-x_0)-Y\partial_t v(\xi_0)t(y-y_0)\big)L_{\xi_0}\phi(\zeta)d\zeta+$$

$$+\int \Gamma_{\xi_0}(\xi,\zeta)\big(a(\zeta)-P^1_{\xi_0}a(\zeta)\big)^2$$

$$\times\Big(\partial_t v(\zeta)-X\partial_v(\xi_0)t(x-x_0)-Y\partial_t v(\xi_0)(y-y_0)\Big)\partial_t\phi(\zeta)d\zeta+$$

$$+\int \Gamma_{\xi_0}(\xi,\zeta)\big(b(\zeta)-P^1_{\xi_0}b(\zeta)\big)^2$$

$$\times\Big(\partial_t v(\zeta)-X\partial_v(\xi_0)t(x-x_0)-Y\partial_t v(\xi_0)(y-y_0)\Big)\partial_t\phi(\zeta)d\zeta,$$

$$F_{1,A}(\xi,\xi_0)=-\partial_t v(\xi_0)\int \Gamma_{\xi_0}(\xi,\zeta)\big(e(\zeta)-e(\xi_0)\big)\phi(\zeta)d\zeta-$$

$$-\frac{1}{(1+\partial_t v(\xi_0)^2)^{1/2}}\int \Gamma_{\xi_0}(\xi,\zeta)f(\zeta)\phi(\zeta)d\zeta-$$

$$-\frac{\partial_t v(\xi_0)^2}{(1+\partial_t v(\xi_0)^2)^{1/2}}\int \Gamma_{\xi_0}(\xi,\zeta)\big(f(\zeta)-f(\xi_0)\big)\phi(\zeta)d\zeta+$$

$$+\left(e(\xi_0)+f(\xi_0)\frac{\partial_t v(\xi_0)}{(1+\partial_t v(\xi_0)^2)^{1/2}}\right)\int \Gamma_{\xi_0}(\xi,\zeta)v(\zeta)\partial_t\phi(\zeta)d\zeta-$$

$$-\left(e(\xi_0)+f(\xi_0)\frac{\partial_t v(\xi_0)}{(1+\partial_t v(\xi_0)^2)^{1/2}}\right)\int \Gamma_{\xi_0}(\xi,\zeta)$$

$$\times\big(t(X\partial_t v(\xi_0)(x-x_0)+Y\partial_t v(\xi_0)(y-y_0))\big)\partial_t\phi(\zeta)d\zeta,$$

$$F_{1,B}(\xi,\xi_0)=-\int \Gamma_{\xi_0}(\xi,\zeta)\big(e(\zeta)-e(\xi_0)\big)(\partial_t v(\zeta)-P^1_{\xi_0}\partial_t v(\zeta))\phi(\zeta)d\zeta-$$

$$-\int \Gamma_{\xi_0}(\xi,\zeta)f(\zeta)\Big((1+\partial_t v(\zeta)^2)^{1/2}-(1+\partial_t v(\xi_0)^2)^{1/2}-$$

$$-\frac{\partial_t v(\xi_0)}{(1+\partial_t v(\xi_0)^2)^{1/2}}(\partial_t v(\zeta)-P^1_{\xi_0}\partial_t v(\zeta))\Big)\phi(\zeta)d\zeta-$$

$$-\frac{\partial_t v(\xi_0)}{(1+\partial_t v(\xi_0)^2)^{1/2}}\int \Gamma_{\xi_0}(\xi,\zeta)\big(f(\zeta)-f(\xi_0)\big)(\partial_t v(\zeta)-P^1_{\xi_0}\partial_t v(\zeta))\phi(\zeta)d\zeta+$$

$$+\int \Gamma_{\xi_0}(\xi,\zeta)e(\zeta)\Big(X\partial_t v(\xi_0)(x-x_0)-Y\partial_t v(\xi_0)(y-y_0)\Big)\phi(\zeta)d\zeta.$$

Proof. – The assertion can be proved applying Theorem 2.1 to

$$w(\zeta) = v(\zeta) - t(X\partial_t v(\xi_0)(x - x_0) - Y\partial_t v(\xi_0)(y - y_0)),$$

and using the fact that

$$Xw(\zeta) = Xv(\zeta) - tX\partial_t v(\xi_0) - aX\partial_t v(\xi_0)(x - x_0),$$

$$Yw(\zeta) = Yv(\zeta) - tY\partial_t v(\xi_0) - aY\partial_t v(\xi_0)(y - y_0),$$

$$\partial_t w(\xi_0) = \partial_t v(\xi_0),$$

$$\partial_t w(\zeta) - \partial_t w(\xi_0) = \partial_t v(\zeta) - P^1_{\xi_0}\partial_t v(\zeta).$$

All term follow straitforward, but A and F_B. Hence we will now consider the sum of A and the second term in F_B applied to w, which we will denote T_F. Note that

$$Lw(\zeta) = Lv(\zeta) - 2a\partial_t v(\xi_0) - bY\partial_t v(\xi_0),$$

hence

$$A + T_F = \int \Gamma_{\xi_0}(\xi, \zeta)\Lambda w(\zeta)\phi(\zeta)d\zeta -$$

$$- \int \Gamma_{\xi_0}(\xi, \zeta)f(\zeta)\Big((1 + \partial_t w(\zeta)^2)^{1/2} - (1 + \partial_t w(\xi_0)^2)^{1/2} -$$

$$- \frac{\partial_t w(\xi_0)}{(1 + \partial_t w(\xi_0)^2)^{1/2}}(\partial_t w(\zeta) - \partial_t w(\xi_0))\Big)\phi(\zeta)d\zeta =$$

$$= \int \Gamma_{\xi_0}(\xi, \zeta)\big(Lv(\zeta) - 2aX\partial_t v(\xi_0) - 2bY\partial_t v(\xi_0) + Mw(\zeta)\big)\phi(\zeta)d\zeta -$$

$$- \int \Gamma_{\xi_0}(\xi, \zeta)f(\zeta)\Big((1 + \partial_t w(\zeta)^2)^{1/2} - (1 + \partial_t v(\xi_0)^2)^{1/2} -$$

$$- \frac{\partial_t v(\xi_0)}{(1 + \partial_t v(\xi_0)^2)^{1/2}}(\partial_t v(\zeta) - P^1_{\xi_0}\partial_t v(\zeta))\Big)\phi(\zeta)d\zeta =$$

(adding and subtracting Mv in the first term, and $f(\zeta)(1 + \partial_t v(\zeta)^2)^{1/2}$ to the second)

$$= \int \Gamma_{\xi_0}(\xi, \zeta)\big(Lv(\zeta) - 2aX\partial_t v(\xi_0) - 2bY\partial_t v(\xi_0) + Mv(\zeta)\big)\phi(\zeta)d\zeta +$$

$$+ \int \Gamma_{\xi_0}(\xi, \zeta)\big(Mw(\zeta) - Mv(\zeta)\big)\phi(\zeta)d\zeta -$$

$$-\int \Gamma_{\xi_0}(\xi,\zeta)f(\zeta)\Big((1+\partial_t v(\zeta)^2)^{1/2}-(1+\partial_t v(\xi_0)^2)^{1/2}-$$

$$-\frac{\partial_t v(\xi_0)}{(1+\partial_t v(\xi_0)^2)^{1/2}}(\partial_t v(\zeta)-P^1_{\xi_0}\partial_t v(\zeta))\Big)\phi(\zeta)d\zeta-$$

$$-\int \Gamma_{\xi_0}(\xi,\zeta)f(\zeta)\Big((1+\partial_t w(\zeta)^2)^{1/2}-(1+\partial_t v(\zeta)^2)^{1/2}\Big)\phi(\zeta)d\zeta.$$

From this relation the expression of A_1 and $F_{B,1}$ follow, since $Mw(\zeta) - Mv(\zeta) - (1+\partial_t w(\zeta)^2)^{1/2} + (1+\partial_t v(\zeta)^2)^{1/2} = e(\zeta)\Big(\partial_t w(\zeta) - \partial_t v(\zeta)\Big) = e(\zeta)\Big(X\partial_t v(\xi_0)(x-x_0) - Y\partial_t v(\xi_0)(y-y_0)\Big)$.

3. REGULARITY RESULTS

In this section we prove Theorems 1.1 and 1.2. First of all we state without proof two regularity results, which can be proved differentiating the representation formulas just stated. The proof of a similar assertion was given in [2], where, however, there is a small mistake on pp. 508-509. The proof of theorems 3.1 and 3.2 can be found in the P.h.d. thesis of A. Montanari and in [3], where analogous Theorems are stated in a more general setting. For simplicity we will denote $D_{\xi_0,1} = X_{\xi_0}$ and $D_{\xi_0,2} = Y_{\xi_0}$ where X_{ξ_0} and Y_{ξ_0} are defined in (20).

THEOREM 3.2. – *Assume that are a, b, e, f and g of class $C^{1,\alpha}_{\Lambda}(\Omega)$ and partially differentiable with respect to t, with derivatives of class $C^{\alpha}_{\Lambda}(\Omega)$. Let v be a solution of $\Lambda v = g$, such that $v \in C^{2,\alpha}_{\Lambda}(\Omega)$ and $\partial_t v \in C^{1,\alpha}_{\Lambda}(\Omega)$. Then $\partial_t v \in C^{2,\alpha}_{\Lambda,loc}(\Omega)$.*

THEOREM 3.3. – *Assume that a, b, e, f and g are of class $C^{1,\alpha}_{\Lambda}(\Omega)$ and let v be a solution of $\Lambda v = g$ such that $v \in C^{2,\alpha}_{\Lambda}(\Omega)$, and $D^2_{\xi_0,i,j}v$ is well defined in Ω. Then $D^2_{\xi_0,i,j}v$ is differentiable with respect to X and Y at the point ξ_0 and the functions $\xi_0 \to XD^2_{\xi_0,i,j}v(\xi_0)$ and $\xi_0 \to YD^2_{\xi_0,i,j}v(\xi_0)$ are of class $C^{\alpha}_{\Lambda,loc}(\Omega)$.*

Proof of Theorem 1.2. – The proof of an analogous theorem can be found in [3], and it is obtained by differentiating the representation formula proved in theorem 2.1.

Let us go back to the study of equation (8). If u is a fixed functions of class C^1, according to (9) we define

$$a = \frac{u_y - u_x u_t}{1+u_t^2} \quad \text{and} \quad b = -\frac{u_x + u_y u_t}{1+u_t^2}.$$

Then the vector fields $X = X_u$ and $Y = Y_u$ defined in (12) are continuous, and it is possible to apply the theory we have just developed to the operator

$$L_u v - M_u v,$$

where

$$L_u v = X^2 v + Y^2 v - (Xa + Yb)\partial_t v$$

and

$$M_u v = q(1 + |Xu|^2 + |Yu|^2)^{3/2}(1 + |\partial_t v|^2)^{1/2}.$$

In particular we will denote $C_\Lambda^{k,\alpha}$ the Lipschitz classes associated to $L_u + M_u$. In the sequel we will also assume that u is of class $C^{2,\alpha}$ and it is a solution of the Levi equation, *i.e.* $L_u u - M_u u = 0$. A direct computation shows that

$$Yu = a \quad \text{and} \quad Xu = -b,$$

so that in this hypothesis a, b and the coefficient of M_u, $f = q(1+|Xu|^2 + |Yu|^2)^{3/2}$ are of class $C_\Lambda^{1,\alpha}(\Omega)$.

PROPOSITION 3.1. – *If* $q \in C^\infty(\Omega)$, $q(\xi) \neq 0$ *for all* $\xi \in \Omega$, $\alpha > \frac{1}{2}$ *and* $u \in C^{2,\alpha}(\Omega)$ *is a solution of* (11), *then* $u \in C^{3,\alpha}_{\Lambda,loc}(\Omega)$.

Proof. – The proof is similar to the one in [2], so that we only explain to adapt it to the present context. As we have already noticed, a, b and f are in $C_\Lambda^{1,\alpha}(\Omega)$. Moreover

$$\partial_t u \in C^{1,\alpha}_{\Lambda,loc}(\Omega)$$

$$\xi_0 \to \partial_t X_{\xi_0} u(\xi_0) \in C^\alpha_{\Lambda,loc}(\Omega) \quad \text{and} \quad \xi_0 \to \partial_t Y_{\xi_0} u(\xi_0) \in C^\alpha_{\Lambda,loc}(\Omega).$$

From this last assertion and Lemma 7.1 in [2] it follows that

$$\exists \partial_t X u \in C^\alpha_{\Lambda,loc}(\Omega) \quad \text{and} \quad \exists \partial_t Y u \in C^\alpha_{\Lambda,loc}(\Omega)$$

This means that $a = Yu$, $b = -Xu$ and f are differentiable with respect to t, with derivative of class $C^\alpha_{\Lambda,loc}(\Omega)$. Hence, from Theorem 3.2 we get

$$\partial_t u \in C^{2,\alpha}_{\Lambda,loc}(\Omega). \tag{24}$$

This, together with the assumption that $u \in C^{2,\alpha}_{\Lambda,loc}(\Omega)$ implies that the function $D_{\xi_0,i,j} u$ are well defined in Ω. Hence, by theorem 3.3 the functions

$\xi_0 \to XD^2_{\xi_0,i,j}v(\xi_0)$ and $\xi_0 \to YD^2_{\xi_0,i,j}v(\xi_0)$ are of class $C^{\alpha}_{\Lambda,loc}(\Omega)$. By Lemma 7.2 in [2] this implies that a, $b \in C^{2,\alpha}_{\Lambda,loc}(\Omega)$, so that

$$u \in C^{3,\alpha}_{\Lambda,loc}(\Omega), \tag{25}$$

and $f \in C^{2,\alpha}_{\Lambda,loc}(\Omega)$.

Now we will prove theorem 1.1:

Proof. – By Proposition 3.1 we can assume that $u \in C^{3,\alpha}_{\Lambda,loc}(\Omega)$, and $\partial_t u \in C^{2,\alpha}_{\Lambda,loc}(\Omega)$ so that we can differentiate both members of equation (11) with respect to X and we get (*see* also the proof of Theorem 7.1 in [2])

$$\begin{aligned} L_u Xu = X\big(q(1+|Xu|^2+|Yu|^2)^{3/2}(1+|\partial_t u|^2)^{1/2}\big)- \\ -(X^2u+Y^2u)\frac{X\partial_t u\partial_t u - Y\partial_t u}{1+(\partial_t u)^2}+ \\ +Y\left(\frac{\partial_t u}{(1+(\partial_t u)^2)^{1/2}}q(1+|Xu|^2+|Yu|^2)^{3/2}\right)- \\ -X\left(\frac{\partial_t u}{(1+(\partial_t u)^2)^{1/2}}q(1+|Xu|^2+|Yu|^2)^{3/2}\right)\partial_t u+ \\ +(XYu-YXu)\frac{X\partial_t u\partial_t u - Y\partial_t u}{1+(\partial_t u)^2}\partial_t u. \end{aligned} \tag{26}$$

Analogously, taking the derivative with respect to Y, we have

$$\begin{aligned} L_u Yu = Y\big(q(1+|Xu|^2+|Yu|^2)^{3/2}(1+|\partial_t u|^2)^{1/2}\big)- \\ -(X^2u+Y^2u)\frac{X\partial_t u + Y\partial_t u\partial_t u}{1+(\partial_t u)^2}- \\ -X\left(\frac{\partial_t u}{(1+(\partial_t u)^2)^{1/2}}q(1+|Xu|^2+|Yu|^2)^{3/2}\right)- \\ -Y\left(\frac{\partial_t u}{(1+(\partial_t u)^2)^{1/2}}q(1+|Xu|^2+|Yu|^2)^{3/2}\right)\partial_t u+ \\ +(XYu-YXu)\frac{X\partial_t u + Y\partial_t u\partial_t u}{1+(\partial_t u)^2}\partial_t u. \end{aligned} \tag{27}$$

Since the second member is of class $C_\Lambda^{1,\alpha}(\Omega)$, then, by theorem 1.2, Xu and Yu are of class $C_\Lambda^{3,\alpha}(\Omega)$, and $u \in_\Lambda^{4,\alpha} (\Omega)$.

We can now differentiate both members of (11) with respect of t, and we get

$$L_u \partial_t u =$$

$$= -2\partial_t Yu X\partial_t u + 2\partial_t Xu Y\partial_t u - \partial_t \big(q(1+|Xu|^2+|Yu|^2)^{3/2}(1+|\partial_t u|^2)^{1/2}\big).$$

Hence if we set

$$e = q(1 + |Xu|^2 + |Yu|^2)^{3/2}(1 + |\partial_t u|^2)^{-1/2}\partial_t u$$

and

$$\Lambda = L_u + e\partial_t,$$

then $\partial_t u$ satisfies the equation

$$\Lambda\partial_t u = -2\partial_t Yu X\partial_t u + 2\partial_t Xu Y\partial_t u -$$

$$-\partial_t\big(q(1 + |Xu|^2 + |Yu|^2)^{3/2}\big)(1 + |\partial_t u|^2)^{1/2}. \tag{28}$$

Since $u \in C^{4,\alpha}_{\Lambda,loc}(\Omega)$ the second member in (28) is of class $C^{1,\alpha}_{\Lambda,loc}(\Omega)$, and the coefficients of the operator are $C^{2,\alpha}_{\Lambda,loc}(\Omega)$ hence $\partial_t u \in C^{3,\alpha}_{\Lambda,loc}(\Omega)$. Thus the second member of (26) and (27) is of class $C^{2,\alpha}_{\Lambda,loc}(\Omega)$, and the coefficients are $C^{3,\alpha}_{\Lambda,loc}(\Omega)$, so that $Xu \in C^{4,\alpha}_{\Lambda,loc}(\Omega)$, $Yu \in C^{4,\alpha}_{\Lambda,loc}(\Omega)$. Hence the second member in (28) is of class $C^{2,\alpha}_{\Lambda,loc}(\Omega)$. This procedure can now be iterated to prove that $u \in C^{k,\alpha}_{\Lambda,loc}(\Omega)$, for every k, and this implies that $u \in C^\infty(\Omega)$.

REFERENCES

[1] E. Bedford and B. Gaveau, Envelopes of holomorphy of certain 2-spheres in C^2, *Am. J. of Math.*, Vol. **105**, 1983, pp. 975-1009.

[2] G. Citti, C^∞ regularity of solutions of a quasilinear equation related to the Levi operator, *Ann. Scuola Norm. Sup. Pisa Cl. Sci* (4), Vol. **23**, 1996, pp. 483-529.

[3] G. Citti and A. Montanari, C^∞ regularity of solutions of an equation of Levi's type in $\mathbb{R}^{2n+1}$, preprint.

[4] G. B. Folland, Subelliptic estimates and function on nilpotent Lie groups, *Arkiv f. Mat.*, Vol. **13**, 1975, pp. 161-207.

[5] G. B. Folland and E. M. Stein, Estimates for the $\bar\partial_b$ Complex and Analysis on the Heisenberg Group, *Comm. Pure Appl. Math.*, Vol. **20**, 1974, pp. 429-522.

[6] A. NAGEL, E. M. STEIN and S. WAINGER, Balls and metrics defined by vector fields I: Basic properties, *Acta Math.*, Vol. **155**, 1985, pp. 103-147.

[7] L. ROTHSCHILD and E. M. STEIN, Hypoelliptic differential operators and nihilpotent Lie groups, *Acta Math.*, Vol. **137**, 1977, pp. 247-320.

[8] Z. SLODKOWSKI and G. TOMASSINI, Weak solutions for the Levi equation and Envelope of Holomorphy, *J. Funct. Anal.*, Vol. **101**, no. 4, 1991, pp. 392-407.

[9] G. TOMASSINI, Geometric Properties of Solutions of the Levi Equation, *Ann. di Mat. Pura e Appl*, Vol. **152**, 1988, pp. 331-344.

[10] C. J. XU, Regularity for quasilinear Second order Subelliptic Equations, *Comm. Pure Appl. Math.,* Vol. **45**, 1992, pp. 77-96.

(Manuscript received November 3, 1995.)