Ann. I. H. Poincaré – AN 19, 4 (2002) 477–504

S0294-1449(01)00082-8/FLA

ASYMPTOTIC BEHAVIOR OF GROUND STATES OF QUASILINEAR ELLIPTIC PROBLEMS WITH TWO VANISHING PARAMETERS

Filippo GAZZOLA [a,1], **James SERRIN** [b]

[a] *Dipartimento di Scienze T.A., via Cavour 84, 15100 Alessandria, Italy*
[b] *School of Mathematics, University of Minnesota, Minneapolis, Minnesota, MN, USA*

Received 27 November 2000

ABSTRACT. – We study the asymptotic behavior of the radially symmetric ground state solution of a quasilinear elliptic equation involving the m-Laplacian. The case of two vanishing parameters is considered: we show that these two parameters have opposite effects on the asymptotic behavior. Moreover the results highlight a suprising phenomenon: different asymptotic are obtained according to whether $n > m^2$ or $n \leqslant m^2$, where n is the dimension of the underlying space. © 2002 Éditions scientifiques et médicales Elsevier SAS

RÉSUMÉ. – Nous étudions le comportement asymptotique de l'état fondamental à symétrie radiale d'une équation elliptique quasilinéaire contenant le m-Laplacien. Le cas de deux paramètres tendant vers 0 est considéré : nous montrons que ces deux paramètres sont en compétition. Les résultats obtenus découvrent un nouveau surprenant phénomène : deux comportements asymptotiques complètement différents sont obtenus suivant une relation entre le paramètre m et la dimension n de l'espace. © 2002 Éditions scientifiques et médicales Elsevier SAS

1. Introduction

Let $\Delta_m u = \operatorname{div}(|\nabla u|^{m-2}\nabla u)$ denote the degenerate m-Laplace operator and consider the quasilinear elliptic equation

$$-\Delta_m u = -\delta u^{m-1} + u^{p-1} \quad \text{in } \mathbb{R}^n, \qquad (P_p^\delta)$$

where $n > m > 1$, $m < p < m^*$, $\delta > 0$ and

$$m^* = \frac{nm}{n-m}.$$

[1] Supported by the Italian MURST project "Metodi Variazionali ed Equazioni Differenziali non Lineari".

By the results in [6,10] (see also [1,4] for earlier results in the case $m = 2$) we know that (P_p^δ) admits a ground state for all p, δ in the given ranges. Here, by a *ground state* we mean a $C^1(\mathbb{R}^n)$ positive distribution solution of (P_p^δ), which tends to zero as $|x| \to \infty$. Since in this paper we only deal with radial solutions of (P_p^δ), from now on by a ground state we shall mean precisely a radial ground state. It is known [14,17] moreover that radial ground states of (P_p^δ) are unique.

Equation (P_p^δ) is of particular interest because of the choice of the power $m - 1$ for the lower order term: if $m = 2$ (i.e. $\Delta_m = \Delta$) this is just the linear case, while for any $m > 1$ the lower order term has the same homogeneity as the differential operator Δ_m, a fact which allows the use of rescaling methods. Moreover, this case is precisely the borderline between compact support and positive ground states, see [7, Section 1.3].

It is our purpose to study the behavior of (radial) ground states of (P_p^δ) as $p \to m^*$, $\delta \to 0$. As far as we are aware, the asymptotic behavior of solutions of (P_p^δ) has been studied previously only for the vanishing parameter $\varepsilon = m^* - p$ and only in the case of *bounded domains*, see [3,8,9,11,15,16] and references therein.

Consider first the case when $\delta = 0$. Then (P_p^δ) becomes

$$-\Delta_m u = u^{p-1} \quad \text{in } \mathbb{R}^n, \tag{P_p^0}$$

which by [13, Theorem 5] admits no ground states (recall $p < m^*$). It is of interest therefore to study the behavior of the ground states u of (P_p^δ) as $\delta \to 0$ and p is fixed: in Theorem 1 below we prove in this case that $u \to 0$ uniformly on $\mathbb{R}^n$ and moreover estimate the rate of convergence. As a side result, the arguments used in the proof of Theorem 1 allow us to show that the corresponding ground states u converge to a Dirac measure concentrated at $x = 0$ when $\delta \to \infty$, see Theorem 9 in Section 4 below.

Next, let $p = m^*$ and $\delta > 0$; then (P_p^δ) becomes

$$-\Delta_m u = -\delta u^{m-1} + u^{m^*-1} \quad \text{in } \mathbb{R}^n, \tag{$P_{m^*}^\delta$}$$

which by the results in [12] again admits no ground states. Thus we next study the behavior of ground states u of (P_p^δ) as $\varepsilon = m^* - p \to 0$ with $\delta > 0$ fixed. We prove in Theorem 2 that u then converges to a Dirac measure concentrated at the origin, namely, $u(0) \to \infty$ and $u(x) \to 0$ for all $x \neq 0$, while also, at the same time, u converges strongly to 0 in any Lebesgue space $L^q(\mathbb{R}^n)$ with $m - 1 \leqslant q < m^*$. Our study also reveals a striking and unexpected phenomenon: the asymptotic behavior is different in the two cases $n \leqslant m^2$ and $n > m^2$; for instance, in the case $m = 2$ (i.e. $\Delta_m = \Delta$) there is a difference of behavior between the space dimensions $n = 3, 4$ and $n \geqslant 5$. More precisely, if $n > m^2$ we show that $u(0)$ blows up asymptotically like $\varepsilon^{-(n-m)/m^2}$ while if $n \leqslant m^2$ it blows up at a stronger rate, essentially $\varepsilon^{-(m-1)/m}$. This phenomenon is closely related with the L^m summability of functions which achieve the best constant in the Sobolev embedding $\mathcal{D}^{1,m} \subset L^{m^*}$, see [18] and (1) below for the explicit form of these functions.

Finally, let both $p = m^*$ and $\delta = 0$; then equation (P_p^δ) reads

$$-\Delta_m u = u^{m^*-1} \quad \text{in } \mathbb{R}^n, \tag{$P_{m^*}^0$}$$

which admits the one-parameter family of ground states

$$U_d(x) = d\left[1 + D\left(d^{\frac{m}{n-m}}|x|^{\frac{m}{m-1}}\right)\right]^{-\frac{n-m}{m}} \quad (d > 0), \tag{1}$$

where $D = D_{m,n} = (m-1)/(n-m)n^{1/(m-1)}$ and $U_d(0) = d$. Since the effects of vanishing $m^* - p$ and δ are in some sense "opposite", it is reasonable to conjecture that there exists a continuous function h, with $h(0) = 0$, such that if $\delta = h(\varepsilon)$, $p = m^* - \varepsilon$, then ground states u of (P_p^δ) converge neither to a Dirac measure nor to 0! In Theorem 4 below we prove the surprising fact that when $n > m^2$ this equilibrium occurs exactly when δ and ε are *linearly related*, $h(\varepsilon) \approx \text{Const}\,\varepsilon$. Moreover in this case the corresponding ground states u then converge uniformly to a suitably concentrated ground state of $(P_{m^*}^0)$, namely a function of the family (1), with the parameter $d = U_d(0)$ representing a "measure of concentration" and depending on the limiting value of the ratio $h(\varepsilon)/\varepsilon$.

Let us heuristically describe the phenomena highlighted by our results. When $p \to m^*$ with δ fixed, the mass of the ground state u of (P_p^δ) tends to concentrate near the point $x = 0$, that is, all other points of the graph are attracted to this point: in order to "let the other points fit near $x = 0$" the maximum level $u(0)$ is forced to blow up. When $\delta \to 0$ with p fixed, the ground state spreads, since now $x = 0$ behaves as a repulsive point, forcing the maximum level to blow down in order "not to break the graph". When both $\varepsilon = m^* - p$ and δ tend to 0 at the "equilibrium velocity" $\delta = h(\varepsilon)$, the point $x = 0$ is neither attractive nor repulsive: in this case, a further striking fact is that the exponential decay of the solution u of (P_p^δ) at infinity reverts to a polynomial decay.

The outline of the paper is as follows. In the next section we state our main results, Theorems 1–5. Then in Section 3 we present background material on radial ground states, including an estimate for the asymptotic decay as $r \to \infty$ of ground states of (P_p^δ), see Theorem 8. This estimate, along with Theorems 6 and 7 in Section 3, seems to be new and may be useful in other contexts. These results allow us to give a simple proof of Theorem 5 while the proofs of Theorems 1–4 are given in subsequent sections.

2. Main results

The existence and uniqueness of radial ground states for equation (P_p^δ) is well known [10,17]. We state this formally as

PROPOSITION 1. – *For all $n > m > 1$, $m < p < m^*$ and $\delta > 0$ equation (P_p^δ) admits a unique radial ground state $u = u(r)$, $r = |x|$. Moreover $u'(r) < 0$ for $r > 0$.*

We start the asymptotic analysis of (P_p^δ) by maintaining p fixed and letting $\delta \to 0$. An important role will be played by the rescaled problem ($\delta = 1$)

$$-\Delta_m v = -v^{m-1} + v^{p-1} \quad \text{in } \mathbb{R}^n. \tag{Q_p}$$

By Proposition 1 there exists a unique (radial) ground state v of (Q_p), so that the constant

$$\beta = v(0) \tag{2}$$

is a well-defined function of the parameters m, n, p.

THEOREM 1. – *For all $\delta > 0$, let u be the unique ground state of (P_p^δ) with $m < p < m^*$. Then $u(0) = \delta^{1/(p-m)}\beta$, while for fixed p and $x \neq 0$ there holds*

$$\frac{u(x)}{u(0)} = 1 - \frac{m-1}{m}\left(\frac{\beta^{p-m}-1}{n}\delta\right)^{\frac{1}{m-1}}|x|^{\frac{m}{m-1}} + \mathrm{o}\left(\delta^{\frac{1}{m-1}}|x|^{\frac{m}{m-1}}\right) \quad \text{as } \delta \to 0. \tag{3}$$

Also, putting $\ell = n(p-m)/m$, there exists $\alpha_{m,n,p} > 0$ independent of δ such that

$$\int_{\mathbb{R}^n} u^\ell = \alpha_{m,n,p} \quad \forall \delta > 0.$$

From Theorem 1 we can also obtain a result which, while slightly beyond the scope of the paper, is nevertheless worth noting. It states that the unique solution of (P_p^δ) for fixed $p < m^*$ tends to a Dirac measure as $\delta \to \infty$, see Theorem 9 in Section 4.

We now maintain $\delta > 0$ fixed and let $p \to m^*$. In order to state our main asymptotic result for this case, it is convenient to introduce the beta function $B(\cdot,\cdot)$ defined by

$$B(a,b) = \int_0^\infty \frac{t^{a-1}}{(1+t)^{a+b}}\,\mathrm{d}t, \quad a, b > 0.$$

Then we put

$$\beta_{m,n} = \left(n\left(\frac{m}{n-m}\right)^2 \frac{B(\frac{n(m-1)}{m}, \frac{n-m^2}{m})}{B(\frac{n(m-1)}{m}, \frac{n}{m})}\right)^{(n-m)/m^2} \quad \text{for } n > m^2,$$

and

$$\gamma_{m,n} = \omega_n \frac{m-1}{m}\left[n\left(\frac{n-m}{m-1}\right)^{m-1}\right]^{n/m} B\left(\frac{n(m-1)}{m}, \frac{n}{m}\right) \quad (\omega_n = \text{ measure } S^{n-1}).$$

We also put $C_{m,n} = D^{-(m-1)(n-m)/m}$, where $D = D_{m,n}$ is given in Eq. (1).

These coefficients allow us to describe the exact behavior of ground states when $n > m^2$: in particular note that $\beta_{m,n} \to \infty$ as $m \uparrow \sqrt{n}$.

THEOREM 2. – *For all $m < p < m^*$, let u be the unique ground state for equation (P_p^δ) with fixed $\delta > 0$. Then, writing $\varepsilon = m^* - p$, we have*

$$\lim_{\varepsilon \to 0}\left[\left(\frac{\varepsilon}{\delta}\right)^{(n-m)/m^2} u(0)\right] = \begin{cases} \beta_{m,n} & \text{if } n > m^2, \\ \infty & \text{if } n \leqslant m^2. \end{cases} \tag{4}$$

Moreover for all $x \neq 0$

$$\lim_{\varepsilon \to 0}\{u(0)u^{m-1}(x)\} \leqslant C_{m,n}|x|^{-(n-m)} \tag{5}$$

uniformly outside of any neighborhood of the origin, while also

$$\lim_{\varepsilon\to 0}\int_{\mathbb{R}^n} u^q = 0 \quad \forall q\in[m-1,m^*), \qquad \lim_{\varepsilon\to 0}\int_{\mathbb{R}^n} u^{m^*} = \gamma_{m,n}. \tag{6}$$

Theorem 2 gives a complete description of the asymptotic behavior of u when $n > m^2$; it leaves open the exact behavior when $n \leqslant m^2$. This latter question is considered in more detail in Section 5.2. The results given there, while not as precise as in the case $n > m^2$, nevertheless provide significant insight into the behavior of $u(0)$ as $\varepsilon \to 0$ beyond that described in the second case of (4). In particular from Lemmas 7 and 8 we have the following additional asymptotic results as $\varepsilon \to 0$.

Let $\delta = 1$. If $n = m^2$, then

$$\left(\frac{\varepsilon}{|\log\varepsilon|}\right)^{(m-1)/m} u(0) \approx 1,$$

while if $m < n < m^2$, then for appropriate positive constants we have

$$\text{Const.}|\log\varepsilon|^{(n-m^2)/m^2} \leqslant \varepsilon^{(m-1)/m}u(0) \leqslant \text{Const.}|\log\varepsilon|^{(n-m)/m^2}.$$

The picture below describes this striking phenomenon; let

$$\mu = \inf\{\gamma > 0;\ \lim_{\varepsilon\to 0}[u(0)\varepsilon^{\gamma}] = 0\},$$

then, $\mu = (m-1)/m$ when $n \leqslant m^2$ and $\mu = (n-m)/m^2$ when $n > m^2$. The figure represents the map $\mu = \mu(n)$ in the case $m = 2$.

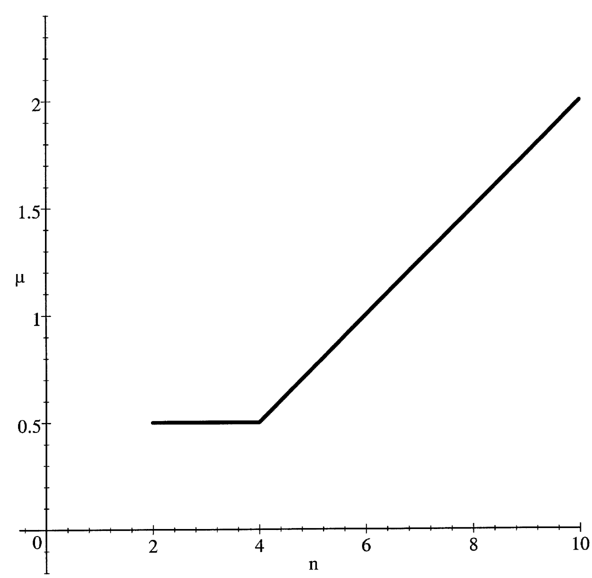

Fig. 1.

Condition (6) shows that, as $\varepsilon \to 0$, not only does u approach a Dirac measure ($u(0) \to \infty$ and $u(|x|) \to 0$ for $|x| \neq 0$), but also that the L^{m^*} norm of u approaches a *non-zero finite limit*. It is a remarkable fact, also, that the limit relation (6) is independent of the value of δ. It is worthwhile to note as well that by (6) and interpolating, the L^q norm of u becomes ∞ if $q > m^*$.

Remark. – The constants in Theorem 2 in the important case $m = 2$ are given by

$$\beta_{2,n} = \left(\frac{4n}{(n-2)^2} \frac{B(\frac{n}{2}, \frac{n-4}{2})}{B(\frac{n}{2}, \frac{n}{2})}\right)^{(n-2)/4}, \qquad \gamma_{2,n} = \frac{\omega_n}{2}[n(n-2)]^{n/2} B\left(\frac{n}{2}, \frac{n}{2}\right),$$

and $C_{2,n} = [n(n-2)]^{(n-2)/2}$.

The results of Theorem 2 can be supplemented with the following asymptotic estimates for the gradient ∇u of a ground state.

THEOREM 3. – *For all $m < p < m^*$, let u be the unique ground state for equation (P_p^δ) with fixed $\delta > 0$. Then for all $x \neq 0$ we have*

$$\lim_{\varepsilon \to 0} \{u(0)|\nabla u(x)|^{m-1}\} \leqslant \left(\frac{n-m}{m-1}\right)^{m-1} C_{m,n}|x|^{1-n} \tag{7}$$

and

$$\lim_{\varepsilon \to 0} \int_{\mathbb{R}^n} |\nabla u|^q = 0 \quad \forall q \in \left(n\frac{m-1}{n-1}, m\right), \qquad \lim_{\varepsilon \to 0} \int_{\mathbb{R}^n} |\nabla u|^m = \gamma_{m,n}. \tag{8}$$

Finally, we may accurately describe the behavior of the ground states of (P_p^δ) when $\varepsilon = m^* - p$ and δ approach zero simultaneously.

THEOREM 4. – *For $\delta > 0$ and $m < p < m^*$, let u be the unique ground state of (P_p^δ). Then for all $d > 0$ there exists a positive continuous function $\tau(\varepsilon) = \tau(\varepsilon, d)$ such that*

(i) $\tau(\varepsilon) \to (d/\beta_{m,n})^{m^2/(n-m)}$ *as $\varepsilon \to 0$ (when $n > m^2$), and $\tau(\varepsilon) \to 0$ as $\varepsilon \to 0$ (when $n \leqslant m^2$).*

(ii) *If $\delta = \varepsilon\tau(\varepsilon)$, $p = m^* - \varepsilon$, then $u(0) = d$. Moreover*

$$u \to U_d \quad \text{as } \varepsilon = m^* - p \to 0$$

uniformly on $\mathbb{R}^n$, where U_d is the function defined in (1).

If $\varepsilon, \delta \to 0$ without respecting the equilibrium behavior $\delta \approx \mathrm{Const}\,\varepsilon$ (in the case $n > m^2$), the central height $u(0)$ of the ground state may either converge to zero or diverge to infinity. We note finally that as soon as the asymptotic behavior of $u(0)$ as $p \to m^*$ is more accurately determined in the case $n \leqslant m^2$ of (4), one also gets a more precise statement of (i): of course, the equilibrium behavior will no longer be $\delta \approx \mathrm{Const}\,\varepsilon$.

To conclude the section, we supply two global estimates for $u(0)$, supplementing the asymptotic conditions (3) and (4).

Table 1

m	n	m^*	p	$\beta(m,n,p)$
1.6	2	8	1.8	$2.11 < \beta < 57.67$
1.2	2	3	1.9	$3.89 < \beta < 37.61$
1.1	2	$2.\bar{4}$	1.6	$5.36 < \beta < 10.72$
1.2	3	2	1.4	$9.1 < \beta < 525.22$

THEOREM 5. – *Let u be a ground state of (P_p^δ). Then*

$$u(0) > \left(\frac{mp}{mn - p(n-m)}\delta\right)^{1/(p-m)}, \tag{9}$$

and, provided that $p < n/(n-1)$,

$$u(0) < \left(\frac{p}{m}\frac{n - m(n-1)}{n - p(n-1)}\delta\right)^{1/(p-m)}. \tag{10}$$

The proof of this result is given in next section. By setting $\delta = 1$ in Theorem 5 we obtain related estimates for the parameter $\beta = v(0)$ in Theorem 1. Also from Theorem 2 we have the following asymptotic formula for β, with $\varepsilon = m^* - p \to 0$,

$$\beta = \beta_{m,n}\varepsilon^{-(n-m)/m^2}\big(1 + \mathrm{o}(1)\big) \quad \text{if } n > m^2;$$

see also Lemmas 5–8 in Sections 5.

Remark. – The condition $p < n/(n-1)$ implies $p < m/(m-1)$, since $n > m$: therefore, the upper bound in (10) is obtained only for values $m < 2$ (because $p > m$) and values p “far” from the critical exponent m^*, that is $m^* - p > n^2(m-1)/(n-m)(n-1)$. However, in the restricted range of values $p < n/(n-1)$, inequality (10) gives useful information about $v(0) = \beta$; we quote here some numerical computations (Table 1).

3. Preliminary results about ground states

In this section we consider the ground state problem for the general equation

$$-\Delta_m u = f(u) \quad \text{in } \mathbb{R}^n, \tag{11}$$

where the function f is assumed only to be continuous on $[0,\infty)$ and to obey the condition

$$f(0) = 0, \quad f(u) < 0 \quad \text{for } u \text{ near } 0. \tag{12}$$

A radial ground state $u = u(r)$, $r = |x|$, of (11) is in fact a C^1 solution of the ordinary differential equation

$$\begin{aligned} &\bigl(|u'|^{m-2}u'\bigr)' + \frac{n-1}{r}|u'|^{m-2}u' + f(u) = 0, \quad r > 0,\\ &u(0) = \alpha > 0, \quad u'(0) = 0 \end{aligned} \tag{13}$$

for some initial value $\alpha > 0$. For our purposes the dimension n may in fact be considered as any real number greater than m.

Put

$$F(u) = \int_0^u f(s)\,\mathrm{d}s \tag{14}$$

and introduce the energy function

$$E = E(r) = \frac{m-1}{m}|u'(r)|^m + F\bigl(u(r)\bigr). \tag{15}$$

The following properties of ground states are well-known [7].

PROPOSITION 2. – *A radial ground state $u = u(r)$ of* (13) *has the properties*

$$\begin{aligned} &\frac{|u'(r)|^{m-1}}{r} \to \frac{f(\alpha)}{n} \quad \textit{as } r \to 0,\\ &r^{n-1}|u'(r)|^{m-1} \to \text{Finite limit} \quad \textit{as } r \to \infty, \end{aligned}$$

$$F(\alpha) = (n-1)\int_0^\infty \frac{|u'(r)|^m}{r}\,\mathrm{d}r$$

and

$$E(r) > 0 \quad \forall r \geqslant 0, \qquad E(r) \to 0 \quad \textit{as } r \to \infty.$$

In the next result we recall a Pohozaev-type identity [12]. [2]

PROPOSITION 3. – *Let $u = u(r)$ be a radial ground state of* (13), *and put*

$$Q(r) = nmF(u) - (n-m)uf(u). \tag{16}$$

Then the functions $r^{n-1}Q(r)$ and $r^{n-1}F(u(r))$ are in $L^1(0,\infty)$, and moreover

$$\int_0^\infty Q(r)r^{n-1}\,\mathrm{d}r = 0. \tag{17}$$

[2] Formula (17) is given in [12] for the case $m = 2$, see (3.7) and put $a = (n-2)/2$; the case for general m moreover is implicit in Section 4, Case (V) of [12].

Remark. – In other terms, the result of Proposition 3 says that the functions $Q(|x|)$ and $F(u(|x|))$ are in $L^1(\mathbb{R}^n)$ and that $\int_{\mathbb{R}^n} Q(|x|)\,\mathrm{d}x = 0$.

For completeness we give a proof of Proposition 3. By direct calculation, using (13), one finds that

$$P(r) = \int_0^r Q(t)t^{n-1}\,\mathrm{d}t, \quad r > 0,$$

where

$$P(r) = (n-m)r^{n-1}u(r)u'(r)|u'(r)|^{m-2} + mr^n E(r).$$

Since $E = \frac{m-1}{m}|u'|^m + F(u(r)) > 0$ and because $f(s) < 0$ for s near 0, we get

$$\bigl|F\bigl(u(r)\bigr)\bigr|,\ E(r) \leqslant \frac{m-1}{m}|u'(r)|^m$$

for all sufficiently large r. Using Proposition 2 then gives $r^{n-1}|u'|^{m-1} \leqslant \text{Const.}$ and

$$r^n\bigl|F\bigl(u(r)\bigr)\bigr|,\ r^n E(r) \leqslant \text{Const.}r^{-(n-m)/(m-1)} \tag{18}$$

for sufficiently large r. Hence $P(r) \to 0$ as $r \to \infty$, which yields

$$\lim_{r\to\infty} \int_0^r Q(t)t^{n-1}\,\mathrm{d}t = 0.$$

But from (18) we get $r^{n-1}|F(u(r))| \in L^1(0,\infty)$, while also $uf(u) < 0$ for all sufficiently large r. Thus the previous equation together with the definition of $Q(r)$ shows in fact that $r^{n-1}Q(r)$ is in $L^1(0,\infty)$ and that (17) holds. This completes the proof. □

Proposition 3 has the following important consequence.

THEOREM 6. – *Suppose there exists* $\gamma > 0$ *such that*

$$nmF(s) - (n-m)sf(s) < 0 \quad \textit{for } 0 < s < \gamma. \tag{19}$$

Then $\alpha > \gamma$.

Proof. – Suppose for contradiction that $\alpha \leqslant \gamma$. Then since $u' < 0$ for $r > 0$, it follows that $u(r) < \gamma$ for all $r > 0$. In turn, by the hypothesis (19) we have $Q(r) = nmF(u) - (n-m)uf(u) < 0$ for all $r > 0$, which contradicts Proposition 3. □

An upper bound for $u(0)$ can also be obtained in some circumstances, as in the following

THEOREM 7. – *Suppose* $f'(s) \geqslant 0$ *whenever* $f(s) > 0$ *and that there exists* $\mu > 0$ *such that*

$$nF(s) - (n-1)sf(s) \geqslant 0 \quad \textit{for } s \geqslant \mu. \tag{20}$$

Then $\alpha < \mu$.

Proof. – We assert that the function $r \mapsto \Phi(r) = r^{-1}|u'(r)|^{m-1}$ is decreasing on $(0,\infty)$. By direct calculation, using (13),

$$r\Phi'(r) = f(u) - n\Phi(r).$$

If $f(u) \leqslant 0$ then $\Phi' < 0$. On the other hand, for all r such that $f(u) > 0$, we have $(f(u) - n\Phi(r))' = f'(u)u' - n\Phi'(r) \leqslant -n\Phi'(r)$, by hypothesis. Consequently

$$(r\Phi')' \leqslant -n\Phi'.$$

By integration this gives $r^{n+1}\Phi'(r) \leqslant r_1^{n+1}\Phi'(r_1)$ on any interval (r_1, r) where $f(u) > 0$. The assertion now follows by an easy argument, once one notes notes that $r^{n+1}\Phi'(r) \to 0$ as $r \to 0$.

Now by Proposition 2 and the assertion, we have

$$F(\alpha) = (n-1)\int_0^\infty \frac{|u'(r)|^m}{r}\,\mathrm{d}r = (n-1)\int_0^\infty \Phi(r)|u'(r)|\,\mathrm{d}r$$
$$< (n-1)\Phi(0)\int_0^\infty |u'(r)|\,\mathrm{d}r = (n-1)\alpha\Phi(0).$$

Since by Proposition 2 we also have $\Phi(0) = f(\alpha)/n$, this gives $nF(\alpha) - (n-1)\alpha f(\alpha) < 0$. The conclusion now follows from the main hypothesis (20).

Using Theorems 6 and 7 it is now easy to obtain the

Proof of Theorem 5. – Equation (P_p^δ) can be written in the form (11), or (13), with

$$f(s) = -\delta s^{m-1} + s^{p-1}, \qquad Q(r) = -\delta m u^m + \frac{mn - p(n-m)}{p}u^p.$$

Hence for this case we can take

$$\gamma = \left(\frac{mp}{mn - p(n-m)}\delta\right)^{1/(p-m)}$$

in (19), giving the first conclusion of Theorem 5 as a consequence of Theorem 6.

Moreover

$$nF(s) - (n-1)sf(s) = -\frac{n - m(n-1)}{m}\delta s^m + \frac{n - p(n-1)}{p}s^p.$$

Thus we can take

$$\mu = \left(\frac{p}{m}\frac{n - m(n-1)}{n - p(n-1)}\delta\right)^{1/(p-m)}$$

in (20), giving the second conclusion as a consequence of Theorem 7.

We conclude the section by showing that radial ground states $u = u(r)$ of (P_ε^δ) have exponential decay as r approaches infinity. This is well-known in the case $m = 2$, see [4, Theorem 1(iv)]: we give here a different proof in the general case $m > 1$.

THEOREM 8. – *Suppose that there exist constants $\delta, \lambda, \rho > 0$ such that f satisfies the inequality*

$$-\delta s^{m-1} \leqslant f(s) \leqslant -\lambda s^{m-1} \quad \text{for } 0 < s < \rho. \tag{21}$$

Then there exist constants $\mu_0, \mu_1, \mu_2, \nu > 0$ (depending on m, n, δ, λ) such that, for r suitably large,

$$u(r) \leqslant \mu_0 \mathrm{e}^{-\nu r} \quad |u'(r)| \leqslant \mu_1 \mathrm{e}^{-\nu r} \quad |u''(r)| \leqslant \mu_2 \mathrm{e}^{-\nu r}. \tag{22}$$

Remark. – For general nonlinearities f in (13), one usually expects polynomial decay at infinity, see [12, Lemma 5.1], [17, Proposition 2.2]. Nevertheless, Theorem 8 is not entirely unexpected, since the nonlinearity (21) has "borderline behavior" which separates compact support and positive ground states, see [7, Section 1.3].

Proof of Theorem 8. – Obviously $u = u(r)$ satisfies (13). Let $R \geqslant 0$ be such that $u(r) \leqslant \rho$ when $r \geqslant R$. Since $u \to 0$ as $r \to \infty$, it is clear that such a value R exists. By Proposition 2 and the right hand inequality of (21) we thus obtain

$$\frac{m-1}{m}|u'(r)|^m > -F\big(u(r)\big) \geqslant \frac{\lambda}{m} u^m(r)$$

for $r \geqslant R$. Therefore,

$$-\frac{u'(r)}{u(r)} > \left(\frac{\lambda}{m-1}\right)^{1/m} \quad \forall r \geqslant R. \tag{23}$$

Integrating this inequality on the interval $[R, r]$ yields the first part of the result, with

$$\mu_0 = \rho \mathrm{e}^{\nu R}, \qquad \nu = \big(\lambda/(m-1)\big)^{1/m}. \tag{24}$$

For the other estimates, we rewrite (13) in the form

$$\big(r^{n-1}|u'(r)|^{m-1}\big)' = r^{n-1} f\big(u(r)\big). \tag{25}$$

Since $f(u) < 0$ for u near 0, it follows that $r^{n-1}|u'(r)|^{m-1}$ is ultimately decreasing, clearly to a non-negative limit as $r \to \infty$ (this is the first result of Proposition 2). By the exponential decay proved above, the limit must be 0. Therefore we can integrate (25) on $[r, \infty)$ for $r \geqslant R$ to obtain, with the help of (21),

$$r^{n-1}|u'(r)|^{m-1} = -\int_r^\infty t^{n-1} f\big(u(t)\big)\,\mathrm{d}t < \delta \int_r^\infty t^{n-1} u^{m-1}(t)\,\mathrm{d}t$$

$$\leqslant \delta \mu_0^{m-1} \int_r^\infty t^{n-1} \mathrm{e}^{-(m-1)\nu t}\,\mathrm{d}t.$$

With $n-1$ integrations by parts, this proves that

$$|u'(r)| \leqslant \mu_1 \mathrm{e}^{-\nu r} \quad \forall r \geqslant R.$$

Finally, we write (13) as

$$(m-1)|u'(r)|^{m-2}u''(r) = \frac{n-1}{r}|u'(r)|^{m-1} - f(u).$$

From the right hand inequality of (21) we get $f(u) \leqslant 0$ for $r \geqslant R$, which shows that $u''(r) > 0$ for all $r \geqslant R$. Further, from the left hand inequality,

$$u''(r) < \frac{n-1}{(m-1)R}|u'(r)| + \frac{\delta}{m-1}\frac{u^{m-1}(r)}{|u'(r)|^{m-2}}.$$

Hence by (23) and by the exponential decay of u and u', this yields

$$0 < u''(r) < \frac{n-1}{(m-1)R}|u'(r)| + \frac{\delta}{m-1}\left(\frac{m-1}{\lambda}\right)^{(m-2)/m} u(r) \leqslant \mu_2 \mathrm{e}^{-\nu r} \quad \forall r \geqslant R.$$

The proof of Theorem 8 is now complete. □

Remarks. – The first estimate of (22) requires only the right hand inequality of (21) for its validity.
It almost goes without saying that the function $f(u) = -\delta u^{m-1} + u^{p-1}$ satisfies (21) for suitable λ, ρ.

4. Proof of Theorem 1

Let $u = u(r)$ be a ground state of (P_p^δ). Define $v = v(r)$ by means of the rescaling

$$v(r) = \delta^{-1/(p-m)} u\left(\frac{r}{\delta^{1/m}}\right), \tag{26}$$

so that v is the unique ground state of the rescaled equation (Q_p). By definition (2) and by (26) one has $u(0) = \delta^{1/(p-m)}\beta$.

Next, from (Q_p) we find, as in (25),

$$\begin{aligned}
|v'(r)|^{m-1} &= \frac{1}{r^{n-1}} \int_0^r s^{n-1}\bigl\{-v^{m-1}(s) + v^{p-1}(s)\bigr\}\, \mathrm{d}s \\
&= \frac{1}{r^{n-1}} \int_0^r s^{n-1}\bigl\{-\beta^{m-1} + \beta^{p-1} + \mathrm{o}(1)\bigr\}\, \mathrm{d}s \\
&= \frac{r}{n}\bigl\{-\beta^{m-1} + \beta^{p-1} + \mathrm{o}(1)\bigr\}
\end{aligned}$$

as $r \to 0$. Taking the $1/(m-1)$ root and integrating from 0 to r then gives

$$v(r) = \beta - \frac{m-1}{m}\left(\frac{\beta^{p-1} - \beta^{m-1}}{n}\right)^{1/(m-1)} r^{m/(m-1)} + o\big(r^{m/(m-1)}\big) \quad \text{as } r \to 0. \tag{27}$$

This, together with (26), yields (3).

The final part of theorem is an almost obvious consequence of (26) and the change of variables $s = \delta^{1/m} r$; in particular $\alpha_{m,n,p} = \int_{\mathbb{R}^n} v^\ell$. □

When $\delta \to \infty$ we can obtain a partial companion result to (3) in Theorem 1.

THEOREM 9. – *For fixed $x \neq 0$ we have*

$$u(x) = o\big(e^{-\nu\delta^{1/m}|x|}\big)$$

as $\delta \to \infty$, where ν is any (positive) number less than $1/(m-1)^{1/m}$.

Proof. – We apply Theorem 7 for ground states of (Q_p). Here $f(s) = -s^{m-1} + s^{p-1}$, so that one can take λ to be any number less than 1 in (21), provided that ρ is chosen appropriately near 0. Thus by Theorem 8 we have

$$v(r) \leqslant \mu_0 e^{-\nu r}$$

for all sufficiently large r, where, see (24), ν is any number less than $1/(m-1)^{1/m}$. Hence, by (26),

$$u(x) = \delta^{1/(p-m)} v\big(\delta^{1/m}|x|\big) \leqslant \mu_0 \delta^{1/(p-m)} e^{-\nu\delta^{1/m}|x|}$$

for all fixed $x \neq 0$ and sufficiently large δ. Finally, taking $\hat{\nu} = \nu - \theta$, with θ small, we get

$$u(x) \leqslant \mu_0 \delta^{1/(p-m)} e^{-\theta\delta^{1/m}|x|} \cdot e^{-\hat{\nu}\delta^{1/m}|x|} = o\big(e^{-\hat{\nu}\delta^{1/m}|x|}\big)$$

as $\delta \to \infty$. The conclusion now follows at once, since clearly by appropriate choice of ν and θ we can assume that $\hat{\nu}$ is any number less than $1/(m-1)^{1/m}$. □

5. Proof of Theorem 2

The argument is delicate, covering a number of pages. For the proof of (4) we need to distinguish the two cases $n > m^2$ and $m < n \leqslant m^2$; this is done in Sections 5.1 and 5.2 below. The proof of (5) and (6) is given in Section 5.3.

We shall prove (4) first, for the case $\delta = 1$, and then obtain the general estimate by means of the rescaling (26).

Thus we assume that $u = u(r)$ satisfies (13) with $f(s) = -s^{m-1} + s^{p-1}$, namely

$$\big(|u'|^{m-2}u'\big)' + \frac{n-1}{r}|u'|^{m-2}u' - u^{m-1} + u^{p-1} = 0 \tag{28}$$

with $u(0)=\alpha$. From the estimate (9) in Theorem 5 we have always $\alpha>1$ (since $p>m$) and, more precisely,

$$\alpha > \left(\frac{mp}{\varepsilon(n-m)}\right)^{1/(p-m)}$$

where

$$p = m^* - \varepsilon.$$

Hence

$$\alpha > \left(\frac{m^2}{n-m}\frac{1}{\varepsilon}\right)^{1/(p-m)},$$

which gives the important condition

$$\omega \equiv \varepsilon\alpha^{p-m} \geqslant K \quad \forall\varepsilon \in (0, m^*-m), \tag{29}$$

where $K = m^2/(n-m)$.

We make a second rescaling

$$w(r) = \frac{1}{\alpha}u\big(\alpha^{-(p-m)/m}r\big), \tag{30}$$

so that if $u=u(r)$ solves (28), then $w=w(r)$ satisfies

$$\begin{cases} (|w'|^{m-2}w')' + \frac{n-1}{r}|w'|^{m-2}w' - \eta w^{m-1} + w^{p-1} = 0, \\ w(0)=1, \quad w'(0)=0, \end{cases} \tag{31}$$

where $\eta = \alpha^{-(p-m)}$. Note that $\eta<1$ since $\alpha>1$, and also, by (29), $\eta\to 0$ as $\varepsilon\to 0$. Now define the modified nonlinearity

$$f_\eta(s) = -\eta s^{m-1} + s^{p-1}$$

and the corresponding functions (see (14) and (16))

$$F_\eta(s) = -\frac{\eta}{m}s^m + \frac{1}{p}s^p, \qquad Q_\eta(r) = -m\eta w^m(r) + \frac{\varepsilon(n-m)}{p}w^p(r). \tag{32}$$

Also, for $r\geqslant 0$ let us define the function

$$z(r) = \big(1 + (1-\eta)^{1/(m-1)}Dr^{m/(m-1)}\big)^{-(n-m)/m} \tag{33}$$

where the constant $D = D_{m,n}$ is given in (1).

We can now prove the following comparison result, closely related to Lemma 2.1 of [11]. [3]

[3] The idea of a uniform upper bound for a scaled function $w(r)$ first appears (for the case $m=2$) in [2].

LEMMA 1. – *We have*

$$w(r) < z(r) \quad \forall r > 0. \tag{34}$$

Proof. – We make use of the function H introduced in Lemma 2.1 in [11]: here however it will be applied without a previous Emden–Fowler inversion. Thus set

$$H(r) = (m-1)r^n|w'(r)|^m - (n-m)r^{n-1}w(r)|w'(r)|^{m-1} + \frac{n-m}{n}r^n w(r) f_\eta\big(w(r)\big).$$

Then by using the fact that w solves (31) we obtain

$$H'(r) = \frac{r^n}{n}\big(m^2\eta w^{m-1}(r) - \varepsilon(n-m)w^{p-1}(r)\big)w'(r).$$

Let R be the unique value of r where

$$w(R) = \left(\frac{m^2}{(n-m)\omega}\right)^{1/(p-m)} \in (0,1);$$

see (29) and recall from Proposition 2 that $w' < 0$ and $w < 1$ for $r > 0$. Hence it is easy to see that H is strictly increasing on $[0, R]$ and strictly decreasing on $[R, \infty)$. Moreover, $H(0) = 0$ and $\lim_{r\to\infty} H(r) = 0$ by Theorem 8. Consequently

$$H(r) > 0 \quad \forall r > 0. \tag{35}$$

Consider the function

$$\Psi(r) = \frac{|w'(r)|^{m-1}}{r w^{n(m-1)/(n-m)}(r)} = \frac{\Phi(r)}{w^{n(m-1)/(n-m)}(r)},$$

where $\Phi(r) = |w'(r)|^{m-1}/r$ (see the proof of Theorem 7). By using (31) again we find that

$$\Psi'(r) = \frac{n}{n-m}\,\frac{1}{r^{n+1}\ w^{m(n-1)/(n-m)}(r)}H(r).$$

From (35) it follows that Ψ is strictly increasing on $[0, \infty)$. Therefore, by Proposition 2 we have

$$\Psi(r) > \lim_{t\to 0}\Psi(t) = \frac{f_\eta(1)}{n} = \frac{1-\eta}{n};$$

hence

$$\frac{|w'(r)|}{w^{n/(n-m)}(r)} > \left(\frac{1-\eta}{n}\right)^{1/(m-1)} r^{1/(m-1)} = \frac{|z'(r)|}{z^{n/(n-m)}(r)} \quad \forall r > 0.$$

The conclusion (34) follows upon integration, and the proof is complete. □

For later use we observe that the function $z = z(r)$ defined in (33) satisfies the equation

$$\big(|z'|^{m-2}z'\big)' + \frac{n-1}{r}|z'|^{m-2}z' + (1-\eta)z^{m^*-1} = 0 \tag{36}$$

(the easiest way to check this is to note from (1) that $z=d^{-1}U_d$ for $d=(1-\eta)^{(n-m)/m^2}$, so that z then satisfies $(P^0_{m^*})$ with the extra coefficient $(1-\eta)$ inserted on the right side).

Now let

$$C_1=C_1(\varepsilon)=\left(\frac{n-m}{m^2}\varepsilon\right)^{\varepsilon/(p-m)}. \tag{37}$$

Then by differential calculus (recalling that $p=m^*-\varepsilon$ and $\eta=\alpha^{-(p-m)}$) we find without difficulty that

$$f_\eta(s)\leqslant C_1\alpha^\varepsilon s^{m^*-1}\quad \forall s>0 \quad\text{and}\quad \lim_{\varepsilon\to 0}C_1=1. \tag{38}$$

This allows us to obtain the following partial converse of Lemma 1.

LEMMA 2. – *There exists a positive function $C_2=C_2(\varepsilon)$ such that* $\lim_{\varepsilon\to 0}C_2=1$ *and*

$$w(r)>C_2\alpha^{\varepsilon/(m-1)}z(r)-\left(C_2\alpha^{\varepsilon/(m-1)}-1\right)\quad \forall r>0. \tag{39}$$

Moreover $C_2\alpha^{\varepsilon/(m-1)}>1$.

Proof. – Eq. (31) may be rewritten as

$$\left(r^{n-1}|w'|^{m-1}\right)'=r^{n-1}f_\eta(w). \tag{40}$$

Integrating on $[0,r]$, and taking into account (38) and Lemma 1, yields

$$r^{n-1}|w'(r)|^{m-1}=\int_0^r t^{n-1}f_\eta\big(w(t)\big)\,\mathrm{d}t<C_1\alpha^\varepsilon\int_0^r t^{n-1}z^{m^*-1}(t)\,\mathrm{d}t$$
$$=\frac{C_1}{1-\eta}\alpha^\varepsilon r^{n-1}|z'(r)|^{m-1},$$

the last equality being obtained by a similar integration of (36) on $[0,r]$. Therefore,

$$|w'(r)|<C_2\alpha^{\varepsilon/(m-1)}|z'(r)|\quad \forall r>0, \tag{41}$$

where

$$C_2=\left(\frac{C_1}{1-\eta}\right)^{1/(m-1)}.$$

Integrating (41) on $[0,r]$ then gives (39).

Finally, from (38) one sees that $C_2\to 1$ as $\varepsilon\to 0$, while by (34) and (39) we infer that

$$\left(C_2\alpha^{\varepsilon/(m-1)}-1\right)(z(r)-1)<0\quad \forall r>0,$$

that is, $C_2\alpha^{\varepsilon/(m-1)}-1>0$ since $z(r)<1$ for $r>0$ by (34) and the fact that $\eta<1$. This completes the proof. □

The following technical lemmas will be crucial in the sequel. To simplify their presentation, we shall think of the functions $w=w(r)$ and $z=z(r)$, given in (30) and (33), to be defined over the space $\mathbb{R}^n$ instead of on $r \geqslant 0$; that is, $w = w(|x|)$ and $z = z(|x|)$. In particular, w then satisfies the partial differential equation

$$-\Delta_m w = f_\eta(w) = -\eta w^{m-1} + w^{p-1}, \quad \eta = \alpha^{-(p-m)}. \tag{42}$$

We observe also that $w(|x|)$ decays exponentially as $|x| \to \infty$, so that the integrals below are well defined.

LEMMA 3. – *We have*

$$c_1\omega \int_{\mathbb{R}^n} w^p \leqslant \int_{\mathbb{R}^n} w^m \leqslant c_2\omega \int_{\mathbb{R}^n} w^p,$$

where $\omega = \varepsilon\alpha^{p-m}$, $p = m^* - \varepsilon$, *and*

$$c_1 = \frac{1}{n}\left(\frac{n-m}{m}\right)^2, \qquad c_2 = \frac{n-m}{m^2}.$$

Proof. – By Proposition 3 applied to the ground state w of (31) we get, with the help of the second part of (32),

$$-m\eta \int_{\mathbb{R}^n} w^m + \frac{\varepsilon(n-m)}{p} \int_{\mathbb{R}^n} w^p = 0, \quad \text{that is,} \quad \int_{\mathbb{R}^n} w^m = \frac{n-m}{mp}\omega \int_{\mathbb{R}^n} w^p.$$

But $p \in (m, m^*)$, so the conclusion follows at once. □

LEMMA 4. – *We have*

$$\int_{\mathbb{R}^n} w^p \geqslant \left(C\alpha^\varepsilon\right)^{-(n-m)/m}, \qquad \int_{\mathbb{R}^n} |\nabla w|^m \geqslant \left(C\alpha^\varepsilon\right)^{-(n-m)/m},$$

where C *is a Sobolev constant for the embedding of* $\mathcal{D}^{1,m}(\mathbb{R}^n)$ *into* $L^{m^*}(\mathbb{R}^n)$.

Proof. – If we multiply (42) by w and integrate by parts, we obtain

$$\int_{\mathbb{R}^n} |\nabla w|^m = -\eta \int_{\mathbb{R}^n} w^m + \int_{\mathbb{R}^n} w^p < \int_{\mathbb{R}^n} w^p. \tag{43}$$

Using (38) and the fact that $C_1 \leqslant 1$ by (37), Eq. (42) can also be written in the form $-\Delta_m w = f_\eta(w) \leqslant \alpha^\varepsilon w^{m^*-1}$. Thus, as before,

$$\int_{\mathbb{R}^n} |\nabla w|^m \leqslant \alpha^\varepsilon \int_{\mathbb{R}^n} w^{m^*} \leqslant C\alpha^\varepsilon \left(\int_{\mathbb{R}^n} |\nabla w|^m\right)^{m^*/m} \tag{44}$$

by the Sobolev inequality. Solving this relation for $\int_{\mathbb{R}^n} |\nabla w|^m$ gives the second inequality of the lemma; the first is then obtained from (43). This completes the proof. □

5.1. The case $n > m^2$

By (33) we see that $z(|x|) \approx |x|^{-(n-m)/(m-1)}$ as $|x| \to \infty$, so $z \in L^m(\mathbb{R}^n)$ if and only if $n > m^2$. This allows us to derive

LEMMA 5. – *Let $n > m^2$. Then there exists $A > 0$ (depending only on m, n) such that*

$$\alpha \leqslant \left(\frac{A}{\varepsilon}\right)^{(n-m)/m} \quad \textit{for all } \varepsilon \in \left(0, \frac{m-1}{n}\frac{m^2}{n-m}\right). \tag{45}$$

Proof. – Define $\hat{z}(|x|)$ to be the function given by (33) with the parameter η *fixed* at the value

$$\hat{\eta} = \frac{(m-1)(n-m)}{n^2 - m(m-1)}.$$

Using (9) with $\delta = 1$, an easy calculation shows that for ε in the range stated in the lemma we have $\eta = \alpha^{-(p-m)} \in (0, \hat{\eta})$. Hence, for the given range of ε, we infer from (34) that

$$\int_{\mathbb{R}^n} w^m \leqslant \int_{\mathbb{R}^n} z^m \leqslant \int_{\mathbb{R}^n} \hat{z}^m \equiv \hat{c}$$

(recall $n > m^2$, and observe specifically that $\hat{c} = \hat{c}(m, n)$).

On the other hand, by Lemmas 3 and 4,

$$\int_{\mathbb{R}^n} w^m \geqslant c_1\omega \int_{\mathbb{R}^n} w^p \geqslant c_1 \left(C\alpha^\varepsilon\right)^{-(n-m)/m}\omega.$$

Combining the two previous lines, and remembering that $\omega = \varepsilon\alpha^{p-m}$, $p = m^* - \varepsilon$, we obtain

$$\alpha^{m^*-m-\varepsilon\frac{n}{m}} \leqslant \frac{A}{\varepsilon}, \tag{46}$$

where $A \equiv (\hat{c}/c_1)C^{(n-m)/m}$ depends only on m, n. Finally, using the given restriction

$$0 < \varepsilon \leqslant \frac{m-1}{n}\frac{m^2}{n-m} \tag{47}$$

(note $m^* - m = m^2/(n-m)$), one derives from (46) that

$$\alpha^{m/(n-m)} \leqslant \frac{A}{\varepsilon};$$

(45) now follows immediately, and the proof is complete. □

Together with the inequality $\alpha > 1$, Lemma 5 implies the important conclusion

$$\alpha^\varepsilon \to 1 \quad \text{as } \varepsilon \to 0. \tag{48}$$

LEMMA 6. – *Let $n > m^2$. Then there exists $K' > 0$ (depending only on m, n) such that*

$$\omega = \varepsilon\alpha^{p-m} \leqslant K' \quad \textit{for all } \varepsilon \in \left(0, \frac{m-1}{n}\frac{m^2}{n-m}\right).$$

Proof. – We have

$$\alpha^{p-m} = \alpha^{m^*-m-\varepsilon\frac{n}{m}} \cdot \alpha^{\varepsilon\frac{n-m}{m}} \leqslant \frac{A}{\varepsilon} \cdot \left(\frac{A}{\varepsilon}\right)^{\varepsilon(\frac{n-m}{m})^2},$$

by (45) and (46). Hence

$$\omega = \varepsilon\alpha^{p-m} \leqslant A \cdot \left(\frac{A}{\varepsilon}\right)^{\varepsilon(\frac{n-m}{m})^2}.$$

It remains to show that the right side is bounded, but this follows directly from the fact that $(1/s)^s$ is bounded ($\leqslant \mathrm{e}^{1/\mathrm{e}}$) on $(0, \infty)$. The proof is complete. □

Remark. – A short calculation, taking into account restriction (47), shows that in fact we can choose $K' = A^{m(n-m+1)/n}\mathrm{e}^{(n-m)^2/\mathrm{e}m^2}$.

We can now complete the proof of (4). Here it is convenient to revert to the original understanding that $w = w(r)$ and $z = z(r)$. We first rewrite the results of Lemmas 1, 2 as

$$0 < z - w < C_3 - 1 \quad \text{for all } r > 0, \tag{49}$$

where $C_3 = C_3(\varepsilon) = C_2\alpha^{\varepsilon/(m-1)} \to 1$ as $\varepsilon \to 0$; of course also $C_3 > 1$ by Lemma 2.

From Proposition 3 applied to equation (31) we obtain

$$\int_0^\infty Q_\eta(r)r^{n-1}\,\mathrm{d}r = 0, \tag{50}$$

where $Q_\eta(r)$ is defined by (32); see the same argument in Lemma 3.

Now by (29) and Lemma 6 we know that $\varepsilon/\eta = \omega \in [K, K']$. Then, since $w \leqslant 1$, it follows from (32) that

$$|Q_\eta(r)| \leqslant \mathrm{Const}\, m\eta w^m \leqslant \mathrm{Const}\, m\hat{\eta}\hat{z}^m,$$

see the proof of Lemma 5. Recalling that $\hat{z}^m \in L^1(\mathbb{R}^n)$, we can therefore apply the Lebesgue dominated convergence theorem to (50) when $\varepsilon \to 0$. Clearly ω converges to some limit $\omega_0 \in [K, K']$, up to a subsequence (in fact we will determine a unique possible value for ω_0, which shows that $\omega \to \omega_0$ on the continuum $\varepsilon > 0$). Moreover by (49) and the fact that $\eta \to 0$ as $\varepsilon \to 0$, we have

$$z(r) \to z_0(r) \equiv \left(1 + Dr^{m/(m-1)}\right)^{-(n-m)/m}$$

pointwise for all $r \geqslant 0$. Consequently there results

$$\int_0^\infty z_0^m(r) r^{n-1}\,\mathrm{d}r = \omega_0 \frac{(n-m)^2}{nm^2} \int_0^\infty z_0^{m^*}(r) r^{n-1}\,\mathrm{d}r.$$

Both $z_0^m r^{n-1}$ and $z_0^{m^*} r^{n-1}$ are in $L^1(0,\infty)$ since $n > m^2$.

By means of the change of variables $s = Dr^{m/(m-1)}$ one obtains

$$\int_0^\infty z_0^m(r) r^{n-1}\,\mathrm{d}r = \frac{m-1}{m} D^{-\frac{m-1}{m}n} B\left(\frac{n(m-1)}{m}, \frac{n-m^2}{m}\right) \tag{51}$$

and

$$\int_0^\infty z_0^{m^*}(r) r^{n-1}\,\mathrm{d}r = \frac{m-1}{m} D^{-\frac{m-1}{m}n} B\left(\frac{n(m-1)}{m}, \frac{n}{m}\right). \tag{52}$$

Hence,

$$\omega_0 = n\left(\frac{m}{n-m}\right)^2 \frac{B(\frac{n(m-1)}{m}, \frac{n-m^2}{m})}{B(\frac{n(m-1)}{m}, \frac{n}{m})}.$$

We can now prove the asymptotic relation (4). Indeed,

$$\varepsilon^{(n-m)/m^2}\alpha = \left(\omega\alpha^\varepsilon\right)^{(n-m)/m^2} \to \omega_0^{(n-m)/m^2} = \beta_{m,n}$$

as $\varepsilon \to 0$ (recall $\alpha^\varepsilon \to 1$), which is just (4) for the case $\delta = 1$. Since for general δ one has $u(0) = \delta^{1/(p-m)}\alpha \approx \delta^{(n-m)/m^2}\alpha$, relation (4) is proved (case $n > m^2$).

5.2. The case $n \leqslant m^2$

Here $z \notin L^m(\mathbb{R}^n)$ and the crucial Lemma 6 does not hold; nevertheless, we can prove the following result.

LEMMA 7. – *Assume that $n \leqslant m^2$. Then then there exists $K' = K'(m,n) > 0$ such that*

$$\varepsilon\alpha^{m/(m-1)} \leqslant K' |\log\varepsilon|^{(n-m)/m(m-1)}.$$

Proof. – We argue as in the proof of Lemmas 5 and 6, with several major changes. Let ℓ be an exponent greater than $n(m-1)/(n-m)$ to be determined later. Then, from (34) we have

$$\int_{\mathbb{R}^n} w^\ell \leqslant \int_{\mathbb{R}^n} z^\ell \leqslant \int_{\mathbb{R}^n} \hat{z}^\ell = \hat{d} < \infty \tag{53}$$

since $\hat{z} \in L^\ell(\mathbb{R}^n)$; here $\hat{d}$ of course depends on ℓ. On the other hand, by Lemmas 3 and 4 we find

$$\int_{\mathbb{R}^n} w^m \geqslant c_1\omega \int_{\mathbb{R}^n} w^p \geqslant c_1\omega\left(C\alpha^\varepsilon\right)^{-(n-m)/m}. \tag{54}$$

Next, integrating (40) over $(0,\infty)$ and taking into account the exponential decay of w and w', as well as (34), we get

$$\int_{\mathbb{R}^n} w^{m-1} = \alpha^{p-m}\int_{\mathbb{R}^n} w^{p-1} \leqslant \alpha^{p-m}\int_{\mathbb{R}^n} \hat{z}^{p-1} = \hat{d}_1\alpha^{p-m}, \tag{55}$$

where we have used the fact that $\hat{z}\in L^{p-1}(\mathbb{R}^n)$ (for $\varepsilon < m/(n-m)$).

By Hölder interpolation,

$$\int_{\mathbb{R}^n} w^m \leqslant \Big(\int_{\mathbb{R}^n} w^{m-1}\Big)^{1-\vartheta}\Big(\int_{\mathbb{R}^n} w^{\ell}\Big)^{\vartheta}, \tag{56}$$

where $\vartheta = 1/(\ell-m+1)\in(0,1)$ since $n\leqslant m^2$. A short calculation shows moreover that

$$\hat{d} = \mathrm{O}\Big(\frac{n-m}{m-1}\ell - n\Big)^{-1} \quad \text{as } \ell\to\frac{n(m-1)}{n-m}. \tag{57}$$

Now we choose ℓ near to but slightly larger than $n(m-1)/(n-m)$, namely

$$\ell = \frac{m-1}{1-|\log\varepsilon|^{-1}}\Big(\frac{n}{n-m} - \frac{1}{|\log\varepsilon|}\Big),$$

with ε so small that $|\log\varepsilon|>1$. Then

$$\vartheta = (\ell-m+1)^{-1} = \frac{n-m}{m(m-1)}\big(1-|\log\varepsilon|^{-1}\big), \quad \text{and}$$
$$\Big(\frac{n-m}{m-1}\ell - n\Big)^{-1} = \frac{|\log\varepsilon|-1}{m}.$$

Inserting (53), (54), (55), (57) into (56) now gives, after a little calculation,

$$\varepsilon\alpha^{(p-m)\vartheta-\varepsilon(n-m)/m} \leqslant A_1|\log\varepsilon|^{\vartheta}$$

where $A_1 = A_1(m,n)$; hence in turn,

$$\varepsilon\alpha^{m/(m-1)-\rho/(m-1)} \leqslant A_1|\log\varepsilon|^{(n-m)/m(m-1)}$$

with $\rho = m|\log\varepsilon|^{-1} + \varepsilon(n-m)$.

For suitably small ε, say $\varepsilon\leqslant\varepsilon_0$, one then obtains (compare Lemma 5)

$$\alpha \leqslant \Big(\frac{A_1}{\varepsilon}\Big)^{2(m-1)/m}. \tag{58}$$

As before this implies that α^{ε} and $\alpha^{1/|\log\varepsilon|}$ are bounded, that is, α^{ρ} is bounded, from which the lemma follows at once, subject of course to the previous restrictions given for ε. □

From (58) it follows that $\alpha^\varepsilon \to 1$ as $\varepsilon \to 0$, just as in the case $n > m^2$. In turn (49) holds exactly as before, with $C_3 \to 1$ as $\varepsilon \to 0$.

For the next conclusion, we shall need a sharper form for the behavior of C_3. First, it is not difficult to verify that the function $C_1 = C_1(\varepsilon)$ defined in (37) satisfies

$$C_1 \leqslant 1 + c\varepsilon|\log\varepsilon|$$

for some constant $c > 0$; we understand here and in what follows that c denotes a generic positive constant, depending only on m and n. Moreover, by (29) we have $\eta < c\varepsilon$, so the function $C_2 = C_2(\varepsilon)$ defined in (41) also satisfies

$$C_2 \leqslant 1 + c\varepsilon|\log\varepsilon|.$$

Finally

$$C_3 = C_2\alpha^{\varepsilon/(m-1)} \leqslant 1 + c\varepsilon|\log\varepsilon| \tag{59}$$

for sufficiently small ε.

Next, let $R > 0$ denote the unique value of r where $z(R) = \nu\varepsilon|\log\varepsilon|$, where $\nu > 0$ is a constant to be determined later; note in particular that $R \to \infty$ as $\varepsilon \to 0$. Now, arguing from (39) and the fact that

$$1 < C_3 < 1 + c\varepsilon|\log\varepsilon|,$$

we infer

$$\begin{aligned} w(r) &> C_3 z(r) - (C_3 - 1)\frac{z(r)}{z(R)} > \left(1 - \frac{C_3 - 1}{\nu\varepsilon|\log\varepsilon|}\right) z(r) \\ &\geqslant \left(1 - \frac{c}{\nu}\right) z(r) \quad \forall r \in [0, R]. \end{aligned}$$

In turn, fixing ν sufficiently large,

$$w(r) \geqslant \frac{1}{2} z(r) \quad \forall r \in [0, R]. \tag{60}$$

We can now prove a companion result to (29); in particular, it shows that Lemma 6 does *not* hold when $n \leqslant m^2$.

LEMMA 8. – *There exists $K_1 = K_1(m, n) > 0$ such that for ε sufficiently small*

$$\varepsilon\alpha^{m/(m-1)} \geqslant K_1|\log\varepsilon|^{(n-m^2)/m(m-1)} \quad \textit{when } m < n < m^2$$

and

$$\varepsilon\alpha^{m/(m-1)} \geqslant K_1|\log\varepsilon| \quad \textit{when } n = m^2.$$

Proof. – Assume first that $n < m^2$. Then for ε sufficiently small there holds

$$
\begin{aligned}
\hat{d}_1 &\geqslant \int_{\mathbb{R}^n} \hat{z}^p \geqslant \int_{\mathbb{R}^n} w^p && \text{by (34)}\\
&\geqslant \frac{c}{\omega}\int_{\mathbb{R}^n} w^m && \text{by Lemma 3}\\
&\geqslant \frac{c}{\omega}\int_{|x|<R} z^m && \text{by (60)}\\
&\geqslant \frac{c}{\omega}\int_1^R \frac{t^{n-1}}{t^{m(n-m)/(m-1)}}\,\mathrm{d}t && \text{by (33)}\\
&= \frac{c}{\omega}\bigl\{R^{(m^2-n)/(m-1)}-1\bigr\}\\
&\geqslant \frac{c}{\omega}(\varepsilon|\log\varepsilon|)^{-(m^2-n)/(n-m)},
\end{aligned}
$$

where the last inequality is obtained by solving $z(R) = \nu\varepsilon|\log\varepsilon|$ (ε small). Rearranging with the help of the relation $\omega = \varepsilon\alpha^{p-m} \leqslant \varepsilon\alpha^{m^2/(n-m)}$ now yields the first statement of the lemma.

If $n = m^2$, the same arguments lead to

$$
\hat{d}_1 \geqslant \frac{c}{\omega}\int_1^R \frac{\mathrm{d}t}{t} = \frac{c}{\omega}\log R \geqslant \frac{c}{\omega}|\log\varepsilon|,
$$

from which the second statement follows at once. □

Lemma 8 shows at once that (4) also holds in the case $m < n \leqslant m^2$, that is whenever $n > m$.

Remark. – As already mentioned in the introduction, more precision in the asymptotic behavior of $u(0)$ is needed in the case $n \leqslant m^2$. We conjecture that also in this case there exists a continuous increasing function $g_{m,n}$ defined on $[0,\infty)$ such that $g_{m,n}(0) = 0$ and $\lim_{\varepsilon\to 0}[g_{m,n}(\varepsilon)u(0)] = 1$.

5.3. Dirac limits

Here we shall complete the demonstration of Theorem 2 by proving conditions (5) and (6). It will be convenient here and in the sequel *not to make* the initial assumption $\delta = 1$, though we continue to write $u(0) = \alpha$.

From Section 5.1 we recall the basic estimate (49); with the help of (59) this can be rewritten in the form

$$
0 < z - w < c\varepsilon|\log\varepsilon|. \tag{61}
$$

Here we wish to scale back to the original function u, this being accomplished by means of (26) and (30). More specifically, in (30) it is necessary to replace u and α respectively by v and β (β as in (2)) because of the initial assumption in Section 5 that $\delta = 1$. The required rescaling is therefore given by

$$w(r) = \frac{1}{\delta^{1/(p-m)}\beta} u\left(\frac{r}{\delta^{1/m}\beta^{(p-m)/m}}\right) = \frac{1}{\alpha} u\left(\frac{r}{\alpha^{(p-m)/m}}\right) \tag{62}$$

where from Theorem 1 we have $\delta^{1/(p-m)}\beta = \alpha$. After a little calculation, (61) then leads to the basic formula

$$0 < z_\alpha - u \leqslant c\alpha\varepsilon|\log\varepsilon|, \tag{63}$$

where

$$\begin{aligned} z_\alpha &= z_\alpha(x) = \alpha z\big(\alpha^{(p-m)/m}|x|\big) \\ &= \alpha/\big[1 + (1-\eta)^{1/(m-1)}\alpha^{(p-m)/(m-1)} D|x|^{m/(m-1)}\big]^{(n-m)/m} \end{aligned} \tag{64}$$

and (33) is used at the last step.

Observe from the left hand inequality of (63) that (recall $\eta \to 0$ as $\varepsilon \to 0$)

$$\alpha^{1/(m-1)}u(x) < \alpha^{1/(m-1)}z_\alpha(x) \to D^{-\frac{n-m}{m}}|x|^{-\frac{n-m}{m-1}} \quad \text{as } \varepsilon \to 0,$$

which immediately yields (5).

To prove (6), let $X = X_R$ denote the Lebesgue space L^{m^*} over the domain $\{|x| < R\}$, and similarly let $X' = X'_R$ be the space L^{m^*} over the domain $\{|x| \geqslant R\}$. By Minkowski's inequality and (63),

$$\big|\|u\|_X - \|z_\alpha\|_X\big| \leqslant \|u - z_\alpha\|_X \leqslant c\alpha\varepsilon|\log\varepsilon|\|1\|_X. \tag{65}$$

In particular, let us make the new choice

$$R = \alpha^{-m/(n-m)+\mu},$$

where $\mu > 0$ is a positive constant to be determined later. Then with the obvious change of variables $s = \alpha^{(p-m)/m}r$, we find

$$\|z_\alpha\|_X^{m^*} = \omega_n\alpha^{\varepsilon n/m} \int_0^{\alpha^{-\varepsilon/m+\mu}} \frac{s^{n-1}\,\mathrm{d}s}{[1 + (1-\eta)^{1/(m-1)}Ds^{m/(m-1)}]^n} \to \gamma_{m,n} \tag{66}$$

as $\varepsilon \to 0$, see (52) and (48) (which as shown in Section 5.2 is valid for all $n > m$). By the same calculation

$$\|z_\alpha\|_{X'}^{m^*} \to 0 \tag{67}$$

as $\varepsilon \to 0$, since the integration is now over the interval $(\alpha^{-\varepsilon/m+\mu}, \infty)$ and the integral is convergent.

Next, one calculates that

$$\|1\|_X = \frac{\omega_n}{n}R^{n/m^*} = \frac{\omega_n}{n}\alpha^{-1+\mu(n-m)/m}$$

in view of the definition of R. We can now determine the limit as $\varepsilon \to 0$ of the quantity

$$\alpha\varepsilon|\log\varepsilon|\|1\|_X = (\omega_n/n)\alpha^{\mu(n-m)/m}\varepsilon|\log\varepsilon|.$$

From Lemmas 6 and 7 it is evident that, whatever the case considered, there exists $\lambda > 0$ (depending only on m, n) such that $\alpha < c\varepsilon^{-\lambda}$, provided ε is small. (One can check that $\lambda = (n-m)/m^2 + 1$ in fact suffices.) Hence

$$\alpha^{\mu(n-m)/m}\varepsilon|\log\varepsilon| \leqslant c\varepsilon^{1-\lambda\mu(n-m)/m}|\log\varepsilon|,$$

which tends to 0 as $\varepsilon \to 0$ if μ is chosen small enough. It now follows at once from (65) and (66) that $\|u\|_X^{m^*} \to \gamma_{m,n}$ as $\varepsilon \to 0$.

We observe finally from the left hand inequality of (63) that

$$\|u\|_{X'}^{m^*} < \|z_\alpha\|_{X'}^{m^*} \to 0$$

by (67). Hence

$$\|u\|_{m^*}^{m^*} = \|u\|_X^{m^*} + \|u\|_{X'}^{m^*} \to \gamma_{m,n},$$

proving the second part of (6).

To obtain the first part, note that integration of (P_p^δ) over $\mathbb{R}^n$ and use of Theorem 8 yields

$$\delta \int_{\mathbb{R}^n} u^{m-1} = \int_{\mathbb{R}^n} u^{p-1}. \tag{68}$$

But from the left inequality of (63) together with a calculation as in (66), we have

$$\int_{\mathbb{R}^n} u^{p-1} \leqslant \int_{\mathbb{R}^n} z_\alpha^{p-1} = \omega_n \alpha^{-1+\varepsilon(n-m)/m} \int_0^\infty \frac{s^{n-1}\,\mathrm{d}s}{[1+(1-\eta)^{1/(m-1)}Ds^{m/(m-1)}]^{(n-m)(p-1)/m}}.$$

Since the integral is uniformly bounded for any ε less than $m/2(n-m)$, we then get

$$\int_{\mathbb{R}^n} u^{p-1} \to 0 \quad \text{as } \varepsilon \to 0.$$

With the help of (68) (and a trivial interpolation) this completes the proof of (6), and therefore of Theorem 2.

6. Proof of Theorem 3

First we prove (8). Multiplying the equation (P_p^δ) by u and integrating over $\mathbb{R}^n$ gives

$$\int_{\mathbb{R}^n} |\nabla u|^m = -\delta \int_{\mathbb{R}^n} u^m + \int_{\mathbb{R}^n} u^p. \tag{69}$$

We now let $\varepsilon \to 0$. The first term on the right approaches 0 by (6).

To treat the second term on the right side of (69), we slightly modify the space X from its meaning in the previous subsection, so that now it represents the Lebesgue space L^p over the domain $\{|x| < R\}$, and similarly for the space X'. Then as in (66) there holds

$$\|z_\alpha\|_X^p = \omega_n \alpha^{\varepsilon(n-m)/m} \int_0^{\alpha^{-\varepsilon/m+\mu}} \frac{s^{n-1}\,\mathrm{d}s}{[1+(1-\eta)^{1/(m-1)} D s^{m/(m-1)}]^{n-\varepsilon(n-m)/m}},$$

the integral being convergent when $\varepsilon < m/(n-m)$. To evaluate the limit of the right side, note first that on the interval $0 < s < \alpha^\mu$ there holds (for small ε)

$$1 < \left[1+(1-\eta)^{1/(m-1)} D s^{m/(m-1)}\right]^{\varepsilon(n-m)/m} < \alpha^{\varepsilon\mu n/(m-1)},$$

so that by (48), uniformly for $s \in (0, \alpha^\mu)$,

$$\left[1+(1-\eta)^{1/(m-1)} D s^{m/(m-1)}\right]^{\varepsilon(n-m)/m} \to 1.$$

Hence as in (66), one obtains $\|z_\alpha\|_X^p \to \gamma_{m,n}$ as $\varepsilon \to 0$. Also as before, $\|z_\alpha\|_{X'}^p \to 0$, so that finally, again arguing as in the previous subsection,

$$\|u\|_p^p = \|u\|_X^p + \|u\|_{X'}^p \to \gamma_{m,n},$$

that is, $\int_{\mathbb{R}^n} u^p \to \gamma_{m,n}$. The second statement in (8) follows at once from (69). In order to prove the first statement in (8), note that by (62) we have

$$\int_{\mathbb{R}^n} |\nabla u|^q = c\alpha^{pq/m} \int_0^\infty |w'(\alpha^{(p-m)/m} r)|^q r^{n-1}\,\mathrm{d}r \quad \forall q \geqslant 1;$$

note also that $z \in \mathcal{D}^{1,q}(\mathbb{R}^n)$ for all $q > n(m-1)/(n-1)$ and that $\|\nabla z\|_q$ remains bounded as $\varepsilon \to 0$: therefore, by (41) and an obvious change of variables, we obtain

$$\int_{\mathbb{R}^n} |\nabla u|^q \leqslant c\alpha^{p(q-n)/m+n} \int_0^\infty |z'(r)|^q r^{n-1}\,\mathrm{d}r \leqslant c\alpha^{p(q-n)/m+n} \to 0 \quad \forall q \in \left(m, \frac{n(m-1)}{n-1}\right)$$

which completes the proof of (8).

It remains to prove (7). By evaluating $z'(r)$ and by using (41) and (59) we obtain

$$\begin{aligned} |w'(r)| \leqslant\ & (1+c\varepsilon|\log\varepsilon|)\frac{n-m}{m-1}(1-\eta)^{1/(m-1)} \\ & \times D\frac{r^{1/(m-1)}}{(1+(1-\eta)^{1/(m-1)} D r^{m/(m-1)})^{n/m}}. \end{aligned} \tag{70}$$

Moreover, according to the “double rescaling” (62) we have

$$|w'(r)| = \frac{1}{\alpha^{p/m}} \left| u'\left(\frac{r}{\alpha^{(p-m)/m}}\right)\right|.$$

Inserting this in (70), using an obvious change of variables and then letting $\varepsilon \to 0$, yields

$$\lim_{\varepsilon\to 0}\{\alpha^{1/(m-1)}|u'(r)|\} \leqslant \left(\frac{n-m}{m-1}\right)^{n/m} n^{(n-m)/m(m-1)} r^{(1-n)/(m-1)},$$

which immediately gives (7) since $\alpha = u(0)$.

7. Proof of Theorem 4

We define

$$\tau(\varepsilon) = \tau(\varepsilon, d) = \frac{1}{\varepsilon}\left(\frac{d}{\beta}\right)^{p-m},$$

where β is given by (2); here β is a (well-defined) continuous function of ε and of course also of m, n. By Theorem 1, when $\delta = \varepsilon\tau(\varepsilon)$ we have

$$u(0) = \delta^{1/(p-m)}\beta = d,$$

proving (ii). Also by Theorem 2 we know that when $n > m^2$ (case $\delta = 1$)

$$\varepsilon^{(n-m)/m^2}\beta \to \beta_{m,n} \quad \text{as } \varepsilon \to 0,$$

so that

$$\tau(\varepsilon) = \left(\frac{d}{\varepsilon^{(n-m)/m^2}\beta}\right)^{p-m} \cdot \varepsilon^{-\varepsilon(n-m)/m^2} \to \left(\frac{d}{\beta_{m,n}}\right)^{m^2/(n-m)}$$

as $\varepsilon \to 0$; similarly, when $n \leqslant m^2$, by Theorem 2 we infer that $\tau(\varepsilon) \to 0$ as $\varepsilon \to 0$. Statement (i) is so proved.

To prove the final statement of the theorem, we first use (63), together with the fact that in the present case $\alpha = u(0) = d$, to infer the fundamental relation

$$|u - z_d| \leqslant cd\varepsilon|\log\varepsilon|. \tag{71}$$

But by (64), and since $\eta \to 0$ as $\varepsilon \to 0$, it now follows that

$$z_d(x) \to d\left[1 + D\left(d^{\frac{m}{n-m}}|x|\right)^{\frac{m}{m-1}}\right]^{-\frac{n-m}{m}} \equiv U_d(x)$$

uniformly for x in $\mathbb{R}^n$; see (1) in the introduction. Together with (71) this completes the proof of (ii).

An easy consequence of the above argument is the following companion result for Theorem 4.

COROLLARY. – *Let $n > m^2$. In place of the condition $\delta = \varepsilon\tau(\varepsilon)$, suppose that $\delta = a\varepsilon$, where a is a positive constant. Then $u \to U_d$ uniformly on $\mathbb{R}^n$ as $\varepsilon = p - m \to 0$, where $d = a^{(n-m)/m^2}\beta_{m,n}$.*

Acknowledgement

The second author wishes to thank Prof. Grozdena Todorova for many valuable and helpful conversations during the preparation of the paper.

REFERENCES

[1] Atkinson F.V., Peletier L.A., Ground states of $-\Delta u = f(u)$ and the Emden–Fowler equation, Arch. Rational Mech. Anal. 93 (1986) 103–127.
[2] Atkinson F.V., Peletier L.A., Emden–Fowler equations involving critical exponents, Nonlinear Anal. TMA 10 (1986) 755–776.
[3] Atkinson F.V., Peletier L.A., Elliptic equations with nearly critical growth, J. Differential Equations 70 (1987) 349–365.
[4] Berestycki H., Lions P.L., Nonlinear scalar field equations, I, Existence of a ground state, Arch. Rational Mech. Anal. 82 (1983) 313–345.
[5] Brezis H., Peletier L.A., Asymptotics for elliptic equations involving critical exponents, in: Partial Differential Equations and Calculus of Variations, Birkhäuser, 1989, pp. 149–192.
[6] Citti G., Positive solutions of quasilinear degenerate elliptic equations in $\mathbb{R}^n$, Rend. Circolo Mat. Palermo 35 (1986) 364–375.
[7] Franchi B., Lanconelli E., Serrin J., Existence and uniqueness of nonnegative solutions of quasilinear equations in $\mathbb{R}^n$, Advances in Math. 118 (1996) 177–243.
[8] García Azorero J.P., Peral Alonso I., On limits of solutions of elliptic problems with nearly critical exponent, Comm. Partial Differential Equations 17 (1992) 2113–2126.
[9] Gazzola F., Critical growth quasilinear elliptic problems with shifting subcritical perturbation, Diff. Int. Eq. 14 (2001) 513–528.
[10] Gazzola F., Serrin J., Tang M., Existence of ground states and free boundary problems for quasilinear elliptic operators, Adv. Diff. Eq. 5 (2000) 1–30.
[11] Knaap M.C., Peletier L.A., Quasilinear elliptic equations with nearly critical growth, Comm. Partial Differential Equations 14 (1989) 1351–1383.
[12] Ni W.M., Serrin J., Nonexistence theorems for quasilinear partial differential equations, Rend. Circolo Mat. Palermo (Centenary Supplement), Series II 8 (1985) 171–185.
[13] Ni W.M., Serrin J., Existence and nonexistence theorems for ground states of quasilinear partial differential equations. The anomalous case, Accad. Naz. dei Lincei, Atti dei Convegni 77 (1986) 231–257.
[14] Pucci P., Serrin J., Uniqueness of ground states for quasilinear elliptic operators, Indiana Univ. Math. J. 47 (1998) 501–528.
[15] Rey O., Proof of two conjectures of H. Brezis and L.A. Peletier, Manuscripta Math. 65 (1989) 19–37.
[16] Rey O., The role of Green's function in a nonlinear elliptic equation involving the critical Sobolev exponent, J. Funct. Anal. 89 (1990) 1–52.
[17] Serrin J., Tang M., Uniqueness of ground states for quasilinear elliptic equations, Indiana Univ. Math. J. 49 (2000) 897–923.
[18] Talenti G., Best constant in Sobolev inequality, Ann. Mat. Pura Appl. 110 (1976) 353–372.