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Some constructive topological properties of function spaces

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SOME CONSTRUCTIVE TOPOLOGICAL PROPERTIES OF FUNCTION SPACES

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1. The theorem of continuity of mappings from recursive separable complete metric spaces into recursive metric spaces has given rise to a lot of works in constructive analysis.

In [6] Moschovakis introduced the notion of traceability and stability of limit which allows, in the framework of recursive metric spaces (and a fortiori in that of constructive metric spaces) to generalize this theorem.

In the framework of constructive analysis according to the Leningrad school, i.e. in a constructive metalanguage, V. P. Chernov has developed a new notion of constructive topological spaces, the notion of sheaf-space (see [2]). Using this notion, a general formulation of the continuity theorem is obtained, which goes beyond both the separable case and the case of constructive metric spaces. One can reasonably think that the framework introduced by V. P. Chernov allows to include most of complete metric spaces of classical functional analysis, if not all of them.

In particular, a certain precision of this theory, which is presented here allows a constructive study of non-separable metric spaces. The matter is the notion of locally vaguely separable space (see [4]) to which Chernov's results apply. In this definition, the algorithmic condition contained in traceability is replaced by a topological condition.

2. Let us recall that a sheaf-space is a 6-tuple $< \mathfrak{M}, B, I, \sigma, \mathfrak{O}, \mathfrak{L} >$ of constructive objects in which :

- $\mathfrak{M}$ is a space, i.e. a constructive set fitted with an equality relation ;

- $< B, I >$ is a family of subsets of $\mathfrak{M}$ called neighbourhoods and indexed by the constructive set I ;

- σ is a transitive relation on I called succession of indexes, such that $i_1 \ \sigma \ i_2 \ \Rightarrow \ (B)_{i_1} \subset (B)_{i_2}$; a sequence α of consecutive successive indexes is called convergent if one can construct a point X of the space $\mathfrak{M}$ such that the sequence of neighbourhoods indexed by α is decreasing with respect to inclusion and is a fundamental system of neighbourhoods of X ; in that case, it is said that X is a limit of α or that α converges with limit X ; the set of convergent sequences is denoted $\Gamma^{=}$ and is endowed with an equality relation defined by equivalence of the corresponding systems of neighbourhoods ;

- $\mathcal{O}$ is a mapping from $\Gamma^{=}$ into $\mathfrak{M}$ such that for every point X, $\mathcal{O}$ (X) converges with limit X ;

- $\mathcal{L}$ is a mapping from $\Gamma^{=}$ into $\mathfrak{M}$ such that for every α in $\Gamma^{=}$, $\mathcal{L}$ (α) is a limit of α and for every point X of $\mathfrak{M}$, $\mathcal{L}(\mathcal{O}(X)) = X$.

A locally vaguely separable space is a sheaf-space in which every neighbourhood is vaguely separable, i.e. ; for every neighbourhood $(B)_i$ one can construct a triple $\langle \mathfrak{X}, \phi, \mathfrak{T} \rangle$ where $\mathfrak{X}$ is a sheaf-space, ϕ a mapping from $\mathfrak{X}$ into $(B)_i$ and $\mathfrak{T}$ a sequence of points of $\mathfrak{X}$ such that

(i) $\phi(\mathfrak{X})$ is dense in $(B)_i$;

(ii) for every point X of $\mathfrak{X}$ it cannot not be found a point X_1 of $\mathfrak{X}$ such that $\phi(X_1) = \phi(X)$ and a sequence φ extracted from $\mathfrak{T}$ converging with limit X_1 in $\mathfrak{X}$ (see fig. 1).

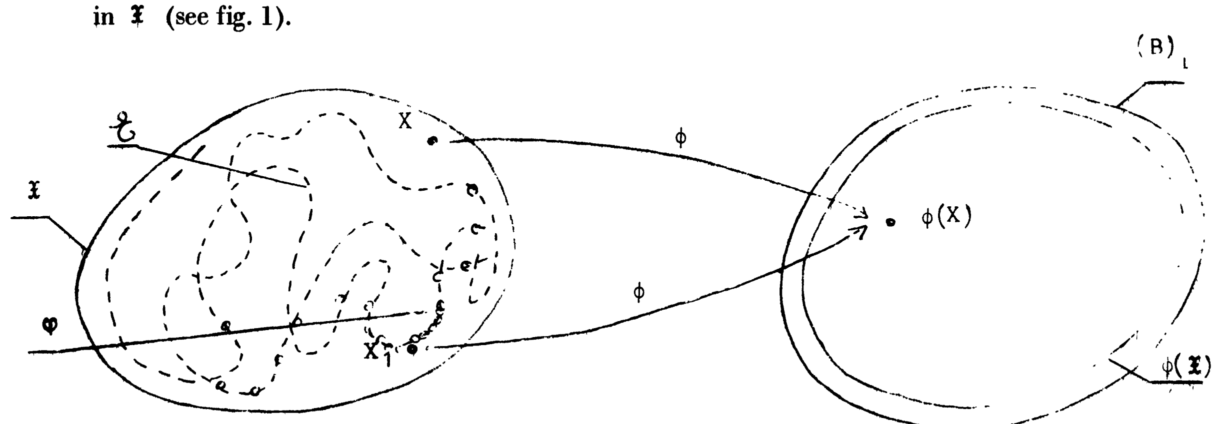


fig. 1

The class of locally vaguely separable spaces naturally contains the class of constructive separable metric spaces. It also contains constructive non-separable topological spaces to which apply the continuity theorem stated above and related theorems.

3. One cannot construct a metrizable constructive translation of a classical non separable metric function space (see [5]).

Let us suppose, that in the case of a classical normed space E_0 the constructive translation of the classical statement «the real number λ is the norm of the element X of E_0» is denoted λ $\underline{\text{norme}}_E$ X. If the space E_0 possesses a family of elements, indexed by the real numbers, such that the members of the family have pairwise a constant positive distance (which is a usual situation in the non separable function spaces of classical analysis), then we obtain

(1) $\neg \forall X \ \exists \lambda \ (\lambda \ \underline{\text{norme}}_E X)$

from which follows the impossibility of a metrizable constructive translation of E_0.

However, it is possible to hope

(2) $\forall X \neg\neg \exists \lambda (\lambda \ \underline{\text{norme}}_E X)$

either for all elements of E_0 or, at least, for a dense subspace of E_0. In that case and if, moreover, the classical space E_0 is complete, one can define a constructive translation E of the space E_0 that would be a sheaf-space.

This applies particularly to spaces of almost periodic functions on the real line (a.p. functions on $\mathbb{R}$).

4. Let us consider the following norms of classical analysis (see, for instance [1]), represented by the generic letter G in the sequel :

U the uniform norm on $\mathbb{R}$ to which corresponds the space of Bohr's a.p. functions,

S_1^p the Stepanhof's norms to which correspond the spaces of Stepanhof's a.p. functions (here p is a real number with $p \geq 1$),

W_p the Weyl's norms p to which correspond the spaces of Weyl's a.p. functions.

B_p the Besicovich's norms p to which correspond the spaces of Besicovich's a.p. functions,

N_1 the norm of absolute convergence of the Fourier series of an a.p. function,

N_∞ the norm of the supremem of the modulus of Fourier coefficients of an a.p. function,

N_p the l_p-norm of the Fourier coefficients of an a.p. function.

λ $\underline{\text{G-norme}}$ X denotes the constructive translation of the classical statement « λ is the G-norm of the element X».

The classical existence of these norms appeals to the notion of almost-periodicity, The constructive translation of the classical statement «the function f is almost periodic» is the formula :

(3) $\forall \varepsilon (\varepsilon > 0 \rightarrow \exists L \ \forall x \ \exists \zeta (x < \zeta < x + L \ \& \ \forall x_1 (|f(x_1+\zeta) - f(x_1)| \leq \varepsilon)))$

where ε, L, x, x_1 and ζ are real numbers. This formula can be written in the form

(4) $\forall n \ \exists S \ P(f, S, n)$ where n is a natural integer S a kind of constructive objets and P a negative formula, using the constructive interpretation (see [7]) of subformula

$$\forall x \quad \exists \zeta (x < \zeta < x + L \ \& \ \forall x_1 (|f(x_1+ \zeta) - f(x_1)| \leq \varepsilon))$$

In the sequel, the expression « f is almost periodic» will mean that formula (3) (or (4)) holds.

By analogy with formulae (1) and (2), we shall consider the formulae

(5) $\neg\neg\forall n \ \exists S \ P(f, S, n)$

(6) $\forall n \ \neg\neg \exists S \ P(f, S, n)$

defining respectively quasi-almost-periodicity and pseudo-almost-periodicity.

It is established in [5] that every trigonometric polynomial is quasi-almost-periodic. From this results

(7) $\forall P \neg\neg \exists \lambda$ (λ G-norme P)

where P is a variable for trigonometric polynomials. But the set of trigonometric polynomials endowed with the G-norm is a non separable (in the meaning given in point 3) normed space. As in formula (1) we deduce that

(8) $\neg\forall P \exists \lambda$ (λ G-norme P)

From formula (7), analogeous to formula (2), we deduce that we may endow the set of constructive trigonometric polynomials with a constructive uniform structure, realising the constructive translation of the classical normed space of trigonometric polynomials endowed with the G-norm. This constructive uniform space is completed by the standard process of constructive completion and can be fitted with a structure of sheaf-space (see [3]). The resulting space, denoted $\mathfrak{P}_G$, is considered as the constructive translation of the space of almost-periodic functions in sense G since, classically, the subspace of trigonometric polynomials is dense in this space.

5. So spaces $\mathfrak{P}_G$ are non metrizable sheaf-spaces since they are non-separable. In fact, we have effective non separability in the sense that for every sequence of elements of $\mathfrak{P}_G$ one can construct an element, far enough from the elements of the sequence. A detailed analysis of the property of quasi-almost-periodicity for trigonometric polynomials allows to show that every space $\mathfrak{P}_G$ is locally vaguely separable (for details, see [5]). From that we deduce, in particular, that every mapping from a space $\mathfrak{P}_G$ into a constructive metric space (in fact a larger class of spaces, see [2]) is continuous at every point where it is defined. In particular, so is the case for numerical functions defined over a space $\mathfrak{P}_G$.

Classically one can classify the different spaces of almost-periodic functions by topological inclusion, i.e. continuous one-one mappings. This classification holds constructively : the proof uses the local vague separability. But we obtain that the constructive spaces $\mathfrak{P}_{B_p}$ and $\mathfrak{P}_{W_p}$ coincide (for same p).

Constructively a second classification appears, which is relative to the properties of support parts that are always G_δ subsets of spaces $\mathfrak{P}_G$. Let us recall that a support part is the domain of definition in the space of a mapping, i.e. of an algorithm, saturated for the equality relation over the elements of the space.

Among spaces $\mathfrak{P}_G$, we observe two contradictory properties of support parts :

(D) Every non empty support part is (effectively) dense in the space.

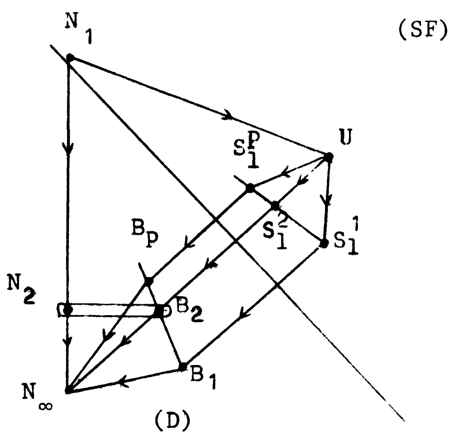
(SF) The non elementhood of a point to a neighbourhood of the space $\mathfrak{P}_G$ is recursively enumerable.

Property (D) yields that every mapping from the space into a constructive metric space is constant. In particular, the algebraic dual space of each space $\mathfrak{P}_G$ which satisfies property (D) is reduced to $\{0\}$. Such is the case, for instance, of the space $\mathfrak{P}_{B_2}$ the classical prototype of which is a hilbertian space.

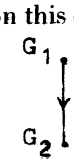
For each space $\mathfrak{P}_G$ satisfying property (SF), one can construct non trivial linear forms and, in particular, one can define integrals on these spaces.

This second classification corresponds to the following classical properties : for the classical prototypes of spaces $\mathfrak{P}_G$ satisfying property (SF) two elements are equal for the G-norm if and only if they are equal almost everywhere ; for the classical prototypes of spaces $\mathfrak{P}_G$ satisfying property (D), two equal elements for the G-norm may be different over sets of infinite measure.

The following diagram sums up both classifications :



on this diagram,



means that one can construct a continuous one-one mapping from G_1 into G_2.

6. Two kinds of applications of these theories can be given :

First for the spaces of almost periodic functions on the real line.

Every element of the space $\mathfrak{P}_U$ defines a pseudo-almost-periodic function, i.e. a function satisfying formula (6). However, one can construct an element of $\mathfrak{P}_U$ which defines a non almost-periodic function, i.e. which defines a function f such that

(9) $\neg\, \forall n \exists S \; P\,(f,\, S,\, n)$

For spaces $\mathfrak{P}_{N_1}$, $\mathfrak{P}_U$ and $\mathfrak{P}_{S_1^P}$, the existence of integrals allows to establish the unicity theorem for the Fourier coefficients of the elements. Let us remark that Fourier coefficients are not defined as ordinary constructive real numbers because of the inclusions between spaces $\mathfrak{P}_G$ and the fact that the «largest» one satisfies property (D), while elements of each classical prototype space of almost-periodic functions do possess Fourier coefficients.

Secondly an other application concerns other function spaces. Until now, the author can indicate that the notion of locally vaguely separable space applies to the spaces of arithmetic almost-periodic functions which constitute an other family of classical non separable metric spaces. The constructive translation obtained according to the scheme indicated here leads to analogeous results including the classification with respect to properties (D) and (SF).

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