Ann. I. H. Poincaré – AN 19, 6 (2002) 927–960

S0294-1449(02)00107-5/FLA

UNIQUENESS AND STABILITY OF REGIONAL BLOW-UP IN A POROUS-MEDIUM EQUATION

UNICITÉ ET STABILITÉ DE L'EXPLOSION RÉGIONALE DANS UNE ÉQUATION DES MILIEUX POREUX

Carmen CORTÁZAR [a,1], **Manuel DEL PINO** [b,2], **Manuel ELGUETA** [a,1]

[a] *Departamento de Matemáticas, Pontificia Universidad Católica de Chile, Casilla 306, Correo 22, Santiago, Chile*

[b] *Departamento de Ingeniería Matemática, Universidad de Chile, and Centro de Modelamiento Matemático, UMR2071 CNRS-UChile, Casilla 170, Correo 3, Santiago, Chile*

Received 22 January 2001, revised 28 June 2001

ABSTRACT. – We study the blow-up phenomenon for the porous-medium equation in $\mathbf{R}^N$, $N \geqslant 1$,

$$u_t = \Delta u^m + u^m,$$

$m > 1$, for nonnegative, compactly supported initial data. A solution $u(x,t)$ to this problem blows-up at a finite time $\bar{T} > 0$. Our main result asserts that there is a finite number of points $x_1, \dots, x_k \in \mathbf{R}^N$, with $|x_i - x_j| \geqslant 2R^*$ for $i \neq j$, such that

$$\lim_{t \to \bar{T}} (\bar{T} - t)^{\frac{1}{m-1}} u(t,x) = \sum_{j=1}^{k} w_*(|x - x_j|).$$

Here $w_*(|x|)$ is the unique nontrivial, nonnegative compactly supported, radially symmetric solution of the equation $\Delta w^m + w^m - \frac{1}{m-1} w = 0$ in $\mathbf{R}^N$ and R^* is the radius of its support. Moreover $u(x,t)$ remains uniformly bounded up to its blow-up time on compact subsets of $\mathbf{R}^N \setminus \bigcup_{j=1}^{k} \bar{B}(x_j, R^*)$. The question becomes reduced to that of proving that the ω-limit set in the problem $v_t = \Delta v^m + v^m - \frac{1}{m-1} v$ consists of a single point when its initial condition is nonnegative and compactly supported.

E-mail addresses: ccortaza@mat.puc.cl (C. Cortázar), delpino@dim.uchile.cl (M. del Pino), melgueta@mat.puc.cl (M. Elgueta).

[1] Supported by Fondecyt grant 1000625.

[2] Supported by grants Lineas Complementarias Fondecyt 8000010 and Fondap Matematicas Aplicadas.

AMS classification: 35K65; 35B40

RÉSUMÉ. – Nous étudions le phénomène d'explosion pour l'équation des milieux poreux dans $\mathbf{R}^N$, $N \geqslant 1$,

$$u_t = \Delta u^m + u^m,$$

$m > 1$, pour une donnée initiale positive ou nulle, à support compact. Une solution $u(x,t)$ de ce problème explose en temps fini $\bar{T} > 0$. Notre principal résultat établit qu'il existe un nombre fini de points $x_1, \dots, x_k \in \mathbf{R}^N$, with $|x_i - x_j| \geqslant 2R^*$ avec $i \neq j$, tels que

$$\lim_{t \to \bar{T}} (\bar{T} - t)^{\frac{1}{m-1}} u(t,x) = \sum_{j=1}^{k} w_*(|x - x_j|).$$

Ici $w_*(|x|)$ est l'unique solution non triviale, positive ou nulle, à support compact et à symétrie radiale de l'équation $\Delta w^m + w^m - \frac{1}{m-1} w = 0$ dans $\mathbf{R}^N$ et R^* est le rayon de son support. De plus, $u(x,t)$ reste uniformément bornée jusqu'à son temps d'explosion sur des sous-ensembles compacts de $\mathbf{R}^N \setminus \bigcup_{j=1}^{k} \bar{B}(x_j, R^*)$. La question est ramenée à la démonstration que l'ensemble ω-limite du problème $v_t = \Delta v^m + v^m - \frac{1}{m-1} v$ est constitué d'un seul point quand sa donnée initiale est positive ou nulle et à support compact.

1. Introduction

This paper deals with the description of the blow-up phenomenon in the porous-medium equation in $\mathbf{R}^N$, $N \geqslant 1$,

$$u_t = \Delta u^m + u^m, \tag{1.1}$$

$$u(x,0) = u_0(x) \tag{1.2}$$

where $m > 1$ and $u_0(x)$ is a compactly supported, not identically zero nonnegative function whose regularity will be specified later. The porous-medium equation $u_t = \Delta u^m$ and its variations represent simple dissipative models for quantities which diffuse slowly. In fact this simple degenerate equation exhibits, unlike the heat equation, finite speed of propagation, which amounts in (1.1) to the fact that the support in space variable of the solution remains bounded at all times where the solution is defined. On the other hand, the presence of the superlinear term u^m as a source makes possible finite time blow-up, for instance the space-independent solution $(m-1)^{-\frac{1}{m-1}}(T-t)^{-\frac{1}{m-1}}$. Actually, the (unique) solution of (1.1)–(1.2) always blows-up in finite time. This is the case for any nonzero initial data in the more general problem

$$u_t = \Delta u^m + u^p \tag{1.3}$$

as long as $1 < p < m + \frac{2}{N}$, as established in [12], a generalization of the classical Fujita's result for $m = 1$ [10]. The power $p = m$ is certainly special since diffusion and source share the same nonlinear growth. This gives rise to the interesting phenomenon of *regional blow-up*, meaning this blow-up taking place only in a compact set with nonempty interior. This is in sharp contrast with phenomena typically exhibited for other powers in the source term: In fact, while for $p < m$ blow-up occurs in entire space, as established in [27], for $p > m$ blow-up is expected to occur only at a lower dimensional set, and generically only at isolated points. Fine knowledge is today available on the blow-up profiles of (1.3) when $m = 1$ and $1 < p < \frac{N+2}{N-2}$, a basic problem that has been the object of extensive study, see [9,15,16,24,25,29] and the references therein. Of course if $m = p = 1$ no blow-up occurs at all, so that the phenomenon here described seems quite characteristic of porous-media type equations.

The purpose of this paper is to describe completely the blow-up in (1.1)–(1.2) in the following sense: we show that for any initial condition the solution u develops (exactly) a finite number of similar *spherical hot spots*: more precisely, there is a finite number of disjoint balls with common radii R^* outside which the solution remains uniformly bounded, while inside each of them it develops a common self-similar radially symmetric profile $(\bar{T} - t)^{-\frac{1}{m-1}} w_*(r)$, where r is the distance to the center of these balls and w_* is a strictly positive function. Moreover, we show that one-ball blow-up is stable in the sense that for a given initial data leading to one-ball blow-up, all neighboring data exhibit the same phenomenon, with blow up taking place "approximately" in the same ball. While k-ball blow-up is in general unstable, it becomes stable within the class of initial data leading to blow-up with exactly k balls.

The presence of regional blow-up in this equation was first observed and studied in the case $N = 1$ in [11]. The elliptic problem found when searching by separation of variables a solution of the form

$$u(x,t) = (\bar{T} - t)^{-\frac{1}{m-1}} \theta(x)$$

has been studied for radial symmetry in [1–3,17].

This paper is a continuation of our previous work [4] where the following partial result was established: Let $\bar{T} > 0$ be the time at which blow-up occurs. Let t_n be any sequence $t_n \uparrow \bar{T}$. Then there is a subsequence of t_n which we still denote t_n, and a nontrivial compactly supported solution $w(x)$ of the elliptic equation

$$\Delta w^m + w^m - \frac{1}{m-1} w = 0, \tag{1.4}$$

such that

$$(\bar{T} - t_n)^{\frac{1}{m-1}} u(x, t_n) \to w(x)$$

uniformly. On the other hand, it was established in [2] that the components of the support of w are balls of the same radii and that the solution is radially symmetric inside each of them. This radially symmetric solution turns out to be unique, as established in [3]. Let $BU(u_0)$ be the set of *blow-up points* of u, namely the set of points x for which there are sequences $x_n \to x$ and $t_n \to \bar{T}$ such that $u(x_n, t_n) \to +\infty$. It was also shown in [4] that this set is compact and it is precisely constituted by the union of the supports of all possible limiting w's. The important point unsolved in [4] was whether there is an *actual unique blow-up profile*, rather than oscillation between different limiting configurations.

The question turns out to be rather subtle, and we answer it affirmatively in the following result.

THEOREM 1.1. – *Let $u(x,t)$ be the solution of* (1.1)–(1.2)*, where $u_0(x)$ is compactly supported, continuous and such that $u_0^m \in H^1(\mathbf{R}^N)$. Let $\bar{T} > 0$ be the blow-up time of this solution. Then there are points $x_1, \dots, x_k \in \mathbf{R}^N$ such that*

$$\lim_{t \to \bar{T}} (\bar{T} - t)^{\frac{1}{m-1}} u(t,x) = \sum_{j=1}^{k} w_*(|x - x_j|)$$

uniformly. Here $w_*(|x|)$ is the unique compactly supported, radially symmetric solution of (1.4)*. If R^* is the radius of its support then we also have $|x_i - x_j| \geqslant 2R^*$ for $i \neq j$. Moreover $u(x,t)$ remains uniformly bounded up to its blow-up time on compact subsets of $\mathbf{R}^N \setminus \bigcup_{j=1}^k \bar{B}(x_j, R^*)$. In other words,*

$$BU(u_0) = \bigcup_{j=1}^{k} \bar{B}(x_j, R^*).$$

In light of this result, it is natural to ask for stability of the regional blow-up phenomenon. The one-ball blow-up turns out to be stable in the following sense.

THEOREM 1.2. – *Assume that u_0 compactly supported, with support contained in $B(0,M)$, is such that $k = 1$ in the above theorem, let us say*

$$BU(u_0) = \bar{B}(\bar{x}, R^*).$$

Then, given $\varepsilon > 0$, there exists a $\delta > 0$ such that for any u_1 continuous, compactly supported in $B(0,M)$ which besides satisfies

$$\left\| u_0^m - u_1^m \right\|_{H^1(\mathbf{R}^N)} < \delta,$$

we have that

$$BU(u_0) = \bar{B}(\bar{x}_1, R^*),$$

for some point $\bar{x}_1$ with $|\bar{x} - \bar{x}_1| < \varepsilon$.

In [4] it is established that the solution $u(x,t)$ is decreasing in the radial direction outside the smallest ball which contains the support of u_0. From here and the fact that w_* is radially decreasing, it follows that a sufficient condition for one-ball blow-up to occur is that this support lies inside a ball of radius less than R^*.

Instead, the two-ball blow-up is not stable as the following example shows. Let us fix points x_1 and x_2 with $|x_1 - x_2| > 2R^*$. Then the function

$$u(x,t) = (\bar{T}_1 - t)^{-\frac{1}{m-1}} w_*(|x - x_1|) + (\bar{T}_2 - t)^{-\frac{1}{m-1}} w_*(|x - x_2|)$$

solves Eq. (1.1) for $0 < t < \min\{\bar{T}_1, \bar{T}_2\}$. If $\bar{T}_1 = \bar{T}_2$, then two-ball blow-up takes place, which however disappears as soon as $\bar{T}_1$ and $\bar{T}_2$ differ, no matter how close they are.

This example suggests that one-ball blow-up may actually hold for "generic" initial data. k-ball blow-up is however stable *within* the class of initial data leading to blow-up in k balls, class which may be conjectured to be a codimension $k-1$ manifold.

THEOREM 1.3. – *Assume that u_0 is compactly supported in $B(0, M)$ and such that*

$$BU(u_0) = \bigcup_{i=1}^{k} \bar{B}(x_i, R^*).$$

Let u_1 be another initial condition supported in $B(0, M)$, with

$$BU(u_1) = \bigcup_{i=1}^{k} \bar{B}\left(x_i^1, R^*\right).$$

Then, given $\varepsilon > 0$, there exists a $\delta > 0$ such that if $\|u_0^m - u_1^m\|_{H^1(\mathbf{R}^N)} < \delta$, then $|x_i - x_i^1| < \varepsilon$ for all $i = 1, \ldots, k$.

It is worthwhile mentioning the apparent analogy of this stability-unstability phenomenon with that for single and multiple-point blow-up in (1.3) for $m = 1$, $1 < p < \frac{N+2}{N-2}$, analyzed in the works [9,24,25].

Now that the blow-up set is fully characterized, further questions arise, for instance that of finding the exact behavior of the solution on the boundary of the balls determining the blow up. On the other hand, the solution remains bounded up to the blow-up instant outside these balls, so one may wonder whether the solution keeps evolving in some sense after blow-up occurs. The general question of "continuation after blow-up" has been treated for a class of related equations in [14].

Next we describe the proof of the above results. Let us introduce the change of variables

$$v(x,t) = (\bar{T} - \tau)^{\frac{1}{m-1}} u(x,\tau)\big|_{\tau = \bar{T}(1-\mathrm{e}^{-t})}. \tag{1.5}$$

It is readily checked that v satisfies the equation

$$v_t = \Delta v^m + v^m - \frac{1}{m-1} v, \tag{1.6}$$

$$v(x,0) = \bar{T}^{\frac{1}{m-1}} u_0(x). \tag{1.7}$$

From Proposition 4.1 in [4], we know that given a sequence $t_n \to +\infty$ there is a subsequence, which we denote in the same way, and a nontrivial, compactly supported solution of (1.4) so that

$$v(x,t_n)^m \to w(x)^m \quad \text{as } n \to \infty,$$

both in uniform and H^1-senses. Thus our task in establishing Theorem 1.1 is precisely to prove that the limit $w(x)$ is actually the same along *every* sequence $t_n \to +\infty$. A main feature of Eq. (1.6) is the presence of a Lyapunov functional for it, namely

$$J(z) = \frac{1}{2}\int \left(|\nabla z^m|^2 - z^{2m}\right) \mathrm{d}x + \frac{m}{m^2-1}\int z^{m+1}\,\mathrm{d}x. \tag{1.8}$$

In fact we have that the application $s \mapsto J(v(\cdot,t))$ is decreasing on $t > 0$ and

$$\lim_{t\to+\infty} J\big(v(\cdot,t)\big) = J(w).$$

Here and in what follows the integral symbol without limits specified means integration on the whole $\mathbf{R}^N$. The presence of this functional implies that limit points of the trajectory must be steady states. The problem of uniqueness of asymptotic limits in nonlinear heat equations under the presence of a Lyapunov functional has been analyzed in a number of works. A general result due to L. Simon [28] shows the uniqueness of the limit for uniformly parabolic equations in the case of uniform real analytic data on a compact manifold. Uniform analiticity cannot be lifted in general in this result, at least in the nonautonomous setting, as shown in [26]. Needless to say, the compactly supported setting we deal with makes our situation highly nonanalytic.

Other uniqueness results in parabolic problems, nondegenerate and degenerate, are contained in the works [5–8,13,19,18,22]. In the latter work, a renormalization method based on L. Simon's ideas, used in classifying singularities in an elliptic problem in [20] was adapted to a semilinear heat equation. The general framework of this method is what we will use here. Alternative methods for degenerate equations of porous-medium type, in one and higher dimensions, have been devised in [7,8]. Those techniques do not apply to the nonlinearity of Eq. (1.7), in particular those in [8], based on analiticity, because of the presence of compactly supported steady states. This is explicitly commented in [8] and posed as an open question.

A main technical difficulty arises when the support of the limit w contains balls that are tangent, for this introduces a noise in the analysis which is rather delicate to get rid of.

In the next section we explain in further detail our method. In particular we derive the main results as corollaries of a more general fact, Theorem 2.1 whose proof is carried out by means of the renormalization approach mentioned above, whose basic scheme is set up by Proposition 2.1. The remaining sections of the paper are devoted to the proof of that intermediate result.

2. Proof of the main results

The theorems stated in the previous section will be direct consequences of the following result:

THEOREM 2.1. – *Given numbers $M > 0$, $K > 0$, there exist numbers $T > 0$ and $t^* > 0$ such that given $\varepsilon > 0$, there exists a $\delta > 0$ with the following property: Let v be any globally defined solution of* (1.7) *in* $[0,\infty)$ = *such that*

$$\operatorname{supp}\big(v(\cdot,0)\big) \subset B(0,M), \tag{2.1}$$

$$\big\|v(\cdot,0)^m\big\|_{L^\infty(\mathbf{R}^N)} + \big\|v(\cdot,0)^m\big\|_{H^1(\mathbf{R}^N)} \leqslant K, \tag{2.2}$$

and

$$\lim_{t\to\infty} J\big(v(\cdot,t)\big) = kJ(w_*).$$

Then the following holds: *If* w *is a compactly supported steady state of* (1.7) *such that for some* $t_0 > t^*$

$$\left(\sup_{t_0 \leqslant t < t_0+T} \int \big(v(x,t)^m - w(x)^m\big)\big(v(x,t) - w(x)\big)\,\mathrm{d}x\right)^{1/2} < \delta,$$

then

$$\left(\sup_{t_0 \leqslant t < \infty} \int \big(v(x,t)^m - w(x)^m\big)\big(v(x,t) - w(x)\big)\,\mathrm{d}x\right)^{1/2}\Bigg)^{1/2} < \varepsilon.$$

The proof of this result is based on the following

PROPOSITION 2.1. – *There exist positive numbers* T, δ, C *and* t^**with the following property. Let* $v(x,t)$ *be any solution of* (1.7), *defined in* $0 < t < \infty$ *as in the statement of the above theorem. Consider points* $x_1, \ldots, x_k$ *and set*

$$w(x) = \left(\sum_{i=1}^{k} w_*(|x - x_i|)^m\right)^{1/m}.$$

Assume that $t_0 > t^*$ *is such that*

$$\eta = \left(\sup_{t_0 \leqslant t \leqslant t_0+T} \int \big(v(x,t)^m - w(x)^m\big)\big(v(x,t) - w(x)\big)\,\mathrm{d}x\right)^{1/2} < \delta.$$

Then there exist points $\bar{x}_i$ *with* $|\bar{x}_i - x_i| \leqslant C\eta$ *such that*

$$\left(\sup_{t_0+T \leqslant t \leqslant t_0+2T} \int \big(v(x,t)^m - \bar{w}(x)^m\big)\big(v(x,t) - \bar{w}(x)\big)\,\mathrm{d}x\right)^{1/2} \leqslant \frac{\eta}{2},$$

where

$$\bar{w}(x) = \left(\sum_{i=1}^{k} w_*(|x - \bar{x}_i|)^m\right)^{1/m}.$$

Let us see how Theorem 2.1 follows from this assertion.

Proof of Theorem 2.1. – Let ε be given and let us write w as

$$w(x) = \sum_{i=1}^{k} w_*(|x - x_i|) = \left(\sum_{i=1}^{k} w_*(|x - x_i|)^m\right)^{1/m}$$

for points x_i with $|x_i - x_j| \geqslant 2R^*$. Let $\delta_0 < \delta$, with δ the number predicted by Proposition 2.1. Assume that for some t_0 we have

$$\eta_1 \equiv \sup_{t_0 \leqslant t \leqslant t_0+T} \left(\int \big(v(x,t)^m - w(x)^m\big)\big(v(x,t) - w(x)\big)\,\mathrm{d}x\right)^{1/2} < \delta_0$$

where T is the number given by Proposition 2.1. We find then that there are points $x_{11},\dots,x_{k1}$ with $|x_i - x_{i1}| \leqslant C\eta_1$ such that

$$\eta_2 \equiv \left(\sup_{t\in[t_0+T,t_0+2T]} \int \big(v(x,t)^m - w_1(x)^m\big)\big(v(x,t) - w_1(x)\big)\,\mathrm{d}x\right)^{1/2} \leqslant \frac{\eta_1}{2}$$

where

$$w_1(x) = \left(\sum_{i=1}^{k} w_*(|x - x_{i1}|)^m\right)^{1/m}.$$

Since $\eta_2 \leqslant \eta_1/2 < \delta$, we can apply again Proposition 2.1 to find points x_{i2} with now $|x_{i2} - x_{i1}| \leqslant C\frac{\eta_1}{2}$ such that

$$\eta_3 = \left(\sup_{t\in[t_0+2T,t_0+3T]} \int \big(v(x,t)^m - w_2(x)^m\big)\big(v(x,t) - w_2(x)\big)\,\mathrm{d}x\right)^{1/2} \leqslant \frac{\eta_2}{2} \leqslant \frac{\eta_1}{4},$$

where

$$w_2(x) = \left(\sum_{i=1}^{k} w_*(|x - x_{i2}|)^m\right)^{1/m}.$$

Iterating this procedure we find a sequence x_{ij}, $j = 1, 2, \dots,$ such that $|x_{ij} - x_{i(j-1)}| \leqslant C\frac{\eta}{2^j}$ and

$$\left(\sup_{t\in[t_0+jT,t_0+(j+1)T]} \int_{\mathbf{R}^N} \big(v(x,t)^m - w_j(x)^m\big)\big(v(x,t) - w_j(x)\big)\,\mathrm{d}x\right)^{1/2} \leqslant \frac{\eta_1}{2^j}$$

with $x_{i0} = x_i$ and

$$w_j(x) = \left(\sum_{i=1}^{k} w_*(|x - x_{ij}|)^m\right)^{1/m}.$$

The following fact is easily checked: there exists a constant $D > 0$ depending only on $m > 1$ such that for all nonnegative numbers a, b, c one has

$$\big(a^m - c^m\big)(a - c) \leqslant D\big\{\big(a^m - b^m\big)(a - b) + \big(b^m - c^m\big)(b - c)\big\}.$$

Now let t be any number greater than t_0. Then $t \in (t_0 + jT, t_0 + (j+1)T]$ for some j and

$$\begin{aligned} g(t) &\equiv \int_{\mathbf{R}^N} \big(v(x,t)^m - w(x)^m\big)\big(v(x,t) - w(x)\big)\,\mathrm{d}x \\ &\leqslant D\bigg\{\sup_{\bar t\in[t_0+jT,t_0+(j+1)T]} \int \big(v(x,\bar t)^m - w_j(x)^m\big)\big(v(x,\bar t) - w_j(x)\big)\,\mathrm{d}x \\ &\qquad + \int \big(w(x)^m - w_j(x)^m\big)\big(w(x) - w_j(x)\big)\,\mathrm{d}x\bigg\}. \end{aligned}$$

Now, we have that w_* is uniformly Hölder continuous for any $m > 1$, hence

$$\left|\left(w(x)^m - w_j(x)^m\right)\left(w(x) - w_j(x)\right)\right| \leqslant C \max_i |x_{ij} - x_i|^a$$

for some numbers $a, C > 0$. It follows that

$$g(t) \leqslant A\left\{\left(\frac{\eta_1}{2^j}\right)^2 + \max_i |x_{ij} - x_i|^a\right\},$$

for certain positive constants A, a. Finally,

$$|x_{ij} - x_i| \leqslant C \sum_{l=1}^{\infty} \eta_1 2^{-l} = C\eta_1,$$

from where it follows that

$$g(t) \leqslant A\left\{(\eta_1)^2 + C^a \eta_1^a\right\} < A\left\{\delta_0^2 + C^a \delta_0^a\right\} < \varepsilon,$$

provided that δ_0 was chosen sufficiently small. This concludes the proof. □

Proof of Theorem 1.1. – From Proposition 4.1 in [4], there is a sequence $t_n \to \infty$ such that $v(x, t_n + \tau) \to w(x)$ for some nontrivial solution of (1.4), uniformly in x and for τ in bounded intervals. We recall that the space support of v is contained inside a ball independent of the time variable. It follows that, given $\varepsilon > 0$, there exists a number $t_0 > 0$ such that

$$\eta_1 = \left(\sup_{t \in [t_0, t_0+T]} \int \left(v(x,t)^m - w(x)^m\right)\left(v(x,t) - w(x)\right) \mathrm{d}x\right)^{1/2} < \delta,$$

where T and δ are the numbers given by Theorem 2.1. Thus,

$$\left(\int \left(v(x,t)^m - w(x)^m\right)\left(v(x,t) - w(x)\right) \mathrm{d}x\right)^{1/2} < \varepsilon$$

for all $t \geqslant t_0$. Since ε is arbitrary, we have actually established that

$$\lim_{t \to +\infty} \int \left(v(x,t)^m - w(x)^m\right)\left(v(x,t) - w(x)\right) \mathrm{d}x = 0,$$

hence w is the unique limit point of the trajectory $v(\cdot, t)$, and the proof of the theorem is complete. □

Proof of Theorems 1.2 and 1.3. – Assume we are in the situation of Theorem 1.3 for the initial condition u_0. Let u_{0n} be a sequence of initial conditions such that u_{0n}^m converges uniformly and in H^1-sense to u_0^m, and with supports contained in some common ball. Let $\bar{T}_n$ be the blow-up time for u_n, the solution with initial condition u_{0n} and $\bar{T}$ that for u_0. We claim that $\bar{T}_n \to \bar{T}$. Let T^* be a limit point for $\bar{T}_n$. After scaling, we end up with the following situation: There is a globally defined solution $\tilde{v}$ of (1.7) with $\tilde{v}(0,x) = T^* u_0(x)$ which converges as $t \to \infty$ to a nontrivial steady state. But this implies, scaling back, that the blow-up time for u cannot be other than T^*. Hence $T^* = \bar{T}$. Assume now additionally that u_n as well as u have k-ball blow-up for any n. Let v_n be the solution of (1.7) defined as

$$v_n(x,s) = (\bar{T}_n - t)^{\frac{1}{m-1}} u_n(x,t)|_{t=\bar{T}_n(1-\mathrm{e}^{-s})}.$$

Then $J(v_n(\cdot,t)) \to kJ(w_*)$ as $t \to \infty$. Since $\bar{T}_n \to \bar{T}$, we will have, on the one hand, $v(x,t)$ close to $w(x)$ for all $t > t_0$, while we can also make, by continuity, $v_n(x,t)$ as close as we wish in compact intervals in time to v. As a consequence, v_n will also be close to w in an interval of length T after t_0, so that Theorem 2.1 applies in this situation as well, so proving the stated result. Finally, for Theorem 1.2 it suffices to observe that for $k = 1$ in the above situation, and an arbitrary sequence of initial conditions converging to u_0, one has that if $J(v(\cdot,\bar{t})) < \frac{3}{2}J(w_*)$ then $J(v_n(\cdot,\bar{t})) < \frac{3}{2}J(w_*)$ for all sufficiently large n. It follows then that $J(v_n(\cdot,t)) \to J(w_*)$ as $t \to +\infty$, hence one-ball blow-up holds for any initial condition close to u_0 and Theorem 1.3 applies in this situation. □

The remaining of this paper will be devoted to the proof of Proposition 2.1. To do this, we will restate it in the next section in a more convenient form.

3. Preliminaries and a key inequality

Our task in what follows is to prove Proposition 2.1. After an indirect argument, it is easy to see that this result follows from the following

PROPOSITION 3.1. – *There exist positive numbers T and C such that if $\tilde{v}_n$ is a sequence of solutions of Eq.* (1.7) *defined on* $0 \leqslant t < +\infty$, *satisfying the constraints* (2.1), (2.2), *and such that for some sequence $t_n \to +\infty$, setting*

$$v_n(x,t) \equiv \tilde{v}_n(x, t_n + t),$$

$$w_n(x) = \left(\sum_i w_*(|x - x_{in}|)^m\right)^{1/m},$$

one has

$$\eta_n \equiv \left(\sup_{t\in[0,T]} \int \bigl(v_n(x,t)^m - w_n(x)^m\bigr)\bigl(v_n(x,t) - w_n(x)\bigr)\,\mathrm{d}x\right)^{1/2} \to 0 \tag{3.1}$$

as $n \to \infty$. Assume besides that for each n,

$$\lim_{t\to\infty} J\bigl(v_n(\cdot,t)\bigr) = kJ(w_*),$$

where J is the Lyapunov functional (1.8). *Then there exist points $\bar{x}_{in}$ with $|\bar{x}_{in} - x_{in}| \leqslant C\eta_n$ and*

$$\left(\sup_{t\in[T,2T]}\int \big(v_n(x,t)^m - \bar{w}_n(x)^m\big)\big(v_n(x,t) - \bar{w}_n(x)\big)\,\mathrm{d}x\right)^{1/2} \leqslant \frac{\eta_n}{2} \tag{3.2}$$

for all n sufficiently large, and where

$$\bar{w}_n(x) = \left(\sum_i w_*(|x - \bar{x}_{in}|)^m\right)^{1/m}.$$

Let v_n be a sequence as in the statement of the above result. Then we have that $v_n(x,t)^m - w_n(x)^m \to 0$ in L^∞ and H^1-senses in $\mathbf{R}^N$, uniformly locally in time $t \in [0,\infty)$. In fact, the estimates leading to convergence up to subsequences to steady states derived in [4] depend only on bounds for the initial condition and on the radius of the smallest ball containing the initial condition (in fact the size support of the solution at all times turns out to depend only on this radius). For $T > 0$ fixed, which we will choose later, we use in what follows the following notation:

$$\phi_n(x,t) \equiv \frac{v_n(x,t) - w_n(x)}{\eta_n}. \tag{3.3}$$

Then ϕ_n satisfies the equation

$$\frac{\partial \phi_n}{\partial t} = m\Delta\big(\tilde{w}_n^{m-1}\phi_n\big) + m\tilde{w}_n^{m-1}\phi_n - \frac{1}{m-1}\phi_n + \eta_n^{-1}z_n, \tag{3.4}$$

where

$$\tilde{w}_n(x,t)^{m-1} \equiv \int_0^1 \big(w_n(x) + t\big(v_n(x,t) - w_n(x)\big)\big)^{m-1}\,\mathrm{d}t \tag{3.5}$$

and

$$z_n(x) = \sum_i w_*(|x - x_{in}|) - w_n(x).$$

Let us observe that, by definition of the number η_n we have that

$$\left(m\int w_n(x,t)^{m-1}\phi_n(x,t)^2\,\mathrm{d}x\right)^{1/2} \leqslant 1$$

for all $t \in [0,T]$. Notice also that z_n is supported only near the boundary of the balls. Indeed, in the limit w_n must converge (up to subsequences) to a steady state w of Eq. (1.4), hence the distance between the centers x_{in} must be in the limit no less than $2R^*$.

These facts suggest that on interior sets of the support of the limit $w(x)$ of the w_n, which is the uniform limit of w_n, we should see convergence in certain sense of ϕ_n to a solution of the degenerate parabolic equation

$$(\phi)_t = m\Delta\left(w^{m-1}\phi\right) + mw^{m-1}\phi - \frac{1}{m-1}\phi. \tag{3.6}$$

The proof of Proposition 2.2 has as its key element the analysis of this convergence, and in particular in finding the form of its limit. Thus we assume in what follows the validity of the conditions of Proposition 2.2. We introduce some notation, and establish in Lemma 3.1 below an inequality which will play a crucial role in the analysis of the next sections. Following the notation of the previous section we write $w_{in} = w_*(|x - x_{in}|)$, so that $w_n = (\sum_{i=1}^k w_{in}^m)^{1/m}$. Let us also consider the functions

$$\psi_n = \frac{v_n^m - w_n^m}{\eta_n}, \tag{3.7}$$

$$G_n = \frac{1}{\eta_n^2}\left\{\frac{v_n^{m+1}}{m+1} - \frac{w_n^{m+1}}{m+1} - w_n^m(v_n - w)\right\}, \tag{3.8}$$

$$H_n = \frac{(m-1)(v_n^{m+1} - w_n^{m+1}) + (m+1)(v_n w_n^m - v_n^m w_n)}{\eta_n^2}. \tag{3.9}$$

It is easily checked the existence of constants C_1 and C_2, depending only on m such that the following inequalities hold:

$$C_1 G_n \leqslant \frac{(v_n^m - w_n^m)(v_n - w_n)}{\eta_n^2} \leqslant C_2 G_n. \tag{3.10}$$

We observe that the already defined quantity

$$z_n = \sum_{i=1}^k w_{in} - w_n, \tag{3.11}$$

measures the "overlap" of the supports of the w_{in}.

We also note that there exists a constant $C > 0$, depending on k and m, such that

$$w_n \leqslant \sum_{i=1}^k w_{in} \leqslant C w_n. \tag{3.12}$$

In particular $0 \leqslant z_n \leqslant C w_n$ for another constant C.

The following relation among the above defined objects may be regarded as the key step in the proof of Proposition 2.2.

LEMMA 3.1. – *The following relation holds*:

$$\frac{1}{(m-1)\eta_n^2}\int (v_n^m - w_n^m) z_n \leqslant -\int \frac{\partial}{\partial t} G_n + \frac{1}{m^2-1}\int H_n - \frac{m}{(m^2-1)\eta_n^2}\int w_n^m z_n.$$

Proof. – Let J be the Lyapunov functional for (1.6) given by (1.8). Let us set

$$I = 2\left(J(v_n) - \sum_{i=1}^{k} J(w_{in})\right).$$

Then $I \geqslant 0$ and we have

$$I = \int \left(\left|\nabla v_n^m\right|^2 - \left|\nabla\left(w_n^m\right)\right|^2\right) - \int \left(v_n^{2m} - w_n^{2m}\right)$$
$$+ \frac{2m}{m^2-1}\int\left(v_n^{m+1} - \sum_{i=1}^{k} w_{in}^{m+1}\right) + \int\left(\sum_{i\neq j} \nabla w_{in}\cdot\nabla w_{jn}\right) - \sum_{i\neq j} w_{in}^m w_{jn}^m.$$

After integrating by parts and using the equations we get

$$I = -\int \left(\Delta v_n^m + \Delta w_n^m\right)\left(v_n^m - w_n^m\right) - \int\left(v_n^{2m} - w_n^{2m}\right)$$
$$+ \frac{2m}{m^2-1}\int\left(v_n^{m+1} - \sum_{i=1}^{k} w_{in}^{m+1}\right) - \frac{1}{m-1}\int \sum_{i\neq j} w_{in}^m w_{jn}.$$

Thus

$$I = -\int v_{nt}\left(v_n^m - w_n^m\right) - \frac{1}{m-1}\int\left(v_n + \sum_{i=1}^{k} w_{in}\right)\left(v_n^m - w_n^m\right)$$
$$- \frac{1}{m-1}\int \sum_{i\neq j} w_i^m w_j + \frac{2m}{m^2-1}\int\left(v_n^{m+1} - \sum_{i=1}^{k} w_{in}^{m+1}\right).$$

So we obtain, after recalling that $\eta_n^2 \frac{\partial G_n}{\partial t} = v_{nt}(v_n^m - w_n^m)$,

$$I = -\eta_n^2 \frac{\partial G_n}{\partial t} + \frac{1}{m-1}\int\left(v_n w_n^m - v_n^m \sum_{i=1}^{k} w_{in}\right) + \frac{1}{m+1}\int\left(v_n^{m+1} - \sum_{i=1}^{k} w_{in}^{m+1}\right).$$

Since the functional J is nonincreasing along trajectories we obtain, after some algebraic manipulations and recalling the definition of H_n and z_n, that

$$0 \leqslant -\frac{\partial G_n}{\partial t} + \frac{1}{m^2-1}\int H_n + \frac{1}{\eta_n^2}\int \frac{w_n^{m+1} - (\sum_{i=1}^{k} w_{in}^{m+1})}{m+1}$$
$$- \frac{1}{m-1}\frac{1}{\eta_n^2}\int\left(v_n^m - w_n^m\right)\left(\sum_{i=1}^{k} w_{in} - w_n\right) - \frac{1}{m-1}\int w_n^m \frac{z_n}{\eta_n^2}.$$

It follows from Lemma A.1, in the appendix, that

$$\left(w_n^{m+1} - \sum_{i=1}^{k} w_{in}^{m+1}\right) \leqslant \frac{1}{m-1}\left(w_n^m z_n\right). \tag{3.13}$$

Hence

$$0 \leqslant -\frac{\partial G_n}{\partial t} + \frac{1}{m^2-1}\int H_n - \frac{1}{m-1}\frac{1}{\eta_n^2}\int \left(v_n^m - w_n^m\right)\left(\sum_{i=1}^k w_{in} - w_n\right) - \frac{m}{m^2-1}\int w_n^m \frac{z_n}{\eta_n^2}$$

which yields the desired result. □

The result just proven plays an essential role in the analysis to follow in the next sections. In Section 4 we will obtain estimates which will lead in particular to convergence of ϕ_n in a suitable sense in Section 5. An important step is to obtain a proper control on the size of $\eta_n^{-1} z_n$, which is measuring the degree of superposition of the balls constituting the support of w_n. In Section 6 we will express the limit ϕ of the ϕ_n's in terms of eigenvalues and eigenfunctions of the elliptic operator in Eq. (3.6). The fact that the operator does not degenerate in the interior of the support of w yields strong smooth convergence there. However a finer understanding of the behavior of ϕ_n near the boundary of the support of w is needed, and found in Section 7.

It should be remarked that a particularly delicate situation is the case that w has adjacent balls in their support, since the effect of superposition is in fact present up to the limit, and that has to been taken care of. This is perhaps the main difficulty overcame in this paper.

4. Further estimates

In this section we establish, as consequences of Lemma 3.1, some important estimates which will lead in particular to convergence of ϕ_n in a suitable sense. Our first result provides a uniform control on the overlap of the supports of the w_{in} by means of the following estimate for z_n.

LEMMA 4.1. – *There is a $C > 0$ such that for all n we have*

$$\eta_n^{-2}\int w_n^m z_n \leqslant C. \tag{4.1}$$

Proof. – Integrating in time from 0 to T the inequality of Lemma 3.1 we obtain

$$0 \leqslant \int \left(G_n(x,0) - G_n(x,T)\right) + \int_0^T \int H_n - \frac{1}{m-1}\frac{1}{\eta_n^2}\int_0^T \int \left(v_n^m - w_n^m\right)\left(\sum_{i=1}^k w_{in} - w_n\right) - \frac{m}{m^2-1}\int_0^T \int w_n^m \frac{z_n}{\eta_n^2}.$$

Recalling now that, $\int G_n(x,t) \leqslant C$ for all $0 \leqslant t \leqslant T$ and that $H_n \leqslant C G_n$, we obtain

$$\frac{m}{m^2-1}\int_0^T \int w_n^m \frac{z_n}{\eta_n^2} \leqslant C - \frac{1}{m-1}\frac{1}{\eta_n^2}\int_0^T \int \left(v_n^m - w_n^m\right)\left(\sum_{i=1}^k w_{in} - w_n\right),$$

where the constant C depends now also in T. On the other hand,

$$\frac{1}{\eta_n^2}\int_0^T\int\left(v_n^m-w_n^m\right)\left(\sum_{i=1}^k w_{in}-w_n\right)\geqslant I$$

where

$$I=\frac{1}{\eta_n^2}\int_0^T\int_{\{x\,|\,v_n(x)<\sum_{i=1}^k w_{in}(x)\}}\left(v_n^m-w_n^m\right)\left(\sum_{i=1}^k w_{in}-w_n\right).$$

Now, using (3.10) and (3.12), after an application of the mean value theorem we get

$$|I|\leqslant\left|\int_0^T\int\frac{(v_n^m-w_n^m)}{\eta_n^2}(v_n-w_n)\right|^{1/2}\left|\int_0^T\int_{\{x\,|\,v(x)<\sum_{i=1}^k w_{in}(x)\}}\frac{(v_n^m-w_n^m)}{(v-w_n)}\frac{z_n^2}{\eta_n^2}\right|^{1/2}$$

$$\leqslant C\left|\int_0^T\int_{\{x\,|\,v(x)<\sum_{i=1}^k w_{in}(x)\}}m w_n^{m-1}\frac{z_n^2}{\eta_n^2}\right|^{1/2}\leqslant C\left|\int w_n^m\frac{z_n}{\eta_n^2}\right|^{1/2}.$$

Hence

$$\frac{m}{m^2-1}\eta_n^{-2}\int w_n^2 z_n\leqslant C+C\left|\int_0^T\int w_n^m\frac{z_n}{\eta_n^2}\right|^{1/2}.$$

From here the proposition immediately follows. □

Examining the above proof, now applied integrating between 0 and any $t>0$, and using the result just proved we see that we have actually established the following bound.

COROLLARY 4.1. – *We have that there is a constant C such that for all $t>0$ one has*

$$0\leqslant\int\bigl(G_n(x,0)-G_n(x,t)\bigr)$$
$$+\frac{1}{m^2-1}\int_0^t\int H_n+C\int_0^t\left(\int G_n(x,s)\,\mathrm{d}x\right)^{1/2}\mathrm{d}s+Ct+C. \tag{4.2}$$

As a further consequence, we see that since $\int G_n(x,0)\leqslant C$, and as we have said, $|H_n|\leqslant CG_n$, the following fact holds.

COROLLARY 4.2. – *There are constants $a,b>0$ such that for all n, t,*

$$\int G_n(x,t)\leqslant b\mathrm{e}^{at}. \tag{4.3}$$

5. Convergence

We will use here the results of the previous section to establish convergence of the quantities ψ_n and ϕ_n in the appropriate sense. Let us note that ψ_n defined by (3.7) satisfies the equation

$$(\phi_n)_t = \Delta\psi_n + \psi_n - \frac{1}{m-1}\phi_n + \frac{z_n}{(m-1)\eta_n}. \tag{5.1}$$

Integrating (5.1) against ψ_n, recalling that $\frac{\partial}{\partial t}G_n = \psi_n \frac{\partial \phi_n}{\partial t}$, we obtain

$$\begin{aligned}\frac{\partial}{\partial t}\int G_n\,\mathrm{d}x &= \int \frac{\partial \phi_n}{\partial t}\psi_n\,\mathrm{d}x \\ &= -\int |\nabla\psi_n|^2\,\mathrm{d}x + \int \psi_n^2 - \frac{1}{m-1}\int \phi_n\psi_n + \frac{1}{m-1}\int \frac{z_n}{\eta_n}\psi_n. \end{aligned}\tag{5.2}$$

LEMMA 5.1. – *There exists a constant $C > 0$ such that*

$$\int \frac{z_n}{\eta_n}\psi_n - \int \phi_n\psi_n \leqslant C.$$

Proof. – We have

$$\begin{aligned}\int \frac{z_n}{\eta_n}\psi_n &= \int\limits_{\{x/z_n(x)\leqslant(v_n-w_n)(x)\}} \frac{z_n}{\eta_n}\psi_n + \int\limits_{\{x/z_n(x)>(v_n-w_n)(x)\}} \frac{z_n}{\eta_n}\psi_n \\ &\leqslant \int \phi_n\psi_n + \int\limits_{\{x/z_n(x)>(v_n-w_n)(x)\}} \frac{z_n}{\eta_n^2}\left(v_n^m - w_n^m\right) \\ &\leqslant \int \phi_n\psi_n + C\frac{1}{\eta_n^2}\int w_n^m z_n\end{aligned}$$

where the last inequality holds by (3.12). Now the lemma follows by Proposition 4.1. □

As a consequence of the above result, we can establish the following estimate (local in time) for ψ_n, from where convergence in the appropriate sense of ϕ_n will follow.

LEMMA 5.2. – *Given $t > 0$, there exists $C(t) > 0$ such that*

$$\int\limits_0^t \int \left(\left|\nabla\psi_n(\cdot,s)\right|^2 + \left|\psi_n(\cdot,s)\right|^2\right)\mathrm{d}s \leqslant C(t) \tag{5.3}$$

for all n.

Proof. – Let us recall that

$$\int G_n(s,x)\,\mathrm{d}x \leqslant b\mathrm{e}^{as}.$$

Since the function v is bounded we see that $|\psi_n| \leqslant C|\phi_n|$. Hence, using (3.10), we get $\psi_n^2 \leqslant CG_n$. Now integrating relation (5.2) in time, between $s = 0$ and $s = t$ and using

Lemma 5.2 and Corollary 4.2 we get

$$\int_0^t \int \left|\nabla \psi_n(\cdot, s)\right|^2 \leqslant C\mathrm{e}^{at},$$

and the result is thus proven. □

As a consequence of the last result, the sequence ψ_n can be assumed, after passing to a subsequence, weakly convergent in $L^2((0, S); H^1(\mathbf{R}^N))$ for each $S > 0$. Let $\psi(s, x)$ be this limit. Assume now that

$$w(x) = \sum_{i=1}^{k} w_*(|x - x_i|),$$

so that the support of w is the union of disjoint balls $B(x_i, R^*)$, $i = 1, \ldots, k$. Let $\mathcal{A} = \bigcup_{i=1}^{k} B(x_i, R^*)$. Then if we define $\phi = w^{-(m-1)}\psi$ on $\mathcal{B} = (0, \infty) \times \mathcal{A}$ then

$$\int_0^S \int_{\mathcal{A}} \left|\nabla\left(w^{m-1}\phi\right)\right|^2 + \left|w^{m-1}\phi\right|^2 < +\infty$$

for each $S > 0$. In the next lemma we study further this convergence.

LEMMA 5.3. – *The function ϕ is of class C^1 in $\mathcal{B}$ and $\phi_n(t, x) \to \phi(t, x)$ in the uniform C^1-sense over compact subsets of $\mathcal{B}$. Moreover ϕ solves on $\mathcal{B}$ the equation*

$$\phi_t = m\Delta w^{m-1}\phi + mw^{m-1}\phi - \frac{1}{m-1}\phi. \tag{5.4}$$

Besides the map $t \to \int w^{m-1}(x)\phi^2(t, x)\,\mathrm{d}x$ is continuous and

$$\int_{\mathcal{A}} w^{m-1}(x)\phi^2(t_1, x)\,\mathrm{d}x - \int_{\mathcal{A}} w^{m-1}(x)\phi^2(t_2, x)\,\mathrm{d}x$$
$$= 2\int_{t_1}^{t_2} \int_{\mathcal{A}} \left(m\left|\nabla\left(w^{m-1}\phi\right)\right|^2 - m\left(w^{m-1}\phi\right)^2 + \frac{1}{m-1}\phi^2\right)\mathrm{d}x\,\mathrm{d}t. \tag{5.5}$$

Additionally, for any function $\zeta \in C^1(\mathcal{A})$ with $\int_{\mathcal{A}}(w^{m-1}\zeta^2 + |\nabla w^{m-1}\zeta)|^2)\,\mathrm{d}x < +\infty$

$$\int_{\mathcal{A}} w^{m-1}(x)\phi(t_1, x)\zeta(x)\,\mathrm{d}x - \int_{\mathcal{A}} w^{m-1}(x)\phi(t_2, x)\zeta(x)\,\mathrm{d}x$$
$$= \int_{t_1}^{t_2} \int_{\mathcal{A}} \left(m\nabla\left(w^{m-1}\phi\right)\nabla\zeta - mw^{m-1}\phi\zeta + \frac{1}{m-1}\phi\zeta\right)\mathrm{d}x\,\mathrm{d}t. \tag{5.6}$$

Proof. – We recall that ϕ_n satisfies the equation

$$(\phi_n)_t = \Delta(a_n\phi_n) + a_n\phi_n - \phi_n + \frac{z_n}{(m-1)\eta_n} \tag{5.7}$$

where $a_n = \frac{v_n^m - w^m}{v_n - w}$. Then $a_n \to m w^{m-1}$ uniformly. Hence over compacts of $\mathcal{B}$ the coefficient a_n is uniformly positive and bounded, and $z_n \equiv 0$ there for large n. The standard theory for quasilinear nondegenerate parabolic equations, see [21], gives that this convergence is also uniform in the C^1-sense over compacts of $\mathcal{B}$, so that ∇a_n is also bounded there. From Lemma 2.1, it follows that L^2 norm on any parabolic cube compactly contained in $\mathcal{B}$. Again the theory for nondegenerate parabolic equations in [21] provides uniform estimates for $C^{1,\alpha}$ norms over compacts of $\mathcal{B}$, from where C^1 convergence follows.

Now, given $\delta > 0$ let us consider a smooth cut-off $0 \leqslant \eta(x) \leqslant 1$ compactly supported in $\mathcal{A}$ with $\eta(x) = 1$ if $\operatorname{dist}(x, \mathcal{A}) > \delta$ and $|\nabla(x)| \leqslant C/\delta$ for all x. Let us integrate Eq. (5.7) against $\eta^2 \psi_n$ in space and in time between $t = t_1$ and $t = t_2$ to obtain

$$\int G_n(t_1, x)\eta(x)\,\mathrm{d}x - \int G_n(t_2, x)\eta(x)\,\mathrm{d}x$$
$$= \int_{t_1}^{t_2} \int \eta^2 \left(|\nabla \psi_n|^2 - \psi_n^2 + \frac{1}{m-1} \psi_n \phi_n \right) \mathrm{d}x\,\mathrm{d}t + \int_{t_1}^{t_2} \int 2\eta \nabla \eta \psi_n \nabla \psi_n \,\mathrm{d}x$$

where ψ_n and G_n are defined in (3.7) and (3.8). Now we let $n \to \infty$ and get

$$\int w(x)^{m-1} \phi^2(t_1, x)\eta(x)\,\mathrm{d}x - \int w(x)^{m-1} \phi^2(t_2, x)\eta(x)\,\mathrm{d}x$$
$$= \int_{t_1}^{t_2} \int \eta^2 \left(\left|\nabla w^{m-1}\phi\right|^2 - \left(w^{m-1}\phi\right)^2 + \frac{1}{m-1} w^{m-1}\phi^2 \right) \mathrm{d}x\,\mathrm{d}t$$
$$+ \int_{t_1}^{t_2} \int 2\eta \nabla \eta \phi w^{m-1} \nabla \left(w^{m-1}\phi\right) \mathrm{d}x = I + II.$$

Now, let us write $\mathcal{A}_\delta = \{x \in \mathcal{A} \mid \operatorname{dist}(x, \partial\mathcal{A}) \leqslant \delta\}$. Since w^{m-1} vanishes quadratically on the boundary of $\mathcal{A}$, then $w^{(m-1)/2} \leqslant C\delta$ on $\mathcal{A}_\delta$. It follows that

$$|II| \leqslant C \int_{t_1}^{t_2} \int_{\mathcal{A}_\delta} \phi w^{(m-1)/2} \nabla\left(w^{m-1}\phi\right) \mathrm{d}x\,\mathrm{d}t \leqslant C \int_{t_1}^{t_2} \int_{\mathcal{A}_\delta} \left\{ \phi^2 w^{m-1} + \left|\nabla\left(w^{m-1}\phi\right)\right|^2 \right\} \mathrm{d}x.$$

Since the function inside the integral is indeed integrable by construction, it follows that $|II| \to 0$ as we let $\delta \to 0$. Let us recall that for each $t > 0$ $\int G_n(t, x)\,\mathrm{d}x \leqslant C(t)$, from where it follows that $\int \phi^2(t, x) w^{m-1}(x)\,\mathrm{d}x$ is finite. Then, letting δ go to zero in the above equality, relation (5.5) thus follows. The proof of the remaining assertion of the lemma is similar, so that we omit it. □

6. The spectral problem

In this section we will analyze the spectrum of the linear elliptic operator associated to Eq. (5.4). The purpose of this is to find an expansion of ϕ in terms of eigenvalues and eigenfunctions of this operator.

Thus we consider the eigenvalue problem

$$m\Delta\left(w^{m-1}\phi\right)+mw^{m-1}\phi-\frac{1}{m-1}\phi+\lambda\phi=0 \tag{6.1}$$

with ϕ such that $w^{m-1}\phi\in H^1(B)$ and $w^{\frac{m-1}{2}}\phi\in L^2(B)$. Here $w(x)=w_*(|x|)$ and $B=B(0,R^*)$. Equivalently, the problem can be restated as

$$m\Delta\psi+m\psi-\frac{1}{m-1}w^{-(m-1)}\psi+\lambda w^{-(m-1)}\psi=0 \tag{6.2}$$

with $\psi\in H$ where

$$H=H^1(B)\cap L^2_w(B),$$

with

$$L^2_w(B)=\left\{\psi\in L^2(B)\;\Big|\;\int_B w^{-(m-1)}\psi^2<+\infty\right\}.$$

Now we make the following observation: for a given $\lambda\leqslant 0$, the set of distributional solutions $\psi\in H$ of (6.2) is finite-dimensional. In fact, using $\psi\eta$ as a test function, where $\eta\in C^\infty(\Omega)$ is such that $\eta(x)=1$ for $\operatorname{dist}(x,\partial B)>\delta$ and $|\nabla\eta|\leqslant C/\delta$ then

$$\begin{aligned}m\int_B|\nabla\psi|^2\eta+\left(\frac{1}{m-1}-\lambda\right)\int w^{-(m-1)}\psi^2\eta\\ =-m\int_B\psi\nabla\eta\nabla\psi+m\int|\psi|^2\eta.\end{aligned}$$

Now, since w^{m-1} vanishes quadratically on ∂B, as the above lemma states, it follows that $|w^{(m-1)/2}\nabla\eta|\leqslant C$. Thus

$$\begin{aligned}&\left|\int_B\eta\left(m|\nabla\psi|^2+\left(\frac{1}{m-1}-\lambda\right)w^{-(m-1)}\psi^2\right)-m\int\psi^2\eta\right|\\ &\quad\leqslant C\left(\int_{R^*-\delta<|x|<R^*}w^{-(m-1)}|\psi|^2\right)^{1/2}.\end{aligned}$$

Letting $\delta\to 0$ we get

$$\int_B m|\nabla\psi|^2+\left(\frac{1}{m-1}-\lambda\right)\int w^{-(m-1)}\psi^2=m\int\psi^2. \tag{6.3}$$

It follows that the unit ball in L^2-norm of the corresponding eigenspace is bounded in H^1-norm, hence compact, so that the multiplicity of the eigenvalue is finite. Moreover, only a finite number of eigenvalues $\lambda\leqslant 0$ may exist. In fact, if an infinite number of them exist, then one can construct an infinite orthonormal sequence of eigenfunctions that must converge producing a contradiction.

Let $\mathcal{N}$ be the finite-dimensional vector space spanned by all eigenfunctions in H associated to nonpositive eigenvalues. Let us consider the number λ^* given by

$$\lambda^* = \frac{1}{m-1} + m \inf\left\{ \int_B |\nabla\psi|^2 - \int \psi^2 \Big/ \int w^{-(m-1)}\psi^2 = 1, \right.$$
$$\left. \int w^{-(m-1)}\psi\zeta = 0,\ \forall \zeta \in \mathcal{N} \right\}. \quad (6.4)$$

We claim that $\lambda^* > 0$. In fact, assume $\lambda^* \leqslant 0$ and let ψ_n be a minimizing sequence of the above quantity. It follows that ψ_n is bounded in the space H. Let ψ be the weak limit in H of some subsequence. Then

$$\frac{1}{m-1} + m\int_B |\nabla\psi|^2 - m\int \psi^2 \leqslant \lambda^* \leqslant 0.$$

Hence $\psi \neq 0$ and $\int w^{-(m-1)}\psi^2 \leqslant 1$. The definition of λ^* gives then that necessarily $\int w^{-(m-1)}\psi^2 = 1$, and hence ψ attains the infimum. It easily follows that λ^* is an eigenvalue of problem (6.2), with ψ as associated eigenfunction. Since $\int_B w^{-(m-1)}\psi\zeta = 0$ for all $\zeta \in \mathcal{N}$, it follows that $\lambda^* > 0$, a contradiction which proves the desired result.

In what follows we shall label the first (possibly repeated) k_0 negative eigenvalues of problem (6.1) as

$$\lambda_1 \leqslant \lambda_2 \leqslant \cdots \leqslant \lambda_{k_0} < 0,$$

with associated eigenfunctions $\phi_1, \ldots, \phi_{k_0}$, orthonormalized so that

$$\int_B \phi_i\phi_j = \delta_{ij}.$$

We observe also that $\lambda = 0$ is an eigenvalue and that $\frac{\partial w^m}{\partial x_i}$ are associated eigenfunctions for $i = 1, \ldots, N$. Besides, as proved in Lemma A.3 in the appendix, the only eigenfunctions of (6.1) for $\lambda = 0$ are linear combinations of the functions $\frac{\partial w}{\partial x_i}$.

Remark 6.1. – Assume that $\lambda < 0$ is an eigenvalue of problem (6.1) and ϕ an associated eigenfunction. For further purposes, we want to estimate the size of ϕ near the boundary. We have that $u = -\frac{\partial w}{\partial r} > 0$ satisfies

$$m\Delta\left(w^{m-1}u\right) + mw^{m-1}u - \frac{1}{m-1}u + \lambda u < 0$$

in $B(0, R^*) \setminus \{0\}$. Hence, near the boundary, ϕ can be estimated in absolute value by a suitable multiple of u. It follows, from Lemma A.2, that

$$|\phi(x)| \leqslant C(R^* - |x|)^{\frac{3-m}{m-1}} \leqslant Cw(|x|)^{\frac{3-m}{2}},$$

where C depends for instance on $\int w^{m-1}\phi^2$.

Now let $\phi(x,t)$ be the function found in Lemma 4.1. Let B be one of the balls constituting the set $\mathcal{A}$. Let us consider the expansion in B,

$$\phi(x,0) = \sum_{i=1}^{k_0} C_i\phi_i + \sum_{i=1}^{N} D_i \frac{\partial w}{\partial x_i} + \theta(x)$$

where

$$\int_B \theta\phi_i = \int_B \theta \frac{\partial w}{\partial x_j} = 0 \tag{6.5}$$

for all i, j. Now let us consider the function

$$\tilde{\phi}(x,t) = \phi(x,t) - \sum_{i=1}^{k_0} C_i \mathrm{e}^{-\lambda_i t}\phi_i - \sum_{i=1}^{N} D_i \frac{\partial w}{\partial x_i}.$$

Let us observe that

$$\sum C_i^2 + \sum D_i^2 \leqslant \int_B w^{m-1}\phi^2(0)\,\mathrm{d}x \leqslant 1.$$

Clearly $\tilde{\phi}(x,t)$ satisfies the equation

$$(\tilde{\phi})_s = m\Delta w^{m-1}\tilde{\phi} + m w^{m-1}\tilde{\phi} - \frac{1}{m-1}\tilde{\phi}$$

on $B \times (0, \infty)$. Our first claim is that

$$\int_B \tilde{\phi}(\cdot,s)\phi_i = \int_B \tilde{\phi}(\cdot,s)\frac{\partial w}{\partial x_j} = 0$$

for all $s > 0$. In fact, let us set

$$\varphi(s) = \int_B \tilde{\phi}(\cdot,s)\phi_i.$$

The definition of $\tilde{\phi}$ implies $\varphi'(s) == -\lambda_i\varphi(s)$. Since $\varphi(0) = 0$, the claim follows. Now, let us set

$$\eta(s) = \int_B w^{m-1}\tilde{\phi}(\cdot,s)^2.$$

Then

$$\eta'(s) = -2\int_B m\left(\left|\nabla\left(w^{m-1}\tilde{\phi}\right)\right|^2 - m\left(w^{m-1}\tilde{\phi}\right)^2 + \frac{w^{m-1}}{m-1}\tilde{\phi}^2\right)\mathrm{d}x.$$

Now, since $\tilde{\phi}$ satisfies the orthogonality relations (6.5), it follows that

$$\eta'(s) \leqslant -2\lambda^* \eta(s),$$

and hence

$$\eta(s) \leqslant \eta(0)\mathrm{e}^{-2\lambda^* s}$$

where, we recall, $\lambda^* > 0$ and $\eta(0) \leqslant 1$. Finally, linear parabolic regularity implies that exponential decay at this rate for $\tilde{\phi}$ also holds uniformly on compact subsets of B.

We summarize the above considerations in the following proposition, which provides a description of the limiting function $\phi(t,x)$ inside $\mathcal{A}$.

PROPOSITION 6.1. – *We have*

$$\phi(x,t) = \sum_{i=1}^{k} \sum_{j=1}^{k_0} D_{ij} \mathrm{e}^{-\lambda_j t} \phi_i(x - x_j) + \sum_{j=1}^{k} \sum_{i=1}^{N} C_{ij} \frac{\partial w(x - x_j)}{\partial x_i} + \theta(x,t), \tag{6.6}$$

where $\theta(x,t)$ converges to zero as $t \to +\infty$, exponentially uniformly inside compact sets of the set $\mathcal{A}$.

7. Analysis near the boundary

In the last two sections we have found the validity of convergence of ϕ_n to ϕ *essentially* in the interior of the support of the limiting w. Here we will show estimates which provide control of ϕ_n near the boundary of the support of w. As a by-product we shall establish that the exponentially increasing terms in the expansion (6.6) actually vanish identically, and as a further consequence that the contribution of the region near the boundary on the integral of G_n is basically negligible.

The next result estimates the contribution near the boundary to the integral of G_n in terms of a boundary integral for the limiting function. We should mention that here, the key estimate, Lemma 3.1, again plays a role.

LEMMA 7.1. – *There exist numbers A and c, depending only on m, with the following property*: *Given $\varepsilon > 0$ and $0 < r_0 < R^*$, with r_0 sufficiently close to R^* and any $t > 0$, we have that for all n sufficiently large,*

$$\int_{\bigcap_{i=1}^{k}\{|x - x_i| \geqslant r_1\}} G_n(x,t)\,\mathrm{d}x \leqslant A\left[\mathrm{e}^{-ct} + \varepsilon + w_*^{\frac{3m-3}{2}}(r_0) \sum_{i=1}^{k} \sup_{(s,r)\in[0,t]\times[r_0,r_1]} \int_{|x - x_i| = r} \phi^2(x,s)\,\mathrm{d}\sigma\right] \tag{7.1}$$

where $r_1 = \frac{r_0 + R^*}{2}$.

Proof. – Set

$$D_n = \bigcap_{i=1}^{k}\{x / |x - x_{in}| \geqslant r_0\}$$

and define for $x \in D_n$

$$g(x) = w_*^m(ar_0) - \sum_{i=1}^{k} w_*^m(a|x - x_{in}|)$$

where

$$a \equiv \frac{2R^*}{R^* + r_0} > 1.$$

We note that the sets $\{x/a|x - x_{in}| \leqslant R^*\}$ are disjoint for all n sufficiently large. Let us multiply Eq. (5.1) by $\psi_n(x,t)g(x)$ and integrate on the region D_n. Since g vanishes on the boundary of this region, recalling that $(G_n)_t = \psi_n(\phi_n)_t$, an integration by parts yields

$$\begin{aligned} \frac{\partial}{\partial t}\int_{D_n} G_n(x,t)g(x)\,\mathrm{d}x = B_n(t) + \int_{D_n} \frac{\psi_n^2}{2}\Delta g(x)\,\mathrm{d}x \\ + \int_{D_n}\left\{-|\nabla\psi_n|^2 + \psi_n^2 - \frac{1}{m-1}\phi_n\psi_n + \frac{z_n\psi_n}{(m-1)\eta_n}\right\}g(x)\,\mathrm{d}x \end{aligned} \tag{7.2}$$

where

$$B_n(t) = -\frac{a\partial w_*^m}{\partial r}(ar_0)\sum_{i=1}^{k}\int_{|x-x_{in}|=r_0}\frac{\psi_n^2}{2}\,\mathrm{d}\sigma.$$

Now, $\Delta w_*^m = \frac{w_*}{m-1} - w_*^m \geqslant 0$ for $|x| \geqslant r_0$ if r_0 is sufficiently close to R^*, hence we assume $\Delta g(x) \leqslant 0$ on D_n. On the other hand, $g(x) = w_*^m(ar_0)$ on the support of z_n, for all large n, since by definition of z_n its support is contained in the union of the sets $\{r_0 < |x - x_i| < R^*\} \cap \{r_0 < |x - x_j| < R^*\}$ for $i, j = 1, \ldots, k$ and $i \neq j$. Using these observations and (7.2) we get

$$\begin{aligned} \frac{\partial}{\partial t}\int_{D_n} G_n(x,t)g(x)\,\mathrm{d}x \leqslant B_n(t) + \int_{D_n}\left\{\psi_n^2 - \frac{1}{m-1}\phi_n\psi_n\right\}g(x)\,\mathrm{d}x \\ + w_*^m(ar_0)\int_{D_n}\frac{z_n\psi_n}{(m-1)\eta_n}. \end{aligned} \tag{7.3}$$

On the other hand, recalling the definition of ψ_n and that $z_n \geqslant 0$, Lemma 3.1 implies

$$\frac{1}{(m-1)\eta_n}\int z_n\psi_n \leqslant -\int\frac{\partial G_n}{\partial t} + \frac{1}{m^2-1}\int H_n.$$

Now, by definition of the corresponding quantities, we see that

$$\frac{1}{m^2-1}H_n = \frac{1}{m-1}\phi_n\psi_n - \frac{2}{m-1}G_n,$$

hence

$$\frac{1}{(m-1)\eta_n}\int z_n\psi_n \leqslant -\int \frac{\partial G_n}{\partial t} + \frac{1}{m-1}\phi_n\psi_n - \frac{2}{m-1}G_n. \tag{7.4}$$

Then using (7.4) in relation (7.3) we obtain that

$$\int_{D_n} \frac{\partial G_n}{\partial t} g(x) \leqslant B_n(t) + \frac{1}{m-1}\int_{D_n} \phi_n\psi_n\bigl(w_*^m(ar_0) - g(x)\bigr)$$
$$- w_*^m(ar_0)\int \frac{\partial G_n}{\partial t} + \int_{D_n}\left\{\psi_n^2 g(x) - \frac{2w_*^m(ar_0)}{m-1}G_n\right\}. \tag{7.5}$$

Now, given $t^* > 0$ we have that $v_n \to w$ uniformly on $[0, t^*] \times \mathbf{R}^N$. Thus, if r_0 is sufficiently close to R^* we obtain that

$$\psi_n^2 \leqslant \frac{1}{(m-1)}G_n$$

for $|x| \geqslant r_0$ and $0 < t < t^*$ for large n. Hence

$$\psi_n^2 g(x) - \frac{2w_*^m(ar_0)}{m-1}G_n \leqslant -\frac{(w_*^m(ar_0) + g(x))}{2(m-1)}G_n$$

on this region. Also, $w_*^m(ar_0) - g(x) = 0$ on $F_n = \bigcap_{i=1}^k \{a|x - x_{in}| \geqslant R^*\}$. Substituting this information into relation (7.5), we obtain the following differential inequality for all sufficiently large n.

$$Y_n'(t) \leqslant B_n(t) + W_n(t) - cY_n(s), \quad 0 < t < t^*,$$

where c is a positive constant depending only on m and

$$Y_n(t) = \int_{D_n} G_n(x,t)\bigl(w_*^m(ar_0) + g(x)\bigr),$$
$$W_n(t) = \frac{1}{m-1}\int_{D_n\setminus F_n} \phi_n\psi_n\bigl(w_*^m(ar_0) - g(x)\bigr).$$

It follows that

$$Y_n(s) \leqslant Y_n(0)\mathrm{e}^{-ct} + \mathrm{e}^{-ct}\int_0^t \mathrm{e}^{cs}\bigl(B_n(s) + W_n(s)\bigr)\,\mathrm{d}s. \tag{7.6}$$

We will estimate the right hand side of (7.6). First, we see that $B_n(s) \to B(s)$ and $W_n(s) \to W(s)$ uniformly on compact sets where

$$B(s) = -\frac{a\partial w_*^m}{\partial r}(ar_0)\sum_{i=1}^k \int_{|x-x_i|=r_0} m^2 w_*^{2(m-1)}(r_0)\frac{\phi^2}{2}\,\mathrm{d}\sigma,$$

and

$$W(s)=\frac{m}{m-1}\sum_{i=1}^{k}\int\limits_{r_0\leqslant|x-x_i|\leqslant\frac{r_0+R^*}{2}}\phi^2\big(w_*^{m-1}(|x-x_i|)\big)\big(w_*^m(ar_0)-g(x)\big).$$

Now,

$$\big(w^m\big)'(r_0)w^{2m-2}(r_0)=\frac{2}{m-1}\big(w^{\frac{m-1}{2}}\big)'(r_0)w^m(r_0)w^{\frac{3m-3}{2}}(r_0),$$

and $w(ar_0)<w(r_0)$, for r_0 close enough to R^*, so that

$$|B(s)|\leqslant Cw^m(r_0)w^{\frac{3m-3}{2}}(r_0)\sum_{i=1}^{k}\int\limits_{|x-x_i|=r_0}\phi^2\,\mathrm{d}\sigma.$$

On the other hand,

$$|W(s)|\leqslant Cw_*^{m-1}(r_0)w_*^m(ar_0)\left(\frac{R^*-r_0}{2}\right)\sum_{i=1}^{k}\sup_{r\in[r_0,\frac{r_0+R^*}{2}]}\int\limits_{|x-x_i|=r}\phi^2\,\mathrm{d}\sigma.$$

Since $\frac{\partial}{\partial r}w_*^{\frac{m-1}{2}}(R^*)>0$, we have

$$\frac{R^*-r_0}{2}\leqslant Cw_*^{\frac{m-1}{2}}(r_0)$$

for some $C>0$ depending only on m, provided that r_0 is sufficiently close to R^*. From these facts and (7.6) we see that for given $\varepsilon>0$ and $t>0$,

$$Y_n(t)\leqslant Y_n(0)\mathrm{e}^{-ct}+\varepsilon+Cw_*^m(r_0)w_*^{\frac{3m-3}{2}}(r_0)\sup_{[0,t]}\sum_{i=1}^{k}\sup_{r\in[r_0,\frac{r_0+R^*}{2}]}\int\limits_{|x-x_i|=r}\phi^2\,\mathrm{d}\sigma,$$

for all sufficiently large n, where we have used again that $w_*(ar_0)<w_*(r_0)$. Finally, it is easily checked that for some constant C independent of r_0 close to R^*, such that $1\leqslant\frac{w_*(r_0)}{w_*(ar_0)}\leqslant C$. Since $g(x)=w_*^m(ar_0)$ for $|x|\geqslant\frac{r_0+R^*}{2}$, the result of the lemma readily follows. □

We prove next two important consequences of the estimate given by the above lemma: One is that no exponentially increasing terms are present in the expansion (6.6) for ϕ, and that the integrals of G_n near and outside the boundary of the support of w become arbitrarily small as $n\to\infty$.

COROLLARY 7.1. – *In the expansion* (6.6), *we actually have* $D_{ij}=0$ *for* $i=1,\dots,k_0$, $j=1,\dots,k$.

Proof. – To establish this assertion, we consider the estimate provided by Corollary 3.1. We have then that,

$$\int G_n(x,t)\,\mathrm{d}x \leqslant \frac{1}{m^2-1}\int_0^t \int H_n + C\left(\int_0^t \left(\int G_n(x,s)\,\mathrm{d}x\right)^{1/2} \mathrm{d}s + t + 1\right) \tag{7.7}$$

for certain number C independent of t and m. Let us fix a number r_0 close to R^* and as in the previous lemma, set $r_1 = \frac{r_0+R^*}{2}$. We also write

$$\mathcal{A}_{r_1} = \bigcup_{i=1}^{k} \{x/|x-x_i| \leqslant r_1\}.$$

Then there is a constant $D > 0$ depending on r_0 such that

$$|H_n| \leqslant \frac{D}{\eta_n^2}|v_n - w|^3 \quad \text{on } \mathcal{A}_{r_1},$$

from where it follows that

$$\int_0^t \int_{\mathcal{A}_{r_1}} H_n \to 0 \quad \text{as } n \to \infty$$

for each fixed t. We note that for any $\sigma > 0$ and any $t > 0$ one has

$$s^{1/2} \leqslant \sigma s + \sigma^{-1}.$$

Then from relation (7.7), recalling that $|H_n| \leqslant CG_n$, we find that for some $C > 0$ independent of r_0 and σ, and all sufficiently large n,

$$\int_{\mathcal{A}_{r_1}} G_n(x,t)\,\mathrm{d}x \leqslant C\left[\left(\frac{1}{m^2-1} + \sigma\right)\int_0^t \int_{\mathbf{R}^N \setminus \mathcal{A}_{r_1}} G_n + \sigma \int_0^t \int_{\mathcal{A}_{r_1}} G_n + \sigma^{-1} t + 1\right].$$

Then, passing to the limit, recalling that G_n converges uniformly in $\mathcal{A}_{r_1} \times [0,t]$ to $mw^{m-1}\phi^2$, we get

$$\int_{\mathcal{A}_{r_1}} mw^{m-1}\phi^2(x,t)\,\mathrm{d}x - C\sigma \int_0^t \int_{\mathcal{A}_{r_1}} mw^{m-1}\phi^2(x,s)\,\mathrm{d}x\,\mathrm{d}s$$

$$\leqslant \limsup_{n\to\infty} C\left(\int_0^t \int_{\mathbf{R}^N \setminus \mathcal{A}_{r_1}} G_n + \sigma^{-1} t + 1\right). \tag{7.8}$$

Now, from the expression (6.6) for ϕ, we obtain

$$\phi(x,t)=\sum_{i=1}^{k}\sum_{j=1}^{k_0} D_{ij}\mathrm{e}^{-\lambda_j t}\phi_i(x-x_j)+\mathrm{O}(1)$$

with O(1) uniformly bounded in time and space inside $\mathcal{A}_{r_1}$. It follows that

$$\begin{aligned}\int\limits_{\mathcal{A}_{\nabla\infty}} w^{m-1}\phi^2(x,t)\,\mathrm{d}x &=\sum_{i=1}^{k}\int\limits_{B(x_i,r_1)} w^{m-1}\phi^2(x,t)\,\mathrm{d}x\\ &=\sum_{i=1}^{k}\sum_{j=1}^{k_0}\int\limits_{B(x_i,r_1)} w^{m-1}D_{ij}^2\mathrm{e}^{-2\lambda_j t}\phi_i(x-x_j)^2\,\mathrm{d}x+\mathrm{O}(1)\\ &=\sum_{i=1}^{k}\sum_{j=1}^{k_0} D_{ij}^2\mathrm{e}^{-2\lambda_j t}+\mathrm{O}(1).\end{aligned}$$

Thus, if we fix now σ sufficiently small, we get

$$\begin{aligned}&\int\limits_{\mathcal{A}_{r_1}} mw^{m-1}\phi^2(x,t)\,\mathrm{d}x-C\sigma\int\limits_0^t\int\limits_{\mathcal{A}_{r_1}} mw^{m-1}\phi^2(x,s)\,\mathrm{d}x\,\mathrm{d}s\\ &\quad=\bar{C}\sum_{i=1}^{k}\sum_{j=1}^{k_0} D_{ij}^2\mathrm{e}^{2\lambda_j t}+\mathrm{O}(t)\end{aligned}\tag{7.9}$$

where $\bar{C}>0$. On the other hand, from Lemma 7.1 we can find numbers A and c which depend only on m so that

$$\begin{aligned}&\limsup_{n\to\infty}\int\limits_{\mathbf{R}^N\setminus\mathcal{A}_{r_1}} G_n(x,s)\,\mathrm{d}x\\ &\leqslant A\left\{\mathrm{e}^{-cs}+w_*^{\frac{3m-3}{2}}(r_0)\sup\left\{\int\limits_{|x-x_i|=r}\phi^2(x,s)\,\mathrm{d}\sigma/s\in[0,t],\ r\in[r_0,r_1],\ 1\leqslant i\leqslant k\right\}\right\}.\end{aligned}\tag{7.10}$$

We recall that from Remark 6.1 $|\phi(x)|\leqslant Cw(|x|)^{\frac{3-m}{2}}$, hence

$$\begin{aligned}&w^{\frac{3m-3}{2}}(r_0)\sup\left\{\int\limits_{|x-x_i|=r}\phi^2\,\mathrm{d}\sigma/\mid s\in[0,t],\ r\in[r_0,r_1],\ 1\leqslant i\leqslant k\right\}\\ &\quad\leqslant Cw(r_0)^{\frac{m+3}{2}}\sum_{i=1}^{k}\sum_{j=1}^{k_0} D_{ij}^2\mathrm{e}^{2\lambda_j t}+\mathrm{O}(1),\end{aligned}\tag{7.11}$$

where C is independent of r_0. Using relations (7.8), (7.9), (7.10), and (7.11) we get then that for certain constant C

$$\sum_{i=1}^{k}\sum_{j=1}^{k_0} D_{ij}^2 \mathrm{e}^{-2\lambda_j t} \leqslant C w(r_0)^{\frac{m+3}{2}} \sum_{i=1}^{k}\sum_{j=1}^{k_0} D_{ij}^2 \mathrm{e}^{-2\lambda_j t} + \mathrm{O}(t),$$

where C is independent of r_0. Since $w(r_0)$ may be chosen arbitrarily small, we obtain a contradiction from this last relation for all t sufficiently large if any of the D_{ij}'s was not zero. Hence $D_{ij} = 0$ for all i, j, and the proof of the lemma is complete. □

Combining Lemma 7.1 and Corollary 7.1, we get the validity of the following fact.

COROLLARY 7.2. – *Let $\varepsilon > 0$ be given. Then there exist numbers $0 < r_0 < R^*$ and $s^* > 0$ such that for each given $\bar{s} \geqslant s^*$ and all n sufficiently large we have*

$$\sup_{s\in[s^*,\bar{s}]} \int\limits_{\bigcap_{i=1}^{k}\{|x-x_i|>r_0\}} G_n(s,x)\,\mathrm{d}x < \varepsilon.$$

8. Conclusion: the proof of Proposition 3.1

Now we are ready to prove Proposition 3.1.
We define

$$\bar{x}_{in} = \eta_n(C_{i1}, \ldots, C_{iN}).$$

Then $|\bar{x}_{in}| \leqslant C\eta_n$ with $C = C(m)$. Let us write

$$\bar{w}_n(x) = \left(\sum_{i=1}^{k} w_*^m(x - x_i + \bar{x}_{in})\right)^{1/m}.$$

We want to estimate the quantity

$$I_n(s) = \int \big(v_n(x,s)^m - \bar{w}_n(x)^m\big)\big(v_n(x,s) - \bar{w}_n(x)\big)\,\mathrm{d}x.$$

Let us consider $r \in [0, R^*]$, to be determined later, and set

$$\mathcal{A}_r = \bigcap_{i=1}^{k}\{|x - x_i| > r\}.$$

Then

$$\begin{aligned} I_n(s) = &\int\limits_{\mathcal{A}_\delta} (v_n^m - \bar{w}_n^m)(v_n - \bar{w}_n)\,\mathrm{d}x \\ &+ \int\limits_{\mathbf{R}^N\setminus\mathcal{A}_\delta} (v_n^m - \bar{w}_n^m)(v_n - \bar{w}_n)\,\mathrm{d}x = I_n^1(s) + I_n^2(s). \end{aligned}$$

We have

$$I_n^1(s) \leqslant C\left[\int\limits_{\mathcal{A}_r} (v_n^m - w_n^m)(v_n - w_n)\,\mathrm{d}x + \int\limits_{\mathcal{A}_r} (\bar{w}_n^m - w_n^m)(\bar{w}_n - w_n)\,\mathrm{d}x\right].$$

Now, from 3.10, we get

$$\int\limits_{\mathcal{A}_r} (v_n^m - w_n^m)(v_n - w_n)\,\mathrm{d}x \leqslant C\eta_n^2 \int\limits_{\mathcal{A}_r} G_n(x,s)\,\mathrm{d}x,$$

again with $C = C(m)$. Corollary 7.2 then implies that if r is chosen close enough to R^*, depending on m, and $s \geqslant s^*$, with s^* also depending only on m then for all n sufficiently large

$$C\int\limits_{\mathcal{A}_r} G_n(x,s)\,\mathrm{d}x < \frac{1}{8}.$$

Also, it follows from Lemma A.2, in the appendix, that

$$C\left(\bar{w}_n^m - w_n^m\right)(\bar{w}_n - w_n) \leqslant K\eta_n^2$$

with K depending on m and k only. Therefore taking r closer to R^*, if necessary, we get

$$C\int\limits_{\mathcal{A}_r} (\bar{w}_n^m - w_n^m)(\bar{w}_n - w_n)\,\mathrm{d}x \leqslant \frac{\eta_n^2}{8}$$

if n is large enough. Putting these two estimates together we see that if we choose $T \geqslant s^*$, then

$$I_n^1(s) \leqslant \frac{\eta_n^2}{4}$$

for all $s \in [T, 2T]$ provided that n is sufficiently large.

On the other hand, we recall that

$$v_n(x,s) = w_n(x) + \eta_n\phi_n(x,s),$$

which we can write, in view of Proposition 6.1 and Corollary 7.1, as

$$v_n(x,s) = w_n(x) + \eta_n \sum_{j=1}^{N} C_{ij}\frac{\partial w_*}{\partial x_i}(x - x_i) + \eta_n\theta(x,s) + \big(\phi_n(x,s) - \phi(x,s)\big),$$

where θ decays exponentially in compact sets of $\mathcal{A}$.

Now, since r has been already fixed and $\liminf |x_{in} - x_{jn}| \geqslant 2R^*$ as $n \to \infty$ for $i \neq j$, it follows that if $x \in \mathbf{R}^N \setminus A_r$ then

$$w_n(x) = \sum_{i=1}^{k} w_*(x - x_i)$$

and

$$\bar{w}_n(x) = \sum_{i=1}^{k} w_*(x - x_{in} + \bar{x}_{in})$$

if n is sufficiently large. From these observations it follows that

$$\lim_{n\to\infty} \eta_n^{-2} \int\limits_{\mathbf{R}^N\setminus\mathcal{A}_r} (v_n^m - \bar{w}_n^m)(v_n - \bar{w}_n)\,\mathrm{d}x = m \int\limits_{\mathbf{R}^N\setminus\mathcal{A}_r} w^{m-1}(x)\theta^2(x,s)\,\mathrm{d}x$$

uniformly on s on compact subsets of $(0,\infty)$. Since $\theta(x,s)$ decays exponentially we have that there are positive numbers A and a, depending only on m, such that

$$I_n^2(s) \leqslant \eta_n^2 A\mathrm{e}^{-at}.$$

Consequently we have

$$I_n(s) \leqslant \eta_n^2\left(\frac{1}{4} + A\mathrm{e}^{-aT}\right)$$

for all $s \in [T, 2T]$. Making T larger if necessary (depending only on m) we obtain that the quantity between brackets is less than $1/2$. This concludes the proof of the proposition. □

Appendix A

LEMMA A.1. – *With the notation of Section* 3, *we have*

$$\left(w_n^{m+1} - \sum_{i=1}^{k} w_{in}^{m+1}\right) \leqslant \frac{1}{m-1}\left(w_n^m z_n\right). \tag{A.1}$$

Proof. – By homogeneity, it suffices to establish the following general fact on real numbers:

For any nonnegative numbers $h_1,\ldots,h_k$ *such that* $\sum_{i=1}^{k} h_i^m = 1$ *one has*

$$\left(1 - \sum_{i=1}^{k} h_i^{m+1}\right) \leqslant \frac{1}{m-1}\left(\sum_{i=1}^{k} h_i - 1\right). \tag{A.2}$$

To prove (A.2), we set $h = (h_1,\ldots,h_k)$,

$$F(h) = \sum_{i=1}^{k} h_i + (m-1)\sum_{i=1}^{k} h_i^{m+1} \quad \text{and} \quad P(h) = \sum_{i=1}^{k} h_i^m$$

we see that our problem reduces to show that the minimum of the function F, over to the set $S = \{h \mid P(h) = 1, h_1 \geqslant 0, \ldots, h_k \geqslant 0\}$, is greater than m. If a minimum of F over S

is attained at a point h with all coordinates positive, then Lagrange multiplier rule yields the existence of a number λ such that

$$\frac{1}{h_i^{m-1}} + \left(m^2-1\right)h_i = m\lambda \quad \text{for all } i = 1, \ldots, k.$$

For each i, this equation has at most two solutions, which we denote by a and b. Therefore there are integers p and q such that $p+q=k$, $pa^m + qb^m = 1$ and the minimum of F is given by

$$pa + qb + (m-1)\left(pa^{m+1} + qb^{m+1}\right).$$

It is clear that if $a \leqslant b$ this quantity is greater or equal to

$$ka + (m-1)ka^{m+1}$$

with $ka^m = 1$. This last number is greater or equal to m if $k \geqslant 1$. So far we have proved that any minimum attained at the interior of S is greater or equal to m. Studying the possible minima attained at the boundary of S reduces to the same problem with a different value of k. Inequality (A.2) is thus established, and the proof of the lemma is concluded. □

Our next result refers to the behavior of the function w near the boundary of B.

LEMMA A.2. – *Let $w_*(|x|)$ be the unique radially symmetric, compactly supported solution of*

$$\Delta w^m + w^m - \frac{1}{m-1} w = 0,$$

and $B(0, R^*)$ its support. Then there exist positive constants A and B such that

$$A(R^* - r)^{\frac{2}{m-1}} \leqslant w_*(r) \leqslant B(R^* - r)^{\frac{2}{m-1}},$$
$$A(R^* - r)^{\frac{3-m}{m-1}} \leqslant -w_*(r)' \leqslant B(R^* - r)^{\frac{3-m}{m-1}},$$

for all $r \in [0, R^*]$.

Proof. – The function $z = w_*^m$ satisfies

$$z'' + \frac{N-1}{r} z' + z - \frac{1}{m-1} z^{1/m} = 0.$$

Multiplying by z' and integrating from r to R^* we get

$$\frac{z'(r)^2}{2} + \frac{z(r)^2}{2} - \int_r^{R^*} \frac{N-1}{s} z'(s)^2 \, \mathrm{d}s - \frac{m}{m^2-1} z^{\frac{m+1}{m}}(r) = 0.$$

Hence near $r = R^*$,

$$z'(r)^2 \geqslant A z^{\frac{m+1}{m}}(r)$$

and hence $z^{-\frac{m+1}{2m}} z' \leqslant -A^{1/2}$, so that $z^{\frac{m-1}{2m}}(r) \geqslant A(R^* - r)$, whence $w(r) \geqslant A(R^* - r)^{\frac{2}{m-1}}$ near $r = R^*$. On the other hand, again near R^*,

$$-r^{N-1} z'(r) \leqslant C \int_r^{R^*} z^{1/m}(s) s^{N-1}\,\mathrm{d}s,$$

or, since z is decreasing in r,

$$-r^{N-1} z'(r) \leqslant C\big((R^*)^N - r^N\big) z^{1/m}(r).$$

It follows that $-z^{\frac{-1}{m}} z'(r) \leqslant C(R^* - r)$. Thus $z^{\frac{m-1}{m}}(r) \leqslant C(R^* - r)^2$ and hence

$$w(r) \leqslant C(R^* - r)^{\frac{2}{m-1}}$$

near $r = R^*$. Similar estimates for the derivative follow from those for w and the above intermediate computations. Making A smaller and B bigger, if necessary, the lemma is proved. □

The following lemma was used in Section 6. Since its proof follows closely the proof of Lemma 4.2 of [23], we only sketch it here.

LEMMA A.3. – *The only eigenfunctions of* (6.1) *for* $\lambda = 0$ *are linear combinations of the functions* $\frac{\partial w}{\partial x_i}$.

Proof. – The lemma will be proved as soon as we prove that the eigenspace corresponding to $\lambda = 0$ has dimension less or equal to $N - 1$. So let ψ satisfy

$$m\Delta\psi + m\psi - \frac{1}{m-1} w^{-(m-1)}\psi = 0. \tag{A.3}$$

Let μ_k, $e_k(\sigma)$, with $\sigma \in S^{N-1}$, be the eigenvalues and eigenvectors of the Laplace–Beltrami operator on S^{N-1}. We recall that

$$\mu_0 = 0 < \mu_1 = \cdots = \mu_N = N - 1 < \mu_{N+1} \leqslant \cdots.$$

We normalize e_k so that they form a complete orthonormal basis of $L^2(S^{N-1})$. Now we set

$$\varphi_k(r) = \int_{S^{N-1}} \psi(r,\sigma) e_k(\sigma)\,\mathrm{d}\sigma$$

and observe that φ_k satisfies

$$m\varphi_k'' + m\frac{N-1}{r}\varphi_k' + m\varphi_k - \frac{w^{-(m-1)}}{m-1}\varphi_k = m\mu_k\varphi_k \quad \text{on } 0 < r < R^*. \tag{A.4}$$

We note that for φ_0 we have

$$m\varphi_0'' + m\frac{N-1}{r}\varphi_0' + m\varphi_0 - \frac{w^{-(m-1)}}{m-1}\varphi_0 = 0 \quad \text{on } 0 < r < R^*$$

with $\varphi_0'(0) = 0$ and $\varphi_0(R^*) = 0$.

Also setting $z = w^m$ we see that

$$z'' + \frac{N-1}{r}z' + z - \frac{z^{1/m}}{m-1} = 0 \quad \text{on } 0 < r < R^*.$$

According to Proposition 4.1 in [3], the corresponding linearized equation

$$m\varphi'' + m\frac{N-1}{r}\varphi'' + m\varphi - \frac{z^{-\frac{m-1}{m}}}{m-1}\varphi = 0$$

has no nontrivial solutions satisfying $z'(0) = 0$ and $z(R^*) = 0$. We observe that the results of [3] hold the same for the case $p = 1$ in the notation of that paper. This implies $\varphi_0 \equiv 0$ since $z^{-\frac{m-1}{m}} = w^{-(m-1)}$. Therefore, as in [23], the lemma will be proved as soon as we prove that $\varphi_k \equiv 0$ for all $k \geqslant N + 1$. To do this assume for a contradiction that $\varphi_k \neq 0$ for some $k \geqslant N + 1$. Since in this case, as in [23], we can assume that $\varphi_k(r)$ is positive for small values of r, we have that there exists $\rho_k \in (0, R^*]$ such that $\varphi_k(r) > 0$ if $r \in (0, \rho_k)$ and $\varphi_k(\rho_k) = 0$. Multiplying (A.4) by $r^{N-1}(w^m)'$ and integrating from 0 to ρ_k, after integrating by parts twice and using boundary values, one gets

$$\rho_k^{N-1}\varphi_k'(\rho_k)\big(w^m\big)'(\rho_k) + (N - 1 + \mu_k)\int_0^{\rho_k} r^{N-3}\varphi_k(r)\big(w^m\big)'(r)\,\mathrm{d}r = 0.$$

This is a contradiction since $\varphi_k'(\rho_k) \leqslant 0$ and $(w^m)'(r) < 0$ for $r \in (0, R^*)$. The lemma is proved. □

REFERENCES

[1] C. Cortázar, M. Elgueta, P. Felmer, Symmetry in an elliptic problem and the blow-up set of a quasilinear heat equation, Comm. P.D.E. 21 (1996) 507–520.

[2] C. Cortázar, M. Elgueta, P. Felmer, Uniqueness of positive solutions of $\Delta u + f(u) = 0$ in $\mathbf{R}^N$, $N \geqslant 3$, Arch. Rat. Mech. Anal. 142 (1998) 127–141.

[3] C. Cortázar, M. Elgueta, P. Felmer, On a semilinear elliptic problem in $\mathbf{R}^N$ with a non-lipschitzian nonlinearity, Adv. Differential Equations 1 (2) (1996) 199–218.

[4] C. Cortázar, M. del Pino, M. Elgueta, On the blow-up set for $u_t = \Delta u^m + u^m$, $m > 1$, Indiana Univ. Math. J. 47 (1998) 541–561.

[5] C. Cortázar, M. del Pino, M. Elgueta, The problem of uniqueness of the limit in a semilinear heat equation, Comm. Partial Differential Equations 24 (1999) 2147–2172.

[6] E. Feireisl, H. Petzeltova, Convergence to a ground state as threshold phenomenos in nonlinear parabolic equations, Differential Integral Equations 10 (1997) 181–196.

[7] E. Feireisl, F. Simondon, Convergence for degenerate parabolic equations, J. Differential Equations 152 (2) (1999) 439–466.

[8] E. Feireisl, F. Simondon, Convergence for semilinear degenerate parabolic equations in several space dimensions, Preprint.

[9] C. Fermanian Kammerer, F. Merle, H. Zaag, Stability of the blow-up profile of non-linear heat equations from the dynamical system point of view, Math. Ann. 317 (2000) 347–387.

[10] H. Fujita, On the blowing-up of solutions of the Cauchy problem for $u_t = \Delta u + u^{1+\alpha}$, J. Fac. Sci. Univ. Tokyo 13 (1966) 109–124.
[11] V. Galaktionov, On a blow-up set for the quasilinear heat equation $u_t = (u^\sigma u_x)_x + u^{\sigma+1}$, J. Differential Equations 101 (1993) 66–79.
[12] V. Galaktionov, Blow-up for quasilinear heat equations with critical Fujita's exponent, Proc. Roy. Soc. Edinburgh 124A (1994) 517–525.
[13] V. Galaktionov, L.A. Peletier, Asymptotic behaviour near finite-time extinction for the fast difussion equation, Arch. Rat. Mech. Anal. 139 (1997) 83–98.
[14] V. Galaktionov, J.L. Vazquez, Continuation of blowup solutions of nonlinear heat equations in several space dimensions, Comm. Pure Appl. Math. 50 (1) (1997) 1–67.
[15] Y. Giga, R. Kohn, Characterizing blow-up using similarity variables, Indiana Univ. Math. J. 36 (1987) 1–40.
[16] Y. Giga, R. Kohn, Nondegeneracy of blow-up for semilinear heat equations, Comm. Pure Appl. Math. 42 (1989) 845–884.
[17] C. Gui, Symmetry of the blow-up set of a porous medium equation, Comm. Pure Appl. Math. 48 (1995) 471–500.
[18] J. Hale, G. Raugel, Convergence in gradient-like and applications, Z. Angew. Math. Phys. 43 (1992) 63–124.
[19] A. Haraux, P. Polacik, Convergence to a positive equilibrium for some nonlinear evolution equations in a ball, Acta Math. Univ. Comeniane 61 (1992) 129–141.
[20] N. Korevaar, R. Mazzeo, F. Pacard, R. Schoen, Refined asymptotics for constant scalar curvature metrics with isolated singularities, Invent. Mat. 135 (2) (1999) 233–272.
[21] O.A. Ladyzenskaja, V.A. Solonnikov, N.N. Ural'ceva, Linear and Quasilinear Equations of Parabolic Type, in: Translations of Mathematical Monographs, Vol. 23, 1968.
[22] H. Matano, Nonincrease of the lap number of a solution for a one-dimensional semilinear parabolic equation, J. Fac. Sci. Univ. Tokyo 1A 29 (1982) 401–411.
[23] W.-M. Ni, I. Takagi, Locating the peaks of least-energy solutions to a semilinear Neumann problem, Duke Math. J. 70 (1993) 247–281.
[24] F. Merle, H. Zaag, Stability of the blow-up profile for equations of the type $u_t = \Delta u + |u|^{p-1}u$, Duke Math. J. 86 (1997) 143–195.
[25] F. Merle, H. Zaag, Optimal estimates for blowup rate and behavior for nonlinear heat equations, Comm. Pure Appl. Math. 51 (1998) 139–196.
[26] P. Polacik, K.P. Rybakowski, Nonconvergent bounded trajectories in semilinear heat equations, J. Differential Equations 124 (1996) 472–494.
[27] A. Samarskii, V. Galaktionov, V. Kurdyumov, A. Mikhailov, Blow-up in Problems for Quasilinear Parabolic Equations, Nauka, Moscow, 1987, in Russian.
[28] L. Simon, Asymptotics for a class of non-linear evolution equations with applications to geometric problems, Ann. of Math. 118 (1983) 525–571.
[29] J. Velázquez, Characterizing blow-up using similarity variables, Indiana Univ. Math. J. 42 (1993) 445–476.