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SOME STEINHAUS TYPE THEOREMS OVER VALUED FIELDS

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1. Preliminaries :

In this paper K denotes R (the field of real numbers) or C (the field of complex numbers) or a complete, non-trivially valued, non-archimedean field as will be explicitly stated depending on the context.

In the sequel, infinite matrices $A = (a_{nk}), n, k = 1, 2, \ldots$ and sequences $x = \{x_k\}$, $k = 1, 2, \ldots$ have their entries in K. If X, Y are two classes of sequences, we write (X, Y) to denote the class of all infinite matrices $A = (a_{nk}), n, k = 1, 2, \ldots$ for which

$$Ax = \{(Ax)_n\} \in Y \text{ whenever } x = \{x_k\} \in X,$$

where $(Ax)_n = \sum_{k=1}^{\infty} a_{nk} x_k, n = 1, 2, \ldots,$

it being assumed that the series on the right converge. The sequence $Ax = \{(Ax)_n\}$ is called the A-transform of $x = \{x_k\}$. The sequence spaces ℓ_p , $p \geq 1$, ℓ_∞ , c, c_0 are defined as usual i.e.,

$$\begin{aligned}
\ell_p &= \{x = \{x_k\} : \sum_{k=1}^{\infty} |x_k|^p < \infty\}, p \geq 1 ; \\
\ell_\infty &= \{x = \{x_k\} : \sup_{k \geq 1} |x^k| < \infty\} ; \\
c &= \{x = \{x_k\} : \lim_{k \to \infty} x_k = s \quad \text{for some} \quad s \in K\} ; \\
c_0 &= \{x = \{x_k\} : \lim_{k \to \infty} x_k = 0\} .
\end{aligned}$$

Note that $\ell_p \subset c_0 \subset c \subset \ell_\infty$ where $p \geq 1$. For convenience we write $\ell_1 = \ell$. $(\ell, c; P')$ denotes the class of all infinite matrices $A \in (\ell, c)$ such that $\lim_{n \to \infty} (Ax)_n = \sum_{k=1}^{\infty} x_k$ whenever $x = \{x_k\} \in \ell$.

2. The case $K = R$ or C

When $K = R$ or C, it is known ([11]), p. 4, 17) that $A = (a_{nk}) \in (\ell, c)$ if and only if

(1) $\sup_{n,k} |a_{nk}| < \infty$;

and

(2) $\lim_{n\to\infty} a_{nk} = \delta_k$ exists, $k = 1, 2, \ldots$.

We now prove the following

THEOREM 2.1 :

When $K = R$ or $C, A \in (\ell, c; P')$ if and only if (1) holds and (2) holds with

(3) $\delta_k = 1, k = 1, 2, \ldots$.

Proof.

Let $A \in (\ell, c\ ; P')$. Let e_k be the sequence in which 1 occurs in the k^{th} place and 0 elsewhere, $k = 1, 2, \ldots$ i.e.,

$$e_k = \left\{x_i^{(k)}\right\}_{i=1}^{\infty}$$

where

$$x_i^{(k)} = 1, \text{ if } i = k ;$$
$$= 0, \text{ otherwise.}$$

Then $e_k \in \ell, k = 1, 2, \ldots, \sum_{i=1}^{\infty} x_i^{(k)} = 1$ and $(Ae_k)_n = a_{nk}$ so that $\lim_{n\to\infty} a_{nk} = 1$, i.e., $\delta_k = 1, k = 1, 2, \ldots$. Thus (1) and (3) are necessary for $A \in (\ell, c\ ; P')$.

Conversely, let (1) and (3) hold. Let $x = \{x_k\} \in \ell$. In view of (1), $\sum_{k=1}^{\infty} a_{nk}x_k$ converges, $n = 1, 2, \ldots$. Now,

$$(Ax)_n = \sum_{k=1}^{\infty} a_{nk}x_k$$
$$= \sum_{k=1}^{\infty}(a_{nk} - 1)x_k + \sum_{k=1}^{\infty} x_k,$$

this being true since $\sum_{k=1}^{\infty} a_{nk}x_k$ and $\sum_{k=1}^{\infty} x_k$ both converge.

Since $\sum_{k=1}^{\infty}|x_k| < \infty$, given $\varepsilon > 0$, there exists a positive integer N such that

$$(4) \qquad \sum_{k=N+1}^{\infty} |x_k| < \frac{\varepsilon}{2A},$$

where $A = \sup_{n,k} |a_{nk} - 1|$. Since $\lim_{n\to\infty} a_{nk} = 1, k = 1, 2, ..., N$, we can choose a positive integer $N' > N$ such that

$$(5) \qquad |a_{nk} - 1| < \frac{\varepsilon}{2NM} \quad , \quad n \geq N', k = 1, 2, ..., N,$$

where $M > 0$ is such that $|x_k| \leq M, k = 1, 2, ...$ Now, for $n \geq N'$,

$$\begin{aligned} |\sum_{k=1}^{\infty}(a_{nk} - 1)x_k| &\leq \sum_{k=1}^{N} |a_{nk} - 1|\ |x_k| + \sum_{k=N+1}^{\infty} |a_{nk} - 1|\ |x_k| \\ &< N.\frac{\varepsilon}{2NM}.M + A.\frac{\varepsilon}{2A}, \text{ in view of (4) and (5)} \\ &= \varepsilon, \end{aligned}$$

so that $\lim_{n\to\infty} \sum_{k=1}^{\infty}(a_{nk} - 1)x_k = 0$. Thus $\lim_{n\to\infty}(Ax)_n = \sum_{k=1}^{\infty} x_k$.
Consequently $A \in (\ell, c\,; P')$ which completes the proof of the theorem.

When $K = R$ or C, the Steinhaus theorem ([4], p. 187, Theorem 14) can be written conveniently in the form $(c, c\,; P) \cap (\ell_\infty, c) = \emptyset$, where $(c, c; P)$ denotes the class of all infinite matrices $A \in (c, c)$ such that $\lim_{n\to\infty}(Ax)_n = \lim_{k\to\infty} x_k$.

We shall call such type of theorems as "Steinhaus type theorems". Such theorems were considered in [2], [3], [8]. Using Theorem 1, we shall deduce one such theorem.

THEOREM 2.2 :

$(\ell, c\,; P') \cap (\ell_p, c) = \emptyset$ whenever $p > 1$.

Proof. :
Suppose $A = (a_{nk}) \in (\ell, c\,; P') \cap (\ell_p, c)$ where $p > 1$. It is known ([11], p. 4, 16) that $A \in (\ell_p, c)$ whenever $p > 1$, if and only if (2) holds and

$$(6) \qquad \sup_n \sum_{k=1}^{\infty} |a_{nk}|^q < \infty,$$

where $\frac{1}{p} + \frac{1}{q} = 1$. It now follows that $\sum_{k=1}^{\infty} |\delta_k|^q < \infty$, which contradicts the fact that

$\delta_k = 1, k = 1,2,...,$ since $A \in (\ell, c\ ; P')$ and consequently $\sum_{k=1}^{\infty} |\delta_k|^q$ diverges. This establishes our claim.

Remark 2.3.
Since $(\ell_\infty, c) \subset (c,c) \subset (c_0, c) \subset (\ell_p, c)$ where $p > 1$, we have $(\ell, c; P') \cap (X, c) = \emptyset$, when $X = \ell_\infty, c, c_0, \ell_p$ where $p > 1$.

3. The case when K is a complete, non-trivially valued, non-archimedean field.

For concepts and results in Analysis over complete, non-trivially valued, non-archimedean fields, we refer to [1]. In this case, Steinhaus type theorems were considered in [6], [7], [8], [10].

When K is a complete, non-trivially valued, non-archimedean field, one can prove that Theorem 2.1 continues to hold. In this case, if $A = (a_{nk}) \in (\ell, c\ ; P') \cap (\ell_\infty, c)$, then $\lim_{n\to\infty} \sup_{k\geq 1} |a_{nk} - 1| = 0$ (see [6], Theorem 2). So for any $\varepsilon, 0 < \varepsilon < 1$, there exists a positive integer N such that

$|a_{nk} - 1| < \varepsilon, n \geq N, k = 1,2,....$

In particular, $|a_{Nk} - 1| < \varepsilon, k = 1,2,....$

Thus $\lim_{k\to\infty} |a_{Nk} - 1| \leq \varepsilon$ i.e. , $|0 - 1| \leq \varepsilon$ (since $A \in (\ell_\infty, c)$, $\lim_{k\to\infty} a_{nk} = 0,\ n = 1,2,...,$ by Theorem 2 of [6]) i.e. , $1, \leq \varepsilon$, a contradiction on the choice of ε. Consequently we have :

Theorem 3.1
When K is a complete, non-trivially valued, non-archimedean field,
$(\ell, c\ ; P') \cap (\ell_\infty, c) = \emptyset$.

Remark 3.2 :
However, $(\ell, c\ ; P') \cap (c, c) \neq \emptyset$ when K is a complete, non-trivially valued, non-archimedean field, as the following example illustrates.

Consider the infinite matrix

$$A = (a_{nk}) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 1 & -1 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 1 & -2 & 0 & 0 & 0 & \cdots \\ 1 & 1 & 1 & -3 & 0 & 0 & \cdots \\ 1 & 1 & 1 & 1 & -4 & 0 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix}$$

$$
\begin{aligned}
i.e., a_{nk} &= 1, k \leq n-1 \quad ; \\
&= -(n-1)\ , \ k = n \quad ; \\
&= 0, \text{otherwise.}
\end{aligned}
$$

Then $\sup_{n,k} |a_{nk}| \leq 1 < \infty$, $\lim_{n\to\infty} a_{nk} = 1, k = 1, 2, \ldots$ and $\lim_{n\to\infty} \sum_{k=1}^{\infty} a_{nk} = 0$ so that $A \in (\ell, c\ ; P') \cap (c, c)$ (for criterion for $A \in (c, c)$, see [5], [9]). Since $(c, c) \subset (c_0, c) \subset (\ell_p, c)$ where $p > 1$, it follows that $(\ell, c; P') \cap (X, c) \neq \emptyset$, when $X = c, c_0, \ell_p$ where $p > 1$. This indicates a violent departure in when K is a non-archimedean valued field from the case $K = R$ or C.
$(c_0, c\ ; P')$ denotes the class of all infinite matrices $A \in (c_0, c)$ such that $\lim_{n\to\infty} (Ax)_n = \sum_{k=1}^{\infty} x_k$ whenever $x = \{x_k\} \in c_0$. In this context it is worthwhile to note that $\sum_{k=1}^{\infty} x_k$ converges if and only if $\{x_k\} \in c_0$.

Remark 3.3 :
$(c_0, c\ ; P') = (\ell, c\ ; P')$.

Proof.
Adapting the proof of Theorem 2.1, with suitable modifications for the non-archimedean case, we have, $A \in (c_0, c\ ; P')$ if and only if (1) and (3) hold. The result now follows.

4. General remarks

It is to be noted that $\ell_p, p \geq 1, c_0, c, \ell_\infty$ are linear spaces with respect to coordinatewise addition and scalar multiplication irrespective of how K is chosen. When $K = R$ or C, c_0, c, ℓ_∞ are Banach spaces with respect to the norm $\|x\| = \sup_{k\geq 1} |x_k|$ where $x = \{x_k\} \in c_0, c$ or ℓ_∞, while they are non-archimedean Banach spaces under the above norm when K is a complete, non-trivially valued, non-archimedean field.

Whatever be K, ℓ_p is a Banach space with respect to the norm

$$\|x\| = \Big(\sum_{k=1}^{\infty} |x_k|^p\Big)^{1/p}, x = \{x_k\} \in \ell_p.$$

Whatever be K, if $A = (a_{nk}) \in (\ell, c\ ; P')$, then A is bounded and $\|A\| = \sup_{n,k} |a_{nk}|$. However, $(\ell, c; P')$ is not a subspace of $BL(\ell, c)$, i.e. , the space of all bounded linear mappings of ℓ into c, since $\lim_{n\to\infty} 2a_{nk} = 2, k = 1, 2, \ldots$ and consequently $2A \notin (\ell, c\ ; P')$ when $A \in (\ell, c\ ; P')$.

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